NCG with a twist

Thanks to Piotr

→ This is about index theory & spectral triples where we do not use a trace, but a KMS state or weight, today just for periodic actions of $\mathbb{R}$, so actions of $\mathbb{T}$

→ Lots of nice algebras do not have a faithful trace, but do have a faithful KMS functional, Eg O_n, SU_q2.

Plan: work our way up the dimensions; joint work with

dim 1 Carey, ~~Phillips~~, Nest, Neshveyer, ~~[illegible]~~, Tong & (Marcolli)

dim 2 Not me Dabrowski & Sitarz plus many after; Wagner

dim 3 Kröhner, R, Senior

higher? preliminary work with Varilly, Senior, Sitarz, Kaad

dim 1

Setup: A C^*-algebra A, a strongly cts action of $\mathbb{T}$, $\sigma : \mathbb{T} \to Aut(A)$, & later a functional

$$\phi : \operatorname{dom}\phi \subset A \longrightarrow \mathbb{C}$$

which is norm densely defined, positive, norm lower semicontinuous β-KMS weight.

The latter means that if $\mathcal{A} \subset A$ is the dense set of analytic vectors which also lie in $\operatorname{dom}\phi$, we have

$$\phi(ab) = \phi(\sigma_{i\beta}(b)a) \qquad \forall a,b \in \mathcal{A}$$

Here $\sigma_{i\beta}$ is an extension of σ_t to $i\beta \in \mathbb{C}$

Notation

$\sigma_t(a)$

or

$\sigma_z(a)$

$z = x^{it}$

some $x \in (0,\infty)$.

Eg $\mathcal{O}_n = C^*\left(S_i,\ i=1,\dots,n \ : \ S_i^* S_i = Id,\ \sum^n S_i S_i^* = Id \right)$

$$\sigma_t(S_i) = e^{2\pi i t} S_i$$

$$\phi(S_{i_1}\cdots S_{i_j} S^*_{k_1}\cdots S^*_{k_m}) = \delta_{jm}\,\delta_{i_1 k_1}\cdots\delta_{i_j k_m}\frac{1}{n^m}$$

Eg $SU_q2 = C^*(a, b \ ; \ ab = qba,\ bb^* = b^*b,\ aa^* + bb^* = 1,\ a^*a + q^{-2}bb^* =$

$$\sigma_t(a) = q^{-2it}a, \quad \sigma_t(b) = b$$

$$h(a^i b^j b^{*k}) = \frac{\delta_{i0}\,\delta_{jk}\,q^{-k}}{[k+1]_q}$$

$$h(a^{*i} b^j b^{*k}) = \frac{\delta_{i0}\,\delta_{jk}\,q^{-k}}{[k+1]_q}.$$

Both functionals are states not weights, so we can dispense with the weighty consideration of domains.

OK back to generalities

Step 1 Given C^*-alg A & $\sigma : \mathbb{T} \to Aut(A)$ we build a Kas mod.

Let $\Phi : A \to F := A^\sigma$ be the expectation

$$\Phi(a) = \frac{1}{2\pi}\int_0^{2\pi} \sigma_z(a)\, d\theta, \quad z = e^{i\theta}$$

Define an F-valued inner product on A by

$$(a|b) := \Phi(a^* b)$$

& let $X = X_F$ be the C^*-module completion of A for the C^*-module norm $\|x\|_X^2 = \|(x|x)\|_F$

Since we have a circle action, everything in sight is $\mathbb{Z}$-graded. In particular

$$X = \bigoplus_{k\in\mathbb{Z}} X_k = \bigoplus_{k\in\mathbb{Z}} \Phi_k X .$$

We define a (regular, self-adjoint) unbdd operator

$$\mathrm{dom}\, D = \{ x = \sum x_k \in X : \sum_{k\in\mathbb{Z}} k^2 \|x_k\|_X^2 < \infty \}$$

$$D x = \sum_{k\in\mathbb{Z}} k x_k$$

* D is, up to a constant, the generator of σ_t on A. In fact σ_t is unitarily implemented on X, using the invariance of the inner product, implemented by $e^{itD}\cdot e^{-itD}$

