Gelfand representation

From Wikipedia, the free encyclopedia
(Redirected from Gelfand isomorphism)

In mathematics, the **Gelfand representation** in functional analysis (named after I. M. Gelfand) has two related meanings:

- a way of representing commutative Banach algebras as algebras of continuous functions;
- the fact that for commutative C*-algebras, this representation is an isometric isomorphism.

In the former case, one may regard the Gelfand representation as a far-reaching generalization of the Fourier transform of an integrable function. In the latter case, the Gelfand-Naimark representation theorem is one avenue in the development of spectral theory for normal operators, and generalizes the notion of diagonalizing a normal matrix.

Contents

Historical remarks

One of Gelfand's original applications (and one which historically motivated much of the study of Banach algebras[*citation needed*]) was to give a much shorter and more conceptual proof of a celebrated lemma of Norbert Wiener (see the citation below), characterizing the elements of the group algebras $L^1(\mathbf{R})$ and $\ell^1(\mathbf{Z})$ whose translates span dense subspaces in the respective algebras.

The model algebra

For any locally compact Hausdorff topological space X, the space $C_0(X)$ of continuous complex-valued functions on X which vanish at infinity is in a natural way a commutative C*-algebra:

- The structure of algebra over the complex numbers is obtained by considering the pointwise operations of addition and multiplication.
- The involution is pointwise complex conjugation.
- The norm is the uniform norm on functions.

Note that A is unital if and only if X is compact, in which case $C_0(X)$ is equal to $C(X)$, the algebra of all continuous complex-valued functions on X.

Gelfand representation of a commutative Banach algebra

Let A be a commutative Banach algebra, defined over the field **C** of complex numbers. A non-zero algebra homomorphism $\varphi: A \to \mathbf{C}$ is called a *character* of A; the set of all characters of A is denoted by Φ_A.

It can be shown that every character on A is automatically continuous, and hence Φ_A is a subset of the space A^* of continuous linear functionals on A; moreover, when equipped with the relative weak-* topology, Φ_A turns out to be locally compact and Hausdorff. (This follows from the Banach–Alaoglu theorem.) The space Φ_A is compact (in the topology just defined) if[*citation needed*] and only if the algebra A has an identity element.

Given $a \in A$, one defines the function $\widehat{a} : \Phi_A \to \mathbf{C}$ by $\widehat{a}(\phi) = \phi(a)$. The definition of Φ_A and the topology on it ensure that $\widehat{a}$ is continuous and vanishes at infinity[*citation needed*], and that the map $a \mapsto \widehat{a}$ defines a norm-decreasing, unit-preserving algebra homomorphism from A to $C_0(\Phi_A)$. This homomorphism is the *Gelfand representation of A*, and $\widehat{a}$ is the *Gelfand transform* of the element a. In general, the representation is neither injective nor surjective.

In the case where A has an identity element, there is a bijection between Φ_A and the set of maximal proper ideals in A (this relies on the Gelfand–Mazur theorem). As a consequence, the kernel of the Gelfand representation $A \to C_0(\Phi_A)$ may be identified with the Jacobson radical of A. Thus the Gelfand representation is injective if and only if A is (Jacobson) semisimple.

Examples

In the case where $A = L^1(\mathbf{R})$, the group algebra of **R**, then Φ_A is homeomorphic to **R** and the Gelfand transform of $f \in L^1(\mathbf{R})$ is the Fourier transform $\tilde{f}$.

In the case where $A = L^1(\mathbf{R}_+)$, the L^1-convolution algebra of the real half-line, then Φ_A is homeomorphic to $\{z \in \mathbf{C}: \mathrm{Re}(z) \geq 0\}$, and the Gelfand transform of an element $f \in L^1(\mathbf{R})$ is the Laplace transform $\mathcal{L}f$.

The C*-algebra case

As motivation, consider the special case $A = C_0(X)$. Given x in X, let $\varphi_x \in A^*$ be pointwise evaluation at x, i.e. $\varphi_x(f) = f(x)$. Then φ_x is a character on A, and it can be shown that all characters of A are of this form; a more precise analysis shows that we may identify Φ_A with X, not just as sets but as topological spaces. The Gelfand representation is then an isomorphism

$$C_0(X) \to C_0(\Phi_A).$$

The spectrum of a commutative C*-algebra

See also: Spectrum of a C-algebra*

The **spectrum** or **Gelfand space** of a commutative C*-algebra A, denoted $\hat{A}$, consists of the set of *non-zero* *-homomorphisms from A to the complex numbers. Elements of the spectrum are called **characters** on A. (It can be shown that every algebra homomorphism from A to the complex numbers is automatically a *-homomorphism, so that this definition of the term 'character' agrees with the one above.)

