Expanders and Morita-compatible exact crossed-products

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EXPANDERS AND MORITA-COMPATIBLE EXACT CROSSED-PRODUCTS

An expander or expander family is a sequence of finite graphs $X_1, X_2, X_3, \ldots$ which is efficiently connected. A discrete group G which "contains" an expander in its Cayley graph is a counter-example to the Baum-Connes (BC) conjecture with coefficients. Some care must be taken with the definition of "contains". M. Gromov outlined a method for constructing such a group. G. Arjantseva and T. Delzant completed the construction. Any group so obtained is known as a Gromov group (or Gromov monster) and these are the only known examples of a non-exact groups.

The left side of BC with coefficients “sees” any group as if the group were exact. This talk will indicate how to make a change in the right side of BC with coefficients so that the right side also “sees” any group as if the group were exact. This corrected form of BC with coefficients uses the unique minimal exact and Morita compatible intermediate crossed-product. For exact groups (i.e. all groups except the Gromov groups) there is no change in BC with coefficients.

In the corrected form of BC with coefficients any Gromov group acting on the coefficient algebra obtained from an expander is not a counter-example.

Thus at the present time (May, 2014) there is no known counter-example to the corrected form of BC with coefficients.

The above is joint work with E. Guentner and R. Willett.
This work is based on — and inspired by — a result of R. Willett and G. Yu, and is very closely connected to results in the thesis of M. Finn-Sell.

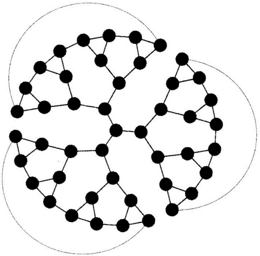
A discrete group Γ which "contains" an expander in its Cayley graph is a counter-example to the usual (i.e. uncorrected) BC conjecture with coefficients.

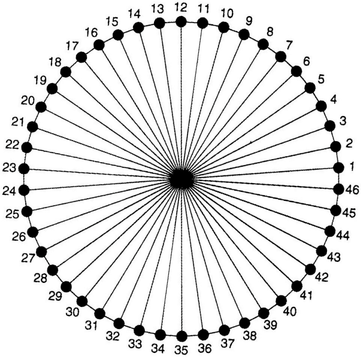
Some care must be taken with the definition of "contains".

An expander or expander family is a sequence of finite graphs

$$X_1, X_2, X_3, \ldots$$

which is *efficiently connected*.





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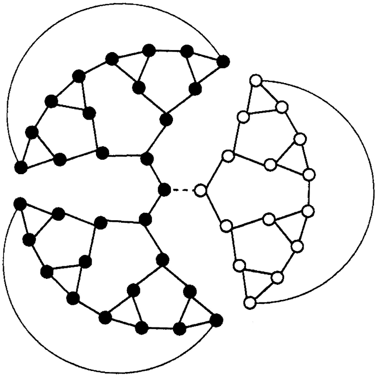

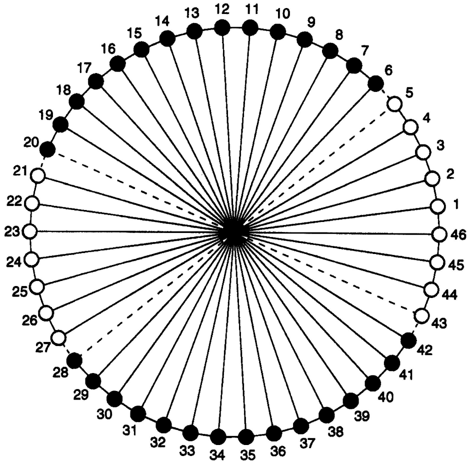

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The isoperimetric constant $h(X)$

For a finite graph X with vertex set V and $|V|$ vertices

$$h(X) =: min\{\frac{|\partial F|}{|F|} \quad | \quad F \subset V \text{ and } |F| \leq \frac{|V|}{2}\}$$

$|F|$ = number of vertices in F. $\quad F \subset V$.
$|\partial F|$ = number of edges in X having one vertex in F and one vetex in $V - F$.

Definition of expander

An expander is a sequence of finite graphs

$$X_1, X_2, X_3, \ldots$$

such that

- Each X_j is conected.
- $\exists$ a positive integer d such that all the X_j are d-regular.
- $|X_n| \to \infty$ as $n \to \infty$.
- $\exists$ a positive real number $\epsilon > 0$ with
  $h(X_j) \geq \epsilon > 0 \qquad \forall j = 1, 2, 3, \ldots.$

G topological group

G is assumed to be :
locally compact, Hausdorff, and second countable.