* $[D,a]$ is a bdd endomorphism of X for any $a\in A$. So all that is required to get a Kas mod is to have

$$a(1+D^2)^{-1/2} \in K(X) \longleftarrow \text{define}$$

This is just

$$\sum_{k\in\mathbb{Z}} (1+k^2)^{-1/2}\, a\Phi_k$$

so compactness holds if & only if all $a\Phi_k$ are compact

$K(X)$ norm closed ideal of $\mathcal{L}(X)$ = adj endos generated by $\Theta_{x,y}$ $x,y\in X$ where $\Theta_{x,y} z = x(y|z)$

<u>Thm</u> $a\Phi_k$ is compact $\forall a\in A$ if & only if

$$X_k X_k^* \triangleleft F \quad \text{is a complemented ideal.}$$

In this case $({}_A X_F, D)$ is an unbdd Kas mod A so defines a class $[D]\in KK^1(A,F)$. In fact it is equivariant & so $[D]\in KK^{1,\mathbb{T}}(A,F)$.

Whether we get a Kas mod or not, we always get a "modular spectral triple"

Step 2 Build a modular spectral triple.

Let $H = L^2(A, \phi)$ & observe that we have a right action of F on H, by mult^n. Let $N = (F^{op})'$. Since $\phi|_F$ is a trace on F, N is semifinite & the dual trace on N is denoted

$$\mathrm{Tr}_\phi : N \to \mathbb{C}$$

It satisfies

$$\mathrm{Tr}_\phi(\Theta_{x,y}) = \phi((y|x)) \qquad x, y \in X \subset H$$

sort of $\Theta_{x,y} z = x(y|z)$

Again we can define $D = \sum_{k \in \mathbb{Z}} k \Phi_k$ on H & we define

$$\phi_D : N \to \mathbb{C}$$
$$\phi_D(T) := \mathrm{Tr}_\phi(e^{-\beta D/2} T e^{-\beta D/2})$$

<u>Lemma</u>: For $0 \le f \in \mathrm{dom}\, \phi|_F$ we have

$$\phi_D(f \Phi_k) \le \phi(f) \qquad \forall k, \text{ always}$$

& if $X_k X_k^* = F$ we get equality.

Consequently $\forall s > 1$ $f(1+D^2)^{-s/2} \in \mathcal{L}^1(N^{\sigma^{\phi_D}}, \phi_D)$ & $f(1+D^2)^{-1/2} \in \mathcal{L}^{1,\infty}(N^{\sigma^{\phi_D}}, \phi_D)$.

Here $\sigma_t^{\phi_D}(T) = e^{i\beta D t} T e^{-i\beta D t}$ & the fixed point algebra

$$M = N^{\sigma^{\phi_D}}$$

has trace $\phi_D|_M$.

What did we get?

$$(\mathcal{A}, \mathcal{H}, \mathcal{D}) \qquad (\mathcal{N}, \overset{\text{weight}}{\phi_D}) \supset (\overset{*}{\mathcal{M}}, \overset{\text{trace}}{\phi_D})$$

1. $[\mathcal{D}, a]$ bdd
2. $\mathcal{D}$ affiliated to $\mathcal{M}$ self-adjoint
3. $f(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{M}, \phi_D)$ $\quad f \in$ ~~$\mathcal{A}\mathcal{A}\mathcal{M}\mathcal{A}$~~ $\mathcal{F} \cap \mathrm{dom}\,\phi$.

Roughly then, this is what we mean by a modular spectral triple.

In an NCG sense they are all 1-dim.
Equivariant Index pairings can be computed, twisted cocycles obtained etc.

<u>dim 2</u> <u>Podles sphere</u>

So our example of SU_q2 comes back now

In this example, the circle action factors as two commuting circle actions.

First let me write down a new vector space basis of SU_q2, given by the matrix elements of irreducible corepresentation

Peter-Weyl $\qquad t^{l}_{ij} \qquad l \in \frac{1}{2}\mathbb{N} \quad i,j \in \{-l, -l+1, \ldots, l\}$

The two circle actions are given on this basis by

$$\sigma_{L,t}(t^{l}_{rs}) = q^{2its} \qquad \sigma_{R,t}(t^{l}_{rs}) = q^{2itr}$$

$$\sigma_t(t^{l}_{r,s}) = q^{2it(r+s)}$$

The ~~f~~ algebra $A = SU_q 2$ breaks up under the left action as

$$A = \oplus B_n$$
$$B_n = \{ a \in A : \sigma_{L,z}(a) = z^n(a) \}$$
$$= \operatorname{span}\{ t^l_{i\, n/2} \}$$

$B_0 = A^{\sigma_L}$ is the Podles sphere.