In particular, the spectrum of a commutative C*-algebra is a locally compact Hausdorff space: In the unital case, i.e. where the C*-algebra has a multiplicative unit element 1, all characters f must be unital, i.e. $f(1)$ is the complex number one. This excludes the zero homomorphism. So $\hat{A}$ is closed under weak-* convergence and the spectrum is actually *compact*. In the non-unital case, the weak-* closure of $\hat{A}$ is $\hat{A} \cup \{0\}$, where 0 is the zero homomorphism, and the removal of a single point from a compact Hausdorff space yields a locally compact Hausdorff space.

Note that *spectrum* is an overloaded word. It also refers to the spectrum σ(*x*) of an element *x* of an algebra with unit 1, that is the set of complex numbers *r* for which *x* - *r* 1 is not invertible in *A*. For unital C*-algebras, the two notions are connected in the following way: σ(*x*) is the set of complex numbers *f*(*x*) where *f* ranges over Gelfand space of *A*. Together with the spectral radius formula, this shows that *Â* is a subset of the unit ball of *A** and as such can be given the relative weak-* topology. This is the topology of pointwise convergence. A net $\{f_k\}_k$ of elements of the spectrum of *A* converges to *f* if and only if for each *x* in *A*, the net of complex numbers $\{f_k(x)\}_k$ converges to *f*(*x*).

If *A* is a separable C*-algebra, the weak-* topology is metrizable on bounded subsets. Thus the spectrum of a separable commutative C*-algebra *A* can be regarded as a metric space. So the topology can be characterized via convergence of sequences.

Equivalently, σ(*x*) is the range of γ(*x*), where γ is the Gelfand representation.

Statement of the commutative Gelfand-Naimark theorem

Let *A* be a commutative C*-algebra and let *X* be the spectrum of *A*. Let

$$\gamma : A \to C_0(X)$$

be the Gelfand representation defined above.

Theorem. The Gelfand map γ is an isometric *-isomorphism from *A* onto $C_0(X)$.

See the Arveson reference below.

The spectrum of a commutative C*-algebra can also be viewed as the set of all maximal ideals *m* of *A*, with the hull-kernel topology. (See the earlier remarks for the general, commutative Banach algebra case.) For any such *m* the quotient algebra *A/m* is one-dimensional (by the Gelfand-Mazur theorem), and therefore any *a* in *A* gives rise to a complex-valued function on *Y*.

In the case of C*-algebras with unit, the spectrum map gives rise to a contravariant functor from the category of C*-algebras with unit and unit-preserving continuous *-homomorphisms, to the category of compact Hausdorff spaces and continuous maps. This functor is one half of a contravariant equivalence between these two categories (its adjoint being the functor that assigns to each compact Hausdorff space *X* the C*-algebra $C_0(X)$). In particular, given compact Hausdorff spaces *X* and *Y*, then *C*(*X*) is isomorphic to *C*(*Y*) (as a C*-algebra) if and only if *X* is homeomorphic to *Y*.

The 'full' Gelfand–Naimark theorem is a result for arbitrary (abstract) noncommutative C*-algebras *A*, which though not quite analogous to the Gelfand representation, does provide a concrete representation of *A* as an algebra of operators.

Applications

One of the most significant applications is the existence of a continuous *functional calculus* for normal elements in C*-algebra *A*: An element *x* is normal if and only if *x* commutes with its adjoint *x**, or equivalently if and only if it generates a commutative C*-algebra C*(*x*). By the Gelfand isomorphism applied to C*(*x*) this is *-isomorphic to an algebra of continuous functions on a locally compact space. This observation leads almost immediately to:

Theorem. Let *A* be a C*-algebra with identity and *x* an element of *A*. Then there is a *-morphism *f* → *f*(*x*) from the algebra of continuous functions on the spectrum σ(*x*) into *A* such that

- It maps 1 to the multiplicative identity of *A*;

- It maps the identity function on the spectrum to *x*.

This allows us to apply continuous functions to bounded normal operators on Hilbert space.

References

- W. Arveson (1981). *An Invitation to C*-Algebras*. Springer-Verlag. ISBN 0-387-90176 Check `|isbn=` value (help).
- Frank F. Bonsall, John Duncan (1973). *Complete Normed Algebras*. Springer-Verlag, New York. ISBN 0-387-06386-2.
- N. Wiener (1932). "Tauberian theorems". *Ann. Of Math. (2)* (Annals of Mathematics) **33** (1): 1–100. doi:10.2307/1968102 (http://dx.doi.org/10.2307%2F1968102). JSTOR 1968102 (http://www.jstor.org/stable/1968102). More than one of `|number=` and `|issue=` specified (help)