(second countable = The topology of G has a countable base.)

Examples

Lie groups	$\mathsf{SL}(n, \mathbb{R})$
p-adic groups	$\mathsf{SL}(n, \mathbb{Q}_p)$
adelic groups	$\mathsf{SL}(n, \mathbb{A})$
discrete groups	$\mathsf{SL}(n, \mathbb{Z})$

G topological group

locally compact, Hausdorff, and second countable

example C_r^*G, the reduced C^* algebra of G

Fix a left-invariant Haar measure dg for G

"left-invariant" $=$ whenever $f\colon G \to \mathbb{C}$ is continuous and compactly supported

$$\int_G f(\gamma g)dg = \int_G f(g)dg \qquad \forall \gamma \in G$$

L^2G Hilbert space

$L^2G = \{u\colon G \to \mathbb{C} \mid \int_G |u(g)|^2 dg < \infty\}$

$\langle u, v\rangle = \int_G \overline{u(g)} v(g) dg \qquad u, v \in L^2G$

$\mathcal{L}(L^2G) = C^*$ algebra of all bounded operators $T\colon L^2G \to L^2G$

$C_cG = \{f\colon G \to \mathbb{C} \mid f$ is continuous and f has compact support$\}$

C_cG is an algebra

$(\lambda f)g = \lambda(fg) \qquad \lambda \in \mathbb{C} \quad g \in G$

$(f+h)g = fg + hg$

Multiplication in C_cG is convolution

$$(f * h)g_0 = \int_G f(g)h(g^{-1}g_0)dg \qquad g_0 \in G$$

$0 \to C_cG \to \mathcal{L}(L^2G)$

Injection of algebras

$f \mapsto T_f$

$T_f(u) = f * u \qquad u \in L^2G$

$(f * u)g_0 = \int_G f(g)u(g^{-1}g_0)dg \qquad g_0 \in G$

$C_r^*G \subset \mathcal{L}(L^2G)$

$C_r^*G = \overline{C_cG} =$ closure of C_cG in the operator norm

C_r^*G is a sub C^* algebra of $\mathcal{L}(L^2G)$

Ordinary BC and BC with coefficients are for topological groups G which are locally compact, Hausdorff, and second countable.

$\underline{E}G$ denotes the universal example for proper actions of G.

EXAMPLE. If Γ is a (countable) discrete group, then $\underline{E}\Gamma$ can be taken to be the convex hull of Γ within $l^2(\Gamma)$.

Example

Give Γ the measure in which each $\gamma \in \Gamma$ has mass one.
Consider the Hilbert space $l^2(\mathit{\Gamma})$.
Γ acts on $l^2(\mathit{\Gamma})$ via the (left) regular representation of Γ.
Γ embeds into $l^2(\mathit{\Gamma})$ $\qquad \mathit{\Gamma} \hookrightarrow l^2(\mathit{\Gamma})$
$\gamma \in \Gamma \qquad \gamma \mapsto [\gamma]$ where $[\gamma]$ is the Dirac function at γ.
Within $l^2(\mathit{\Gamma})$ let Convex-Hull(Γ) be the smallest convex set which contains Γ. The points of Convex-Hull(Γ) are all the finite sums

$$t_0[\gamma_0] + t_1[\gamma_1] + \cdots + t_n[\gamma_n]$$

$$\text{with} \quad t_j \in [0,1] \quad j = 0, 1, \ldots, n \quad \text{and} \quad t_0 + t_1 + \cdots + t_n = 1$$

The action of Γ on $l^2(\mathit{\Gamma})$ preserves Convex-Hull(Γ).
$\Gamma\times$Convex-Hull(Γ) $\longrightarrow$Convex-Hull(Γ)
$\underline{E}\Gamma$ can be taken to be Convex-Hull(Γ) with this action of Γ.

$K_j^G(\underline{E}G)$ denotes the Kasparov equivariant K-homology — with G-compact supports — of $\underline{E}G$.

Definition

A closed subset Δ of $\underline{E}G$ is G-compact if:

1. The action of G on $\underline{E}G$ preserves Δ.
and
2. The quotient space Δ/G (with the quotient space topology) is compact.