Let $\quad H = L^2(B_1, h|_B) \oplus L^2(B_{-1}, h|_B)$

$$\langle b_1 \oplus b_{-1}, c_1 \oplus c_{-1} \rangle = h(b_1^* c_1 + b_{-1}^* c_{-1})$$

B acts on the left by mult^n.

The dual Hopf algebra to $SU_q 2$ is $U_q(su(2))$ with generators k, e, f. Using the duality pairing & the Hopf structure we get left & right actions of $U_q(su(2))$ on $SU_q 2$: for $f \in U_q(su(2))$ & $a \in SU_q 2$

$$f \triangleright a := \sum a_{(1)} \langle f, a_{(2)} \rangle, \quad a \triangleleft f = \sum a_{(2)} \langle f, a_{(1)} \rangle$$

$\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ Sweedler notation & $\langle \cdot, \cdot \rangle$ dual pairing.

The generators act as follows

$$k^n \triangleright t^l_{ij} = q^{nj} t^l_{ij} \qquad t^l_{ij} \triangleleft k^n = q^{ni} t^l_{ij}$$

$$e \triangleright t^l_{ij} = \alpha^l_{j+1} t^l_{i\, j+1} \qquad f \triangleright t^l_{ij} = \alpha^l_j t^l_{ij}$$

$$\alpha^l_j := \sqrt{[l+\tfrac{1}{2}]_q^2 - [j-\tfrac{1}{2}]_q^2}$$

$$[s]_q = \frac{q^{-s}-q^s}{q^{-1}-q}$$

We denote $K \triangleright = \partial_K$ $e \triangleright = \partial_e$ $f \triangleright = \partial_f$
& observe that

$$\partial_K : B_n \to B_n \quad , \quad \partial_e : B_n \to B_{n+2} \quad , \quad \partial_f : B_n \to B_{n-2}$$

This means that we can define (densely) an operator

$$D_B := \begin{bmatrix} 0 & \partial_e \\ \partial_f & 0 \end{bmatrix} : \begin{matrix} L^2(B_1) \\ \oplus \\ L^2(B_{-1}) \end{matrix} \circlearrowleft$$

D_B has eigenvalues $\pm [l + \frac{1}{2}]_q$, $\begin{pmatrix} t^l_{i \frac{1}{2}} \\ \pm t^l_{i -\frac{1}{2}} \end{pmatrix}$

* $[D_B, b]$ bdd for $b \in B$, $= \begin{pmatrix} 0 & q^{1/2} \partial_e(b) \\ q^{-1/2} \partial_f(b) & 0 \end{pmatrix}$
* $(1 + D_B^2)^{-\varepsilon}$ trace class $\forall \varepsilon > 0$
* If we define $\Delta_R t^l_{ij} := q^{2i} t^l_{ij}$ then

$$\Delta_R^{-1} \, b \, \Delta_R = \sigma_{R,-i}(b) =: \sigma_R(b)$$

* $h|_B$ is KMS wrt $\sigma_{R,t}$ &

$$h|_B(b_1 b_2) = h|_B(\sigma_{R,-i}(b_2) b_1)$$

* $Tr(\Delta_R^{1/2} (1 + D_B^2)^{-s/2} \Delta_R^{1/2}) < \infty$ for $Re(s) > 2$, holomorphic there, & has simple pole at $s=2$.

What have we got? (B, H, D_B, γ) $(B(H), Tr(\Delta_R^{1/2} \cdot \Delta_R^{1/2}))$
$\gamma = [1 \; {-1}]$, $\gamma b = b \gamma$, $\gamma D = -D \gamma$ $(B(H)^{\sigma_R}, Tr(\Delta_R^{1/2} \cdot \Delta_R^{1/2}))$
$[D_B, b]$ bdd
D_B affiliated to $B(H)^{\sigma_R}$ selfadjoint
$b(1 + D^2)^{-s/2} \in \mathcal{L}^1(B(H)^{\sigma_R}, Tr(\Delta_R^{1/2} \cdot \Delta_R^{1/2}))$, $b \in B^{\sigma_R}$
So a 2 dim modular spectral triple.

