Monomorphism

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$$Z \underset{g_2}{\overset{g_1}{\rightrightarrows}} X \xrightarrow{f} Y$$

In the context of abstract algebra or universal algebra, a **monomorphism** is an injective homomorphism. A monomorphism from *X* to *Y* is often denoted with the notation $X \hookrightarrow Y$.

In the more general setting of category theory, a **monomorphism** (also called a **monic morphism** or a **mono**) is a left-cancellative morphism, that is, an arrow $f : X \to Y$ such that, for all morphisms $g_1, g_2 : Z \to X$,

$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2 .$$

Monomorphisms are a categorical generalization of injective functions (also called "one-to-one functions"); in some categories the notions coincide, but monomorphisms are more general, as in the examples below.

The categorical dual of a monomorphism is an epimorphism, i.e. a monomorphism in a category *C* is an epimorphism in the dual category C^{op}. Every section is a monomorphism, and every retraction is an epimorphism.

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Relation to invertibility

Left invertible morphisms are necessarily monic: if *l* is a left inverse for *f* (meaning *l* is a morphism and $l \circ f = \mathrm{id}_X$), then *f* is monic, as

$$f \circ g_1 = f \circ g_2 \Rightarrow lfg_1 = lfg_2 \Rightarrow g_1 = g_2 .$$

A left invertible morphism is called a **split mono**.

However, a monomorphism need not be left-invertible. For example, in the category **Group** of all groups and group morphisms among them, if *H* is a subgroup of *G* then the inclusion $f : H \to G$ is always a monomorphism; but *f* has a left inverse in the category if and only if *H* has a normal complement in *G*.

A morphism $f : X \to Y$ is monic if and only if the induced map $f_\star : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$, defined by $f_* h = f \circ h$ for all morphisms $h : Z \to X$, is injective for all *Z*.

Examples

Every morphism in a concrete category whose underlying function is injective is a monomorphism; in other words, if morphisms are actually functions between sets, then any morphism which is a one-to-one function will necessarily be a monomorphism in the categorical sense. In the category of sets the converse also holds, so the monomorphisms are exactly the injective morphisms. The converse also holds in most naturally occurring categories of algebras because of the existence of a free object on one generator. In particular, it is true in the categories of all groups, of all rings, and in any abelian category.

It is not true in general, however, that all monomorphisms must be injective in other categories; that is, there are settings in which the morphisms are functions between sets, but one can have a function that is not injective and yet is a monomorphism in the categorical sense. For example, in the category **Div** of divisible (abelian) groups and group homomorphisms between them there are monomorphisms that are not injective: consider, for example, the quotient map $q : \mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$, where **Q** is the rationals under addition, **Z** the integers (also considered a group under addition), and $\mathbf{Q}/\mathbf{Z}$ is the corresponding quotient group. This is not an injective map, as for example every integer is mapped to 0. Nevertheless, it is a monomorphism in this category. This will follows from the implication $q \circ h = 0 \Rightarrow h = 0$, which we now prove. If $h : G \to \mathbf{Q}$, where G is some divisible group, and $q \circ h = 0$, then $h(x) \in \mathbf{Z},\ \forall\ x \in G$. Now fix some $x \in G$. Without loss of generality, we may assume that $h(x) \geq 0$ (otherwise, choose $-x$ instead). Then, letting $n=h(x)+1$, since G is a divisible group, there exists some $y \in G$ such that $x = ny$, so $h(x) = n\, h(y)$. From this, and $0 \leq h(x) < h(x) + 1=n$, it follows that

$$0 \leq \frac{h(x)}{h(x)+1} = h(y) < 1$$

Since $h(y) \in \mathbf{Z}$, it follows that $h(y) = 0$, and thus $h(x) = 0 = h(-x),\ \forall\ x \in G$. This says that $h = 0$, as desired.