Definition

$$K_j^G(\underline{E}G) = \varinjlim_{\substack{\Delta \subset \underline{E}G \\ \Delta\ G\text{-compact}}} KK_G^j(C_0(\Delta), \mathbb{C}).$$

The direct limit is taken over all G-compact subsets Δ of $\underline{E}G$.

$K_j^G(\underline{E}G)$ is the Kasparov equivariant K-homology of $\underline{E}G$ with G-compact supports.

Ordinary BC

Conjecture

For any G which is locally compact, Hausdorff and second countable

$$\mathrm{K}_j^G(\underline{E}G) \to \mathrm{K}_j(C_r^* G) \qquad j = 0, 1$$

is an isomorphism

Corollaries of BC

Novikov conjecture $=$ homotopy invariance of higher signatures

Stable Gromov Lawson Rosenberg conjecture (Hanke + Schick)

Idempotent conjecture

Kadison Kaplansky conjecture

Mackey analogy (Higson)

Exhaustion of the discrete series via Dirac induction
(Parthasarathy, Atiyah + Schmid, V. Lafforgue)

Homotopy invariance of ρ-invariants
(Keswani, Piazza + Schick)

G topological group
locally compact, Hausdorff, second countable

Examples

Lie groups ($\pi_0(G)$ finite)	$\mathsf{SL}(n, \mathbb{R})$ OK✓
p-adic groups	$\mathsf{SL}(n, \mathbb{Q}_p)$OK✓
adelic groups	$\mathsf{SL}(n, \mathbb{A})$OK✓
discrete groups	$\mathsf{SL}(n, \mathbb{Z})$

Let A be a $G - C^*$ algebra i.e. a C^* algebra with a given continuous action of G by automorphisms.

$$G \times A \longrightarrow A$$

BC with coefficients

Conjecture

For any G which is locally compact, Hausdorff, and second countable and any $G - C^*$ algebra A

$$\mathrm{K}_j^G(\underline{E}G, A) \to \mathrm{K}_j(C_r^*(G, A)) \qquad\qquad j = 0, 1$$

is an isomorphism.

Definition

$$K_j^G(\underline{E}G, A) = \varinjlim_{\substack{\Delta \subset \underline{E}G \\ \Delta \ G\text{-compact}}} KK_G^j(C_0(\Delta)\,,\, A).$$

The direct limit is taken over all G-compact subsets Δ of $\underline{E}G$.

$K_j^G(\underline{E}G, A)$ is the Kasparov equivariant K-homology of $\underline{E}G$ with G-compact supports and with coefficient algebra A.

THEOREM [N. Higson + G. Kasparov] Let Γ be a discrete (countable) group which is amenable or a-t-menable, and let A be any $\Gamma - C^*$algebra. Then

$$\mu\colon K_j^\Gamma(\underline{E}\Gamma, A) \to K_j C_r^*(\Gamma, A)$$

is an isomorphism. $j = 0, 1$

THEOREM [V. Lafforgue] Let Γ be a discrete (countable) group which is hyperbolic (in Gromov's sense), and let A be any $\Gamma - C^*$algebra. Then

$$\mu\colon K_j^{\Gamma}(\underline{E}\Gamma, A)) \to K_j C_r^*(\Gamma, A)$$

is an isomorphism. $j = 0, 1$

$\mathrm{SL}(3, \mathbb{Z})$??????

Basic property of C^* algebra K-theory

SIX TERM EXACT SEQUENCE

Let

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

be a short exact sequence of C^* algebras.

Then there is a six term exact sequence of abelian groups

$$\begin{array}{ccccc} K_0 I & \longrightarrow & K_0 A & \longrightarrow & K_0 B \\ \uparrow & & & & \downarrow \\ K_1 B & \longleftarrow & K_1 A & \longleftarrow & K_1 I \end{array}$$

DEFINITION. G is *exact* if whenever

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

is an exact sequence of $G - C^*$ algebras, then

$$0 \longrightarrow C_r^*(G, I) \longrightarrow C_r^*(G, A) \longrightarrow C_r^*(G, B) \longrightarrow 0$$

is an exact sequence of C^* algebras.