Index pairings (Wagner, Neshveyev & Tuset) & cocycles (Krähmer & Wagner, Neshveyev & Tuset) computed etc.

More recently Krähmer & Wagner computed the "top degree part" of the Chern character of $(\mathcal{B}, \mathcal{H}, D_{\mathcal{B}})$ in twisted cyclic cohomology.

The ~~corresponds to~~ twisting is by $\sigma_R^{-1} = \sigma_{R,-i}^{-1}$. Amazingly

$$\mathrm{Tr}\left(\Delta_R^{-1/2}(1+D^2)^{-s/2}\Delta_R^{-1/2}\right) < \infty \quad \text{for } \mathrm{Re}(s)>2 \;!!$$

It is $HH_2^{\sigma_R^{-1}}(\mathcal{B}) = \mathbb{C}$ which is nontrivial (& dual), not $HH_2^{\sigma_R}(\mathcal{B}) = 0$. A generator was found by U. Krähmer:

$$\varphi(b_0, b_1, b_2) = \varepsilon\left(b_0 (k^{-1}e \triangleright b_1)(k^{-1}f \triangleright b_2)\right)$$

$$b_0, b_1, b_2 \in \mathcal{B}$$

$\varepsilon(t^l_{jk}) = \delta_{jk}$ is the counit of $SU_q 2$

Then

$$\phi(b_0, b_1, b_2) := \operatorname*{Res}_{z=2} \mathrm{Tr}_{\mathcal{H}_{\mathcal{B}}}\left(\Delta_R^{-1} \gamma\, b_0 [D_{\mathcal{B}}, b_1][D_{\mathcal{B}}, b_2](1+D^2)^{-z/2}\right)$$

is a σ_R^{-1} twisted Hochs cocycle &

$$\phi = \text{const}\; \varphi \qquad \boxed{\gamma = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}}$$

So in a homological as well as analytic sense, this is the "correct" object to capture the geometry of the Podlés sphere. There are also Poincaré duality isos, spin structures etc etc etc.

<u>Dim 3</u> So, can we get to dim 3 with SU_q2?

Here is the idea. The left circle action gives

$$({}_A X_B, D_L) \in KK^1(A, B)$$

& Podles gives $({}_B H_B, D_B, \gamma) \in KK^0(B, \mathbb{C})$

(suitably equivariant). Then we can try to put these together using the Kasparov product

Not quite right.... $X \otimes_B H_B$ ⟵ need operator here

$$X_B = \bigoplus X_{B,K}$$ each one finite projective. Gives a canonical connection on each

$$\nabla_K : X_{B,K} \to X_{B,K} \otimes_B \Omega^1(B)$$

Define a map

$$c_B : \Omega^1(B) \otimes_B H_B \longrightarrow H_B$$

$$c_B(b_0 \delta(b_1) \otimes_B \xi) = b_0 [D_B, b_1] \xi$$

Then with $\nabla = \bigoplus \nabla_K$ define

$$D = \left(D_L \otimes \gamma + 1 \otimes D_B + (1 \otimes c_B) \circ \nabla \otimes 1 \right)$$

Van Suijlekom

me Steve & Joe

Works on manifold to produce representative of Kasparov product. Somehow canonical.

Then we get, after unpacking isos etc

$$(\mathcal{A}, H, D)$$

$H = L^2(A, h) \oplus L^2(A, h)$, A acts by left mult^n

$$D = \sum_{n \in \mathbb{Z}} \begin{pmatrix} n\,\bar{\Phi}_{n+1} & q^{n/2}\partial_e \\ q^{n/2}\partial_f & -n\,\bar{\Phi}_{n-1} \end{pmatrix} : \bigoplus \begin{matrix} H_{n+1} \\ H_{n-1} \end{matrix} \circlearrowleft$$

$$= \begin{bmatrix} D_L & 0 \\ 0 & -D_L \end{bmatrix} + \Delta_L^{1/2} \begin{bmatrix} 0 & \partial_e \\ \partial_f & 0 \end{bmatrix}$$

$$= S + T.$$

$$\Delta_L t^l_{rs} := q^{2s} t^l_{rs}$$

(1) $[S, a]$ bdd $\quad [T, a] = \begin{bmatrix} 0 & \partial_e(\sigma_L^{-1/2}(a)) \\ \partial_f(\sigma_L^{-1/2}(a)) & 0 \end{bmatrix} \Delta_L$

not bdd.

