Matrix ring

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In abstract algebra, a **matrix ring** is any collection of matrices forming a ring under matrix addition and matrix multiplication. The set of $n \times n$ matrices with entries from another ring is a matrix ring, as well as some subsets of infinite matrices which form **infinite matrix rings**. Any subrings of these matrix rings are also called matrix rings.

In the case when R is a commutative ring, then the matrix ring $M_n(R)$ is an associative algebra which may be called a **matrix algebra**. In this situation, if M is a matrix and r is in R, then the matrix Mr is the matrix M with each of its entries multiplied by r.

It is assumed throughout that R is an associative ring with a unit $1 \neq 0$, although matrix rings can be formed over rings without unity.

Contents

Examples

- The set of all $n \times n$ matrices over an arbitrary ring R, denoted $M_n(R)$. This is usually referred to as the "full ring of n by n matrices". These matrices represent endomorphisms of the free module R^n.

- The set of all upper (or set of all lower) triangular matrices over a ring.

- If R is any ring with unity, then the ring of endomorphisms of $M = \bigoplus_{i \in I} R$ as a right R module is isomorphic to the ring of **column finite matrices** $\mathbb{CFM}_I(R)$ whose entries are indexed by $I \times I$, and whose columns each contain only finitely many nonzero entries. The endomorphisms of M considered as a left R module result in an analogous object, the **row finite matrices** $\mathbb{RFM}_I(R)$ whose rows each only have finitely many nonzero entries.

- If R is a normed ring, then the condition of row or column finiteness in the previous point can be relaxed. With the norm in place, absolutely convergent series can be used instead of finite sums. For example, the matrices whose column sums are absolutely convergent sequences form a ring. Analogously of course, the matrices whose row sums are absolutely convergent series also form a ring. This idea can be used to represent operators on Hilbert spaces, for example.

- The intersection of the row and column finite matrix rings also forms a ring, which can be denoted by $\mathbb{RCFM}_I(R)$.

- The algebra $M_2(\mathbf{R})$ of 2×2 real matrices is a simple example of a non-commutative associative algebra. Like the quaternions, it has dimension 4 over $\mathbf{R}$, but unlike the quaternions, it has zero divisors, as can be seen from the following product of the matrix units: $E_{11}E_{21} = 0$, hence it is not a division ring. Its invertible elements are nonsingular matrices and they form a group, the general linear group $GL(2,\mathbf{R})$.

- If R is commutative, the matrix ring has a structure of a *-algebra over R, where the involution * on $M_n(R)$ is the matrix transposition.

- Complex matrix algebras $M_n(\mathbf{C})$ are, up to isomorphism, the only simple associative algebras over the field $\mathbf{C}$ of complex numbers. For $n = 2$, the matrix algebra $M_2(\mathbf{C})$ plays an important role in the theory of angular momentum. It has an alternative basis given by the identity matrix and the three Pauli matrices. $M_2(\mathbf{C})$ was the scene of early abstract algebra in the form of biquaternions.

- A matrix ring over a field is a Frobenius algebra, with Frobenius form given by the trace of the product: $\sigma(A,B)=\mathrm{tr}(AB)$.

Structure

- The matrix ring $M_n(R)$ can be identified with the ring of endomorphisms of the free R-module of rank n, $M_n(R) \cong \mathrm{End}_R(R^n)$. The procedure for matrix multiplication can be traced back to compositions of endomorphisms in this endomorphism ring.

- The ring $M_n(D)$ over a division ring D is an Artinian simple ring, a special type of semisimple ring. The rings $\mathbb{CFM}_I(D)$ and $\mathbb{RFM}_I(D)$ are *not* simple and not Artinian if the set I is infinite, however they are still full linear rings.

- In general, every semisimple ring is isomorphic to a finite direct product of full matrix rings over division rings, which may have differing division rings and differing sizes. This classification is given by the Artin–Wedderburn theorem.

- There is a one-to-one correspondence between the two-sided ideals of $M_n(R)$ and the two-sided ideals of R. Namely, for each ideal I of R, the set of all $n\times n$ matrices with entries in I is an ideal of $M_n(R)$, and each ideal of $M_n(R)$ arises in this way. This implies that $M_n(R)$ is simple if and only if R is simple. For $n \geq 2$, not every left ideal or right ideal of $M_n(R)$ arises by the previous construction from a left ideal or a right ideal in R. For example, the set of matrices whose columns with indices 2 through n are all zero forms a left ideal in $M_n(R)$.

- The previous ideal correspondence actually arises from the fact that the rings R and $M_n(R)$ are Morita equivalent. Roughly speaking, this means that the category of left R modules and the category of left $M_n(R)$ modules are very similar. Because of this, there is a natural bijective correspondence between the *isomorphism classes* of the left R-modules and the left $M_n(R)$-modules, and between the isomorphism classes of the left ideals of R and $M_n(R)$. Identical statements hold for right modules and right ideals. Through Morita equivalence, $M_n(R)$ can inherit any properties of R which are Morita invariant, such as being simple, Artinian, Noetherian, prime and numerous other properties as given in the Morita equivalence article.

Properties

- The matrix ring $M_n(R)$ is commutative if and only if $n = 1$ and R is commutative. As an example for 2×2 matrices which do not commute,

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

and $\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$. This example is easily generalized to $n\times n$ matrices.

- For $n \geq 2$, the matrix ring $M_n(R)$ has zero divisors. An example in 2×2 matrices would be

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

- The center of a matrix ring over a ring R consists of the matrices which are scalar multiples of the identity matrix, where the scalar belongs to the center of R.

- In linear algebra, it is noted that over a field F, $M_n(F)$ has the property that for any two matrices A and B, AB=1 implies BA=1. This is not true for every ring R though. A ring R whose matrix rings all have the mentioned property is known as a **stably finite ring** or sometimes **weakly finite ring** (Lam 1999, p. 5).

See also

- Central simple algebra
- Clifford algebra
- Hurwitz's theorem (normed division algebras)

References

- Lam, T. Y. (1999), *Lectures on modules and rings*, Graduate Texts in Mathematics No. 189, Berlin, New York: Springer-Verlag, ISBN 978-0-387-98428-5

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