To go from that implication to the fact that q is an epimorphism, assume that $q \circ f = q \circ g$ for some morphisms $f, g : G \to \mathbf{Q}$, where G is some divisible group. Then $q \circ (f - g) = 0$, where $(f - g) : x \mapsto f(x) - g(x)$. (Since $(f - g)(0) = 0$, and $(f - g)(x + y) = (f - g)(x) + (f - g)(y)$, it follows that $(f - g) \in Hom(G, \mathbf{Q})$). From the implication just proved, $q \circ (f - g) = 0 \Rightarrow f - g = 0 \Leftrightarrow\ \forall\ x \in G, f(x) = g(x) \Leftrightarrow f = g$. Hence q is a monomorphism, as claimed.

Properties

- In a topos, every monic is an equalizer, and any map that is both monic and epic is an isomorphism.
- Every isomorphism is monic.

Related concepts

There are also useful concepts of **regular monomorphism**, **strong monomorphism**, and **extremal monomorphism**. A regular monomorphism equalizes some parallel pair of morphisms. An extremal monomorphism is a monomorphism that cannot be nontrivially factored through an epimorphism: Precisely, if $m=g \circ e$ with e an epimorphism, then e is an isomorphism. A strong monomorphism satisfies a certain lifting property with respect to commutative squares involving an epimorphism.

Terminology

The companion terms *monomorphism* and *epimorphism* were originally introduced by Nicolas Bourbaki; Bourbaki uses *monomorphism* as shorthand for an injective function. Early category theorists believed that the correct generalization of injectivity to the context of categories was the cancellation property given above. While this is not exactly true for monic maps, it is very close, so this has caused little trouble, unlike the case of epimorphisms. Saunders Mac Lane attempted to make a distinction between what he called *monomorphisms*, which were maps in a concrete category whose underlying maps of sets were injective, and *monic maps*, which are monomorphisms in the categorical sense of the word. This distinction never came into general use.

Another name for monomorphism is *extension*, although this has other uses too.

See also

- embedding
- isomorphism
- subobject

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Epimorphism

From Wikipedia, the free encyclopedia

In category theory, an **epimorphism** (also called an **epic morphism** or, colloquially, an **epi**) is a morphism $f : X \to Y$ which is right-cancellative in the sense that, for all morphisms $g_1, g_2 : Y \to Z$,

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & Z \end{tikzcd}$$

$$g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2.$$

Epimorphisms are analogues of surjective functions, but they are not exactly the same. The dual of an epimorphism is a monomorphism (i.e. an epimorphism in a category C is a monomorphism in the dual category C^{op}).

Many authors in abstract algebra and universal algebra define an **epimorphism** simply as an *onto* or surjective homomorphism. Every epimorphism in this algebraic sense is an epimorphism in the sense of category theory, but the converse is not true in all categories. In this article, the term "epimorphism" will be used in the sense of category theory given above. For more on this, see the section on Terminology below.

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Examples

Every morphism in a concrete category whose underlying function is surjective is an epimorphism. In many concrete categories of interest the converse is also true. For example, in the following categories, the epimorphisms are exactly those morphisms which are surjective on the underlying sets:

- **Set**, sets and functions. To prove that every epimorphism $f: X \to Y$ in **Set** is surjective, we compose it with both the characteristic function $g_1: Y \to \{0,1\}$ of the image $f(X)$ and the map $g_2: Y \to \{0,1\}$ that is constant 1.
- **Rel**, sets with binary relations and relation preserving functions. Here we can use the same proof as for **Set**, equipping $\{0,1\}$ with the full relation $\{0,1\}\times\{0,1\}$.
- **Pos**, partially ordered sets and monotone functions. If $f : (X,\leq) \to (Y,\leq)$ is not surjective, pick y_0 in $Y \setminus f(X)$ and let $g_1 : Y \to \{0,1\}$ be the characteristic function of $\{y \mid y_0 \leq y\}$ and $g_2 : Y \to \{0,1\}$ the characteristic function of $\{y \mid y_0 < y\}$. These maps are monotone if $\{0,1\}$ is given the standard ordering $0 < 1$.
- **Grp**, groups and group homomorphisms. The result that every epimorphism in **Grp** is surjective is due to Otto Schreier (he actually proved more, showing that every subgroup is an equalizer using the free product with one amalgamated subgroup); an elementary proof can be found in (Linderholm 1970).
- **FinGrp**, finite groups and group homomorphisms. Also due to Schreier; the proof given in (Linderholm 1970) establishes this case as well.
- **Ab**, abelian groups and group homomorphisms.
- ***K*-Vect**, vector spaces over a field K and K-linear transformations.