(2) Eigs of D^2 are

$$\lambda^2_{l,n} = n^2 + q^n\left([l+\tfrac{1}{2}]_q^2 - [\tfrac{n}{2}]_q^2\right)$$

[$q=1$ limit

replace D_L with $\frac{1}{2}D_L$ & get correct classical limit]

can't change $[\frac{n}{2}]$ to $[n]$ without screwing up the twists.

Now modular operators are typically unbounded on one half of the Hilbert space & VERY bounded on the other. No exceptions here.

So we could try to deal with all this unbddness by just chopping it out.
Crude but effective!

Before looking at this, we point out that Uli Krähmer & Tom Hadfield found the fundamental Hochschild cocycle on $SU_q 2$.

It is twisted by $\sigma^{-1} = \sigma_L^{-1} \circ \sigma_R^{-1}$. Here is the formula.

Define $\int_{[1]} : SU_q 2 \longrightarrow \mathbb{C}$

$$\int_{[1]} a^i b^j b^{*k} = \delta_{i,0}\, \delta_{j,0}\, \delta_{k,0}$$

twisted trace for almost every automorphism

& write $k = q^H$. Then the fun cocycle is

$$\varphi(a_0, a_1, a_2, a_3) = \int_{[1]} \left(k^{-4} \triangleright (a_0 (H \triangle a_1))\right)\left(k^{-3} \triangleright \partial_e(a_1)\right)\left(k^{-1} \triangleright \partial_f(a_3\right.$$

Observe $H \sim D_L, \sim \log \Delta_L$, $\Delta_L \sim k^2$

H ordinary derivation, ∂_e, ∂_f twisted derivations

Really, really subtle beast.

Theorem Let P be projection on $\{n>0\} \cap (\ker T)^{\perp}$

(1) For $\alpha \in SU_q 2$

$$\operatorname*{Res}_{z=3} \operatorname{Tr}\left(P \Delta_R^{-1} \Delta_L \, \alpha \, (1+D^2)^{-z/2}\right) = \text{const} \int_{[1]} \alpha$$

(2) $$\varphi(a_0,a_1,a_2,a_3) = \text{const} \operatorname*{Res}_{z=3} \operatorname{Tr}\left(P \Delta_R^{-1} \Delta_L^{-1} a_0 [D,a_1][D,a_2][D,a_3] (1+D^2)^{-z/2}\right)$$

__Idea:__ Crude bit: use P to kill off anything we don't like

Nice bit: $[D,a_1][D,a_2][D,a_3]$

$$\sim [S,a_1][T,a_2][T,a_3] + \text{bdd } \Delta_L^2$$

bdd bit carries twists as we moved Δ_L's through. Permuting T's & S's gives (up to consts) equivalent cocycles. Similarly permuting ∂_e & ∂_f terms gives equiv cocycles

3 S terms cancels

1 or 3 T terms is off-diagonal

leaves only perms of those above.

Moreover

$$\Delta_L^{-2} a_0 [D,a_1][D,a_2][D,a_3] =_{\text{const}} \left(K^{-4} \triangleright (a_0 (H \triangleright a_1))\right)\left(K^{-3} \triangleright \partial_e(a_2)\right)\left(K^{-1} \triangleright \partial_f(a_3)\right)$$

$+$ rubbish

(ie necessarily in kernel of functional

→ Unbounded commutators appear to be NECESSARY to get the correct twists.

→ Our current approach to summability is, as I said, crude but effective.

More understanding necessary if we are to get the full local index formula.

→ If we let $F_D = D(1+D^2)^{-1/2}$

1. Is (A, H, F_D) a Fred mod?
2. If so, does (A, H, F_D) rep the Kas product?
3. If so, then we have a new type [& very hard to characterise type] unbdd representative of a Fred mod

→ Can we compute a version of the local index formula using this beast?