LEMMA. Let

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

be an exact sequence of $G - C^*$ algebras. Assume that G is exact. Then there is a six term exact sequence of abelian groups

$$\begin{tikzcd}
K_0C_r^*(G, I) \arrow[r] & K_0C_r^*(G, A) \arrow[r] & K_0C_r^*(G, B) \arrow[d] \\
K_1C_r^*(G, B) \arrow[u] & K_1C_r^*(G, A) \arrow[l] & K_1C_r^*(G, I) \arrow[l]
\end{tikzcd}$$

The left side of BC with coefficients “sees” any G as if G were exact.

LEMMA. For any locally compact Hausdorff second countable topological group G and any exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

of $G - C^*$ algebras,
there is a six term exact sequence of abelian groups

$$\begin{tikzcd}
K_0^G(\underline{E}G, I) \arrow[r] & K_0^G(\underline{E}G, A) \arrow[r] & K_0^G(\underline{E}G, B) \arrow[d] \\
K_1^G(\underline{E}G, B) \arrow[u] & K_1^G(\underline{E}G, A) \arrow[l] & K_1^G(\underline{E}G, I) \arrow[l]
\end{tikzcd}$$

QUESTION. Do non-exact groups exist?

ANSWER. If a discrete group Γ "contains" an expander in its Cayley graph, then Γ is not exact.

"contains" $=$ There exists an expander X and a map

$$f\colon X \longrightarrow \mathrm{Cayley\,graph}(\Gamma)$$

such that f is a uniform embedding in the sense of coarse geometry of metric spaces.

Definition of expander

An expander is a sequence of finite graphs

$$X_1, X_2, X_3, \ldots$$

such that

- Each X_j is conected.
- $\exists$ a positive integer d such that all the X_j d-regular.
- $|X_n| \to \infty$ as $n \to \infty$.
- $\exists$ a positive real number $\epsilon > 0$ with $h(X_j) \geq \epsilon > 0 \qquad \forall j = 1, 2, 3, \ldots$.

Precise meaning of "contains"
Let $X_1, X_2, X_3, \ldots$ be an expander.
$\exists$ maps (of sets) $\varphi_1, \varphi_2, \varphi_3, \ldots$

$$\varphi_j\colon \text{vertices}(X_j) \longrightarrow \Gamma \qquad\qquad j = 1, 2, 3, \ldots$$

with

- There is a constant K such that $d(\varphi_j(x), \varphi_j(x')) \leq K d(x, x') \quad \forall j \quad \text{and} \quad \forall x, x' \in X_j$.
- $\underset{n\to\infty}{\text{limit}}(max\{|\varphi_n^{-1}(\gamma)|/|\text{vertices}(X_n)| \quad \gamma \in \Gamma) = 0$.

M.Gromov indicated how a discrete group Γ which “contains” an expander in its Cayley graph might be constructed. Several mathematicians (Silberman, Arjantseva, Delzant etc etc) then worked on the problem of constructing such a Γ.

For a complete proof that such a Γ exists, see the paper of G. Arjantseva and T. Delzant.

If Γ, "contains" an expander in its Cayley graph, then there exists an exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

of $\Gamma - C^*$ algebras,
such that

$$K_0 C_r^*(\Gamma, I) \longrightarrow K_0 C_r^*(\Gamma, A) \longrightarrow K_0 C_r^*(\Gamma, B)$$

is not exact
Since

$$K_0^\Gamma(\underline{E}\Gamma, I) \longrightarrow K_0^\Gamma(\underline{E}\Gamma, A) \longrightarrow K_0^\Gamma(\underline{E}\Gamma, B)$$

is exact, such a Γ is a counter-example to BC with coefficients.

For the construction (given such a Γ) of the relevant exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

of $\Gamma - C^*$ algebras,
see the paper of N.Higson and V. Lafforgue and G. Skandalis.
A is (the closure of) the sub-algebra of $L^\infty(\Gamma)$ consisting of functions which are supported in R-neighborhoods of the expander.

Also, see the thesis of M. Finn-Sell.

Theorem (N. Higson and G. Kasparov)

If Γ is a discrete group which is amenable (or a-t-menable), then BC with coeficients is true for Γ.

Theorem (V. Lafforgue)

If Γ is a discrete group which is hyperbolic (in Gromov's sense), then BC with coefficients is true for Γ.

Possible Happy Ending

A possible happy ending is :
If G is exact, then BC with coefficients is true for G.

PROBLEM. Is BC (i.e.ordinary BC $=$ BC without coefficients) true for $SL(3, \mathbb{Z})$?

