The von Neumann Algebras of Quantum Permutation Groups

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From permutations to quantum permutations

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- **To quantize S_n:** Replace S_n with the **Hopf algebra** $C(S_n)$, then "deform" $C(S_n)$ to get a genuine quantum group.
- Let $U : S_n \hookrightarrow M_n(\mathbb{C})$; $U(g) = [u_{ij}(g)] \in M_n(\mathbb{C})$ be the "permutation matrix" representation.
- Easy to check: The coordinate functions $u_{ij} : S_n \to \mathbb{C}$ generate $C(S_n)$, obviously commute, and satisfy:

$$u_{ij} = u_{ij}^* = u_{ij}^2 \text{ and } U = [u_{ij}] \text{ is unitary in } M_n(C(S_n)).$$

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Hopf algebra maps Δ, κ, ϵ on $C(S_n)$ encode group structure of S_n:

$$\underbrace{\Delta u_{ij} = \sum_k u_{ik} \otimes u_{kj},}_{\text{coproduct } \Delta f(x,y)=f(xy)} \qquad \underbrace{\kappa(u_{ij}) = u_{ji}}_{\text{co-inverse } \kappa f(x)=f(x^{-1})} \quad , \quad \underbrace{\epsilon(u_{ij}) = \delta_{ij}}_{\text{co-unit } \epsilon f=f(e)}$$

The quantum permutation group S_n^+

Definition/Theorem (Wang 1998)

Consider the universal unital C*-algebra

$$A_s(n) = C^*\Big(\{v_{ij}\}_{i,j=1}^n \mid V = [v_{ij}] \text{ is unitary } \& \ v_{ij} = v_{ij}^2 = v_{ij}^*\Big),$$

and endow $A_s(n)$ with a Hopf C*-algebra structure just like $C(S_n)$:

$$\text{(coproduct) } \Delta : A_s(n) \to A_s(n) \otimes A_s(n); \quad \Delta v_{ij} = \sum_k v_{ik} \otimes v_{kj},$$

$$\text{(co-inverse) } \kappa : A_s(n) \to A_s(n)^{\mathrm{op}}; \quad \kappa(v_{ij}) = v_{ji},$$

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- Note: $\exists$ natural quotient maps $A_s(n) \twoheadrightarrow C(S_n) \implies S_n < S_n^+$.
- Terminology "quantum permutation group" is justified: S_n^+ is the universal quantum automorphism group acting on $C(X_n)$.

The reduced operator algebras on S_n^+

- S_n^+ has a unique **Haar integral**. I.e., a state $h : A_s(n) \to \mathbb{C}$, which is Δ-invariant:

$$(h \otimes \mathrm{id})\Delta(x) = (\mathrm{id} \otimes h)\Delta(x) = h(x)1.$$

- Do the GNS construction:

$L^2(S_n^+) = L^2(A_S(n), h), \quad \lambda : A_s(N) \to \mathcal{B}(L^2(S_n^+))$ GNS representation.

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- Heuristic Model: $A_s(N) = C^*_{\mathrm{full}}(\widehat{S_n^+})$, $C_{\mathrm{red}}(S_n^+) = C^*_{\mathrm{red}}(\widehat{S_n^+})$ and $L^\infty(S_n^+) = \mathcal{L}(\widehat{S_n^+})$, where $\widehat{S_n^+}$ is the dual discrete quantum group.

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- We will mainly focus on the structure of $L^\infty(S_n^+)$.

Some known results on S_n^+ and $L^\infty(S_n^+)$

- (Wang 1998) If $1 \leq n \leq 3$, $A_s(n) \cong C(S_n)$. I.e., $S_n = S_n^+$.
- (Wang 1998) If $n \geq 4$, $A_s(n)$, $C_{\text{red}}(S_n^+)$, and $L^\infty(S_n^+)$) are non-commutative and infinite dimensional.

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Question (Banica+Collins 2008)

What can be said about $L^\infty(S_n^+)$ $n \geq 5$? Is it a II_1-factor?

Approximation, factoriality and fullness for $L^\infty(S_n^+)$

In the non-injective regime $n \geq 5$:

Theorem (B. 2011)

If $n \geq 8$, $L^\infty(S_n^+)$ is a **full type II_1-factor**. *Moreover, $L^\infty(S_n^+)$ has the* **Haagerup property (HP)** *for all $n \geq 5$.*

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Note:

- A finite vN. algebra (M, τ) has the **HP** if $\exists$ a net of τ-preserving, normal, UCP maps $\Phi_t : M \to M$ s.t.
  1. $\forall t, \Phi_t : L^2(M, \tau) \to L^2(M, \tau)$ is **compact**,
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- A II_1-factor (M, τ) is **full (or non-Gamma)** if for any sequence

$$\{x_n\}_n \subset \mathcal{U}(M) \quad \text{s.t. } \|x_n y - y x_n\|_2 \to 0 \ \forall y \in M,$$
$$\implies \|x_n - \tau(x_n)1\|_2 \to 0.$$

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- Classical examples of vN. algebras with above properties are $\mathcal{L}(\mathbb{F}_n)$, $n \geq 2$ (or $\mathcal{L}(\Gamma)$ for any non-amenable i.c.c. hyperbolic group Γ).
- Factoriality/fullness remains open when $5 \leq n \leq 7$!

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A d-**dimensional unitary representation** of S_n^+ is a unitary operator $W = [w_{ij}] \in M_d(A_S(n))$ s.t. $\Delta w_{ij} = \sum_{k=1}^d w_{ik} \otimes w_{kj}$ $\forall i, j$.

