Bounded operator

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In functional analysis, a branch of mathematics, a **bounded linear operator** is a linear transformation *L* between normed vector spaces *X* and *Y* for which the ratio of the norm of *L*(*v*) to that of *v* is bounded by the same number, over all non-zero vectors *v* in *X*. In other words, there exists some *M* > 0 such that for all *v* in *X*

$$\|Lv\|_Y \leq M\|v\|_X.$$

The smallest such *M* is called the operator norm $\|L\|_{\mathrm{op}}$ of *L*.

A bounded linear operator is generally not a bounded function; the latter would require that the norm of *L*(*v*) be bounded for all *v*, which is not possible unless *Y* is the zero vector space. Rather, a bounded linear operator is a locally bounded function.

A linear operator on a metrizable vector space is bounded if and only if it is continuous.

Contents

Examples

- Any linear operator between two finite-dimensional normed spaces is bounded, and such an operator may be viewed as multiplication by some fixed matrix.

- Many integral transforms are bounded linear operators. For instance, if
  $$K : [a, b] \times [c, d] \to \mathbb{R}$$
  is a continuous function, then the operator L, defined on the space $C[a, b]$ of continuous functions on $[a, b]$ endowed with the uniform norm and with values in the space $C[c, d]$, with L given by the formula
  $$(Lf)(y) = \int_a^b K(x, y) f(x)\, dx,$$
  is bounded. This operator is in fact compact. The compact operators form an important class of bounded operators.

- The Laplace operator
  $$\Delta : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$
  (its domain is a Sobolev space and it takes values in a space of square integrable functions) is

bounded.

- The shift operator on the l^2 space of all sequences (x_0, x_1, x_2...) of real numbers with $x_0^2 + x_1^2 + x_2^2 + \cdots < \infty$,
$$L(x_0, x_1, x_2, \dots) = (0, x_0, x_1, x_2, \ldots)$$
is bounded. Its operator norm is easily seen to be 1.

Equivalence of boundedness and continuity

As stated in the introduction, a linear operator L between normed spaces X and Y is bounded if and only if it is a continuous linear operator. The proof is as follows.

- Suppose that L is bounded. Then, for all vectors v and h in X with h nonzero we have
$$\|L(v+h) - Lv\| = \|Lh\| \leq M\|h\|.$$
Letting h go to zero shows that L is continuous at v. Moreover, since the constant M does not depend on v, this shows that in fact L is uniformly continuous (Even stronger, it is Lipschitz continuous.)

- Conversely, it follows from the continuity at the zero vector that there exists a $\delta > 0$ such that $\|L(h)\| = \|L(h) - L(0)\| \leq 1$ for all vectors h in X with $\|h\| \leq \delta$. Thus, for all non-zero v in X, one has
$$\|Lv\| = \left\| \frac{\|v\|}{\delta} L\left(\delta \frac{v}{\|v\|}\right) \right\| = \frac{\|v\|}{\delta} \left\| L\left(\delta \frac{v}{\|v\|}\right) \right\| \leq \frac{\|v\|}{\delta} \cdot 1 = \frac{1}{\delta}\|v\|.$$
This proves that L is bounded.

Linearity and boundedness

Not every linear operator between normed spaces is bounded. Let X be the space of all trigonometric polynomials defined on [−π, π], with the norm

$$\|P\| = \int_{-\pi}^{\pi} |P(x)|\, dx.$$

Define the operator L:X→X which acts by taking the derivative, so it maps a polynomial P to its derivative P'. Then, for

$$v = e^{inx}$$

with n=1, 2, ..., we have $\|v\| = 2\pi$, while $\|L(v)\| = 2\pi n \to \infty$ as n→∞, so this operator is not bounded.

It turns out that this is not a singular example, but rather part of a general rule. Any linear operator defined on a finite-dimensional normed space is bounded. However, given any normed spaces X and Y with X infinite-dimensional and Y not being the zero space, one can find a linear operator which is not continuous from X to Y.

That such a basic operator as the derivative (and others) is not bounded makes it harder to study. If, however, one defines carefully the domain and range of the derivative operator, one may show that it is a closed operator. Closed operators are more general than bounded operators but still "well-behaved" in many ways.

Further properties

The condition for L to be bounded, namely that there exists some M such that for all v

$$\|Lv\| \leq M\|v\|,$$

is precisely the condition for L to be Lipschitz continuous at 0 (and hence, everywhere, because L is linear).

A common procedure for defining a bounded linear operator between two given Banach spaces is as follows. First, define a linear operator on a dense subset of its domain, such that it is locally bounded. Then, extend the operator by continuity to a continuous linear operator on the whole domain (see continuous linear extension).

Properties of the space of bounded linear operators

- The space of all bounded linear operators from U to V is denoted by $B(U,V)$ and is a normed vector space.
- If V is Banach, then so is $B(U,V)$,
- from which it follows that dual spaces are Banach.
- For any A in $B(U,V)$, the kernel of A is a closed linear subspace of U.
- If $B(U,V)$ is Banach and U is nontrivial, then V is Banach.

Topological vector spaces

The boundedness condition for linear operators on normed spaces can be restated. An operator is bounded if it takes every bounded set to a bounded set, and here is meant the more general condition of boundedness for sets in a topological vector space (TVS): a set is bounded if and only if it is absorbed by every neighborhood of 0. Note that the two notions of boundedness coincide for locally convex spaces.

This formulation allows one to define bounded operators between general topological vector spaces as an operator which takes bounded sets to bounded sets. In this context, it is still true that every continuous map is bounded, however the converse fails; a bounded operator need not be continuous. Clearly, this also means that boundedness is no longer equivalent to Lipschitz continuity in this context.

A converse does hold when the domain is pseudometrisable, a case which includes Fréchet spaces. For LF spaces, a weaker converse holds; any bounded linear map from an LF space is sequentially continuous.

See also

- Operator algebra
- Operator theory
- Unbounded operator

References

- Kreyszig, Erwin: *Introductory Functional Analysis with Applications*, Wiley, 1989

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