STOP!!!! HOLD EVERYTHING!!!!

Consider the result of Rufus Willett and Guoliang Yu:

Theorem

Let Γ be the Gromov group and let A be the Γ - C^* algebra obtained by mapping an expander to the Cayley graph of Γ. Then

$$\mathrm{K}_j^{\Gamma}(\underline{E}\Gamma, A) \to \mathrm{K}_j(C^*_{max}(\Gamma, A)) \qquad j = 0, 1$$

is an isomorphism.

This theorem indicates that for non-exact groups the right side of BC with coefficients has to be reformulated.

For exact groups (i.e. all groups except the Gromov groups) no change should be made in the statement of BC with coefficients.

With G fixed, $\{G - C^*$ algebras$\}$ denotes the category whose objects are all the $G - C^*$ algebras.

Morphisms in $\{G - C^*$ algebras$\}$ are $*$-homomorphisms which are G-equivariant.

$\{C^*$ algebras$\}$ denotes the category of C^* algebras.
Morphisms in $\{C^*$ algebras$\}$ are $*$-homomorphisms.

A crossed-product is a functor , denoted $A \mapsto C^*_\tau(G, A)$
from $\{G - C^*$ algebras$\}$ to $\{C^*$ algebras$\}$

$$C^*_\tau \colon \{G - C^* \text{algebras}\} \longrightarrow \{C^* \text{algebras}\}$$

"intermediate" = "between the max and the reduced crossed-product"

For an intermediate crossed-product C^*_τ there are surjections:

$$C^*_{max}(G, A) \longrightarrow C^*_\tau(G, A) \longrightarrow C^*_r(G, A)$$

Denote by $\tau(G, A)$ the kernel of the surjection $C^*_{max}(G, A) \longrightarrow C^*_\tau(G, A)$

$$0 \longrightarrow \tau(G, A) \longrightarrow C^*_{max}(G, A) \longrightarrow C^*_\tau(G, A) \longrightarrow 0$$

is exact.

Denote by $\epsilon(G, A)$ the kernel of $C^*_{max}(G, A) \longrightarrow C^*_r(G, A)$.

$$0 \longrightarrow \epsilon(G, A) \longrightarrow C^*_{max}(G, A) \longrightarrow C^*_r(G, A) \longrightarrow 0$$

is exact.

An intermediate crossed-product C^*_τ is then a function τ which assigns to each $G - C^*$ algebra A a norm closed ideal $\tau(G, A)$ in $C^*_{max}(G, A)$ such that :

- For each $G - C^*$ algebra A, $\tau(G, A) \subseteq \epsilon(G, A)$.
- For each morphism $A \to B$ in $\{G - C^* \text{ algebras}\}$ the resulting $*$-homomorphism $C^*_{max}(G, A) \to C^*_{max}(G, B)$ maps $\tau(G, A)$ to $\tau(G, B)$.

$$C^*_\tau(G, A) = C^*_{max}(G, A) \,/\, \tau(G, A)$$

An intermediate crossed-product C^*_τ is exact if whenever

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence in $\{G - C^*$ algebras$\}$ the resulting sequence in $\{C^*$ algebras$\}$

$$0 \longrightarrow C^*_\tau(G, A) \longrightarrow C^*_\tau(G, B) \longrightarrow C^*_\tau(G, C) \longrightarrow 0$$

is exact.

Equivalently :
An intermediate crossed-product C^*_τ is exact if whenever

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence in $\{G - C^*$ algebras$\}$ the resulting sequence in $\{C^*$ algebras$\}$

$$0 \longrightarrow \tau(G, A) \longrightarrow \tau(G, B) \longrightarrow \tau(G, C) \longrightarrow 0$$

is exact.

Set $H_G = L^2(G) \oplus L^2(G) \oplus \ldots \qquad \mathcal{K}_G = \mathcal{K}(H_G)$

An intermediate crossed-product C^*_τ is Morita compatible if for any $G - C^*$ algebra A the natural isomorphism of C^* algebras

$$C^*_{max}(G, A \otimes \mathcal{K}_G) = C^*_{max}(G, A) \otimes \mathcal{K}_G$$

descends to give an isomorphism of C^* algebras

$$C^*_\tau(G, A \otimes \mathcal{K}_G) = C^*_\tau(G, A) \otimes \mathcal{K}_G$$

QUESTION. Given G, does there exist a unique minimal intermediate crossed-product which is exact and Morita compatible?