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Usual constructions: Direct sum $W^1 \oplus W^2$, tensor products $W^1 \boxtimes W^2 = [w^1_{ij} w^2_{kl}] \in M_{d_1 d_2}(A_s(n))$, conjugate representation $\overline{W} = [w^*_{ij}]$, unitary equivalence $\sim$ and irreducibility.

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Theorem (Banica 1999)

$\exists$ a maximal family of inequivalent finite dimensional irreducible unitary reps. $\{W^x\}_{x=0}^\infty$, where $W^x = [w^x_{ij}] \in M_{d_x}(A_s(n))$, such that

- $W^0 = 1_{A_s(N)}$, $V \cong W^0 \oplus W^1$,
- $W^x \sim \overline{W^x}$, *$(x \geq 0)$.*
- $W^1 \boxtimes W^x \sim W^{x+1} \oplus W^x \oplus W^{x-1}$, $(x \geq 1)$ **"fusion rules"**.

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Peter-Weyl decomposition of L^2:
$L^2(S_n^+) = \bigoplus_{x \geq 0} L^2_x(S_n^+)$, $\quad L^2_x(S_n^+) = span\{\Lambda_h(w^x_{ij}) : 1 \leq i, j \leq d_x\}$.

The Haagerup property

To study the HP, we search for a simple class of NUCP maps on $L^\infty(S_n^+)$: To each $\psi \in \ell^\infty(\mathbb{N}_0)$, associate

$$M_\psi = \bigoplus_{x \geq 0} \psi(x) \mathrm{id}_{L^2(S_n^+)} \in \mathcal{B}(L^2(S_n^+)).$$

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For $x \in \mathbb{N}_0$, consider the **character** $\chi_x = (Tr \otimes id)W^x \in A_s(n)$. *Then $\psi \in \ell^\infty(\mathbb{N}_0)$ is a radial multiplier iff $\exists$ a state $\psi \in C^*\big(\chi_x : x \in \mathbb{N}_0\big)^*$ s.t.*

$$\psi(x) = \frac{\varphi(\chi_x)}{\dim W^x}.$$

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$$\psi(x) = \frac{\varphi(\chi_x)}{\dim W^x}.$$

- But since $W^1 \boxtimes W^x \sim W^{x+1} \oplus W^x \oplus W^{x-1} \implies \chi_1\chi_x = \chi_{x+1} + \chi_x + \chi_{x-1}$ $\implies C^*(\chi_x : x \in \mathbb{N}_0) = C^*(1, \chi_1)$ - **commutative**!

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- Taking dirac measures δ_t $(4 < t < n)$ yields a net of radial multipliers M_{ψ_t} s.t. $0 < \psi_t(x) \leq C(t/n)^x$ and $\lim_{t\to n} M_{\psi_t} = \text{id}$ pointwise $\implies$ HP.

Factoriality and fullness

- Consider the irrep. $W^1 = [w^1_{ij}] \sim V \ominus 1$, acting on $\mathbb{C}^{n-1}$ with ONB $\{e_i\}_{i=1}^{n-1}$. **Observe:** $L^\infty(S_n^+) = \{\lambda(w^1_{ij})\}''$.

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- Consider tensor powers of fundamental rep. $V^{\boxtimes x}$ and write W^x as the **subrepresentation** $W^x = Q_x V^{\boxtimes x} Q_x \subset V^{\boxtimes x}$, where

$$Q_x = Q_x^* = Q_x^2 \in \mathrm{Mor}(V^{\boxtimes x}, V^{\boxtimes x}) = \left\{S \in M_{n^x}(\mathbb{C}) \mid V^{\boxtimes x}S = SV^{\boxtimes x}\right\}.$$

- (Banica 1999) If $n \geq 4$, $\mathrm{Mor}(V^{\boxtimes x}, V^{\boxtimes x}) \cong TL_{2x}(\sqrt{n})$, the **Temperley-Lieb** planar algebra at index $\sqrt{n}$.

$$TL_{2x}(\sqrt{n}) = C^*\left(1, f_1, \ldots, f_{2x-1} \;\middle|\; \substack{f_i^* = f_i = f_i^2,\ f_i f_{i\pm 1} f_i = \frac{1}{\sqrt{n}} f_i, \\ f_i f_j = f_j f_i \text{ when } |i-j| \geq 2}\right)$$

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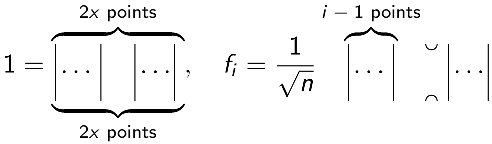

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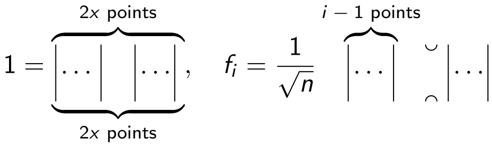

- By considering the **Jones-Wenzl projection**

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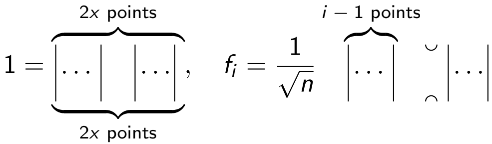

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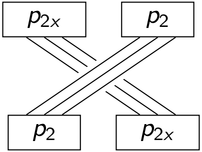
- Bounding $T|_{(\mathbb{C}1)^\perp}$ from below amounts to showing that the **flip**

maps 

$: H_1 \otimes H_x \to H_x \otimes H_1$ are **"far from being intertwiners".**

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- Is $L^\infty(S_n^+)$ a **prime** factor?
- (joint with B. Collins) Is it possible to construct matrix matrix models for $L^\infty(S_n^+)$?
  $\rightsquigarrow$ **Connes' embedding property**, and **free entropy dimension** estimates.