PROPOSITION. (E. Kirchberg, P.Baum& E.Guentner& R.Willett) For any locally compact Hausdorff second countable topological group G there exists a unique minimal intermediate crossed-product which is exact and Morita compatible.

Denote the unique minimal intermediate exact and Morita compatible crossed-product by C^*_{exact}.

Reformulation of BC with coefficients.

CONJECTURE. Let G be a locally compact Hausdorff second countable topological group, and let A be a $G-C^*$ algebra, then

$$K_j^G(\underline{E}G, A) \longrightarrow K_j(C^*_{exact}(G, A)) \qquad\qquad j = 0, 1$$

is an isomorphism.

Theorem (PB and E. Guentner and R. Willett)

Let Γ be a Gromov group and let A be the Γ - C^* algebra obtained by mapping an expander to the Cayley graph of Γ. Then

$$\mathrm{K}_j^{\Gamma}(\underline{E}\Gamma, A) \to \mathrm{K}_j(C^*_{exact}(\Gamma, A)) \qquad j = 0, 1$$

is an isomorphism.

Implications of the corrected conjecture:

Novikov (homotopy invariance of higher signatures) and stable Gromov-Lawson-Rosenberg are implied by the corrected conjecture.

Kadison-Kaplansky : If Γ is a torsion-free discrete group, then in $C_r^*\Gamma$ there are no idempotent elements (other than 0 and 1).

Kadison-Kaplansky is not implied by the corrected conjecture.

Implied by validity of the corrected conjecture:

If Γ is a torsion-free discrete group, then in the Banach algebra $l^1\Gamma$ there are no idempotent elements (other than 0 and 1).

$l^1\Gamma \subset C_r^*\Gamma$

Question. Is it possible for a torsion-free discrete group Γ to have no idempotent elements (other than 0 and 1) in $l^1\Gamma$ — and to have idempotent elements (other than 0 and 1) in $C_r^*\Gamma$?

$$F\Gamma := \left\{ \sum_{\gamma \in \Gamma} \lambda_\gamma [\gamma] \;\middle|\; \begin{array}{l} \text{order } (\gamma) < \infty \\ \lambda_\gamma \in \mathbb{C} \end{array} \right\}$$

Finite Formal Sums

$F\Gamma$ is a vector space over $\mathbb{C}$

$$\left(\sum_{\gamma \in \Gamma} \lambda_\gamma [\gamma] \right) + \left(\sum_{\gamma \in \Gamma} \mu_\gamma [\gamma] \right) = \sum_{\gamma \in \Gamma} (\lambda_\gamma + \mu_\gamma) [\gamma]$$

$$\lambda \left(\sum_{\gamma \in \Gamma} \lambda_\gamma [\gamma] \right) = \sum_{\gamma \in \Gamma} \lambda \lambda_\gamma [\gamma] \qquad \lambda \in \mathbb{C}$$

$F\Gamma$ is a Γ-module

$$\Gamma \times F\Gamma \to F\Gamma$$

$$g \in \Gamma \qquad \sum_{\gamma \in \Gamma} \lambda_\gamma [\gamma] \in F\Gamma$$

$$g \left(\sum_{\gamma \in \Gamma} \lambda_\gamma [\gamma] \right) = \sum_{\gamma \in \Gamma} \lambda_\gamma [g\,\gamma\,g^{-1}]$$

$H_j(\Gamma; F\Gamma) :=$

the j-th homology group of Γ with coefficients the Γ-module $F\Gamma$

$j = 0, 1, 2, \ldots$

Remark

This is standard homological algebra, and is pure algebra (*i.e.* Γ is a discrete group and $F\Gamma$ is a non-topologized module over Γ).

$$\mathrm{ch} : K_*^{\mathrm{top}}(\Gamma) \to H_*(\Gamma; F\Gamma)$$

$$\mathrm{ch} : K_j^{\mathrm{top}}(\Gamma) \to \bigoplus_{\ell} H_{j+2\ell}(\Gamma; F\Gamma)$$

j = 0,1

Proposition

$$K_j^{\mathrm{top}}(\Gamma) \bigotimes_{\mathbb{Z}} \mathbb{C} \to \bigoplus_{\ell} H_{j+2\ell}(\Gamma; F\Gamma)$$

is an isomorphism of vector spaces over $\mathbb{C}$.

$\underline{E}\Gamma$ denotes the universal example for proper actions of Γ.
$\underline{E}\Gamma$ can be taken to be the convex hull of Γ within $l^2(\mathit{\Gamma})$.

Example

Give Γ the measure in which each $\gamma \in \Gamma$ has mass one.
Consider the Hilbert space $l^2(\mathit{\Gamma})$.
Γ acts on $l^2(\mathit{\Gamma})$ via the (left) regular representation of Γ.
Γ embeds into $l^2(\mathit{\Gamma})$ $\qquad \mathit{\Gamma} \hookrightarrow l^2(\mathit{\Gamma})$
$\gamma \in \Gamma \qquad \gamma \mapsto [\gamma]$ where $[\gamma]$ is the Dirac function at γ.
Within $l^2(\mathit{\Gamma})$ let Convex-Hull(Γ) be the smallest convex set which contains Γ. The points of Convex-Hull(Γ) are all the finite sums

$$t_0[\gamma_0] + t_1[\gamma_1] + \cdots + t_n[\gamma_n]$$

$$\text{with} \quad t_j \in [0,1] \quad j = 0, 1, \ldots, n \quad \text{and} \quad t_0 + t_1 + \cdots + t_n = 1$$

The action of Γ on $l^2(\mathit{\Gamma})$ preserves Convex-Hull(Γ).
$\Gamma \times$Convex-Hull(Γ) $\longrightarrow$Convex-Hull(Γ)
$\underline{E}\Gamma$ can be taken to be Convex-Hull(Γ) with this action of Γ.

X topological space

$\Gamma \times X \to X$ continuous action of Γ on X

$\tilde{X} := \{(\gamma, x) \in \Gamma \times X \mid \gamma x = x\}$

$\tilde{X} \subset \Gamma \times X$

$\Gamma \times \tilde{X} \to \tilde{X}$

$g(\gamma, x) = (g\,\gamma\,g^{-1}, gx) \qquad g \in \Gamma,\ (\gamma, x) \in \tilde{X}$

Lemma

$$H_j(\Gamma; F\Gamma) = H_j(\widetilde{\underline{E}\Gamma}/\Gamma; \mathbb{C})$$

$j = 0, 1, 2, \ldots$

$$K_*^{\mathrm{top}}(\Gamma) := \{(M,E)\}/\sim$$

$$\mathrm{ch} : K_*^{\mathrm{top}}(\Gamma) \to H_*(\Gamma; F\Gamma)$$

$$\mathrm{ch}(M,E) := \tilde{\epsilon}_*(\mathrm{ch}_\Gamma(E) \cap \mathrm{td}_\Gamma(M))$$

$$\mathrm{ch}_\Gamma(E) \cap \mathrm{td}_\Gamma(M) \in H_*(\tilde{M}/\Gamma; \mathbb{C})$$

$\mathrm{ch}_\Gamma(E) \cap \mathrm{td}_\Gamma(M)$ is the Atiyah-Singer formula for $\mathrm{Index}_\Gamma(D_E)$

(M, E)

Action of Γ on M is proper

$$\Downarrow$$

$\exists$ a continuous Γ-equivariant map $\epsilon : M \to \underline{E}\Gamma$

$\tilde{\epsilon} : \tilde{M} \to \widetilde{\underline{E}\Gamma}$

$\tilde{\epsilon} : \tilde{M}/\Gamma \to \widetilde{\underline{E}\Gamma}/\Gamma$

$\tilde{\epsilon}_* : H_*(\tilde{M}/\Gamma; \mathbb{C}) \to H_*(\widetilde{\underline{E}\Gamma}/\Gamma; \mathbb{C}) = H_*(\Gamma; F\Gamma)$

$\mathrm{ch} : K_*^{\mathrm{top}}(\Gamma) \to H_*(\Gamma; F\Gamma)$

$\mathrm{ch}(M, E) = \tilde{\epsilon}_*(\mathrm{ch}_\Gamma(E) \cap \mathrm{td}_\Gamma(M))$

$\mathrm{ch}_\Gamma(E) \cap \mathrm{td}_\Gamma(M) \in H_*(\tilde{M}/\Gamma; \mathbb{C})$

How is $\mathrm{ch}_\Gamma(E) \cap \mathrm{td}_\Gamma(M)$ defined ?

Two methods:

(1) Spectral triple + cyclic cohomology

(2) Classical algebraic topology

$$\mathrm{ch} : K_*^{\mathrm{top}}(\Gamma) \to H_*(\Gamma; F\Gamma)$$

$$\mathrm{ch} : K_j^{\mathrm{top}}(\Gamma) \to \bigoplus_{\ell} H_{j+2\ell}(\Gamma; F\Gamma)$$

j = 0,1

Proposition

$$K_j^{\mathrm{top}}(\Gamma) \bigotimes_{\mathbb{Z}} \mathbb{C} \to \bigoplus_{\ell} H_{j+2\ell}(\Gamma; F\Gamma)$$

is an isomorphism of vector spaces over $\mathbb{C}$.

$K_j^{\Gamma}(\underline{E}\Gamma)$ denotes the Kasparov equivariant K-homology — with Γ-compact supports — of $\underline{E}\Gamma$.

Definition

A closed subset Δ of $\underline{E}\Gamma$ is Γ-compact if:

1. The action of Γ on $\underline{E}\Gamma$ preserves Δ.

and

2. The quotient space Δ/Γ (with the quotient space topology) is compact.

Definition

$$K_j^\Gamma(\underline{E}\Gamma) = \varinjlim_{\substack{\Delta \subset \underline{E}\Gamma \\ \Delta\ \Gamma\text{-compact}}} KK_\Gamma^j(C_0(\Delta), \mathbb{C}).$$

The direct limit is taken over all Γ-compact subsets Δ of $\underline{E}\Gamma$.

$K_j^\Gamma(\underline{E}\Gamma)$ is the Kasparov equivariant K-homology of $\underline{E}\Gamma$ with Γ-compact supports.

$$\tau : K_j^{\mathrm{top}}(\Gamma) \to K_j^{\Gamma}(\underline{E}\Gamma)$$
$$(M, E) \mapsto \epsilon_*[D_E]$$

where

$\epsilon : M \longrightarrow \underline{E}\Gamma$ is (as above) a continuous Γ-equivariant map

and

$[D_E] \in KK_{\Gamma}^{j}(C_0(M), \mathbb{C})$ is the element in the Kasparov equivariant K-homology of M determined by D_E.

$j = 0, 1$

Theorem (P.B. + N. Higson + T. Schick)

$$\tau : K_j^{\mathrm{top}}(\Gamma) \to K_j^{\Gamma}(\underline{E}\Gamma)$$

is an isomorphism $j = 0, 1$

Gromov's principle

There is no statement about all finitely presentable discrete groups which is both non-trivial and true.

Theorem

$$K_j^{\mathrm{top}}(\Gamma) \xrightarrow[\cong]{} K_j^{\Gamma}(\underline{E}\Gamma)$$

$$\mathbb{C} \bigotimes_{\mathbb{Z}} K_j^{\mathrm{top}}(\Gamma) \xrightarrow[\cong]{} \bigoplus_{\ell} H_{j+2\ell}(\Gamma; F\Gamma)$$

$$j = 0, 1$$

Question. Does this theorem violate Gromov's principle ?

Theorem (PB and R. Willett)

Let Γ be a Gromov group and let A be the Γ - C^* algebra obtained by mapping an expander to the Cayley graph of Γ. Then

$$\mathrm{K}_j^{\Gamma}(\underline{E}\Gamma, A) \to \mathrm{K}_j(C^*_{exact}(\Gamma, A)) \qquad j = 0, 1$$

is an isomorphism.

Talks given using this file:
January 7, 2011, Joint Mathematics Meetings, New Orleans, Special Session on Expanders
January 4, 2012, Joint Mathematics Meetings, Boston, Special Session on Generalized Cohomology Theories in Engineering Practice
February 27, 2012, IMPAN Non-commutative geometry seminar
March 8, 2012, University of Warsaw, Cathedra Mathematical Methods in Physics
September 8, 2012, University of New Brunswick, Non-Commutative Geometry Workshop
November 7, 2012, University of Hawaii, Non-Commutative Geometry Seminar
November 15, 2012, Tohoku University, Geometry Seminar
December 14, 2012, Australian National University, Canberra, Baum Fest
January 16, 2013, University of Nijmegen, Mathematical Physics Seminar
May 4, 2013, NCGOA, Vanderbilt University, Nashville
June 25, 2013, Fields Institute, Marc Rieffel 75, Toronto