- **Mod**-R, right modules over a ring R and module homomorphisms. This generalizes the two previous examples; to prove that every epimorphism f: $X \to Y$ in **Mod**-R is surjective, we compose it with both the canonical quotient map g_1: $Y \to Y/f(X)$ and the zero map g_2: $Y \to Y/f(X)$.
- **Top**, topological spaces and continuous functions. To prove that every epimorphism in **Top** is surjective, we proceed exactly as in **Set**, giving {0,1} the indiscrete topology which ensures that all considered maps are continuous.
- **HComp**, compact Hausdorff spaces and continuous functions. If f: $X \to Y$ is not surjective, let y in Y-fX. Since fX is closed, by Urysohn's Lemma there is a continuous function g_1:$Y \to [0,1]$ such that g_1 is 0 on fX and 1 on y. We compose f with both g_1 and the zero function g_2: $Y \to [0,1]$.

However there are also many concrete categories of interest where epimorphisms fail to be surjective. A few examples are:

- In the category of monoids, **Mon**, the inclusion map $\mathbf{N} \to \mathbf{Z}$ is a non-surjective epimorphism. To see this, suppose that g_1 and g_2 are two distinct maps from **Z** to some monoid M. Then for some n in **Z**, $g_1(n) \neq g_2(n)$, so $g_1(-n) \neq g_2(-n)$. Either n or $-n$ is in **N**, so the restrictions of g_1 and g_2 to **N** are unequal.
- In the category of algebras over commutative ring **R**, take $\mathbf{R}[\mathbf{N}] \to \mathbf{R}[\mathbf{Z}]$, where $\mathbf{R}[\mathbf{G}]$ is the group ring of the group **G** and the morphism is induced by the inclusion $\mathbf{N} \to \mathbf{Z}$ as in the previous example. This follows from the observation that **1** generates the algebra $\mathbf{R}[\mathbf{Z}]$ (note that the unit in $\mathbf{R}[\mathbf{Z}]$ is given by **0** of **Z**), and the inverse of the element represented by **n** in **Z** is just the element represented by **-n**. Thus any homomorphism from $\mathbf{R}[\mathbf{Z}]$ is uniquely determined by its value on the element represented by **1** of **Z**.
- In the category of rings, **Ring**, the inclusion map $\mathbf{Z} \to \mathbf{Q}$ is a non-surjective epimorphism; to see this, note that any ring homomorphism on **Q** is determined entirely by its action on **Z**, similar to the previous example. A similar argument shows that the natural ring homomorphism from any commutative ring R to any one of its localizations is an epimorphism.
- In the category of commutative rings, a finitely generated homomorphism of rings $f : R \to S$ is an epimorphism if and only if for all prime ideals P of R, the ideal Q generated by $f(P)$ is either S or is prime, and if Q is not S, the induced map $\mathrm{Frac}(R/P) \to \mathrm{Frac}(S/Q)$ is an isomorphism (EGA IV 17.2.6).
- In the category of Hausdorff spaces, **Haus**, the epimorphisms are precisely the continuous functions with dense images. For example, the inclusion map $\mathbf{Q} \to \mathbf{R}$, is a non-surjective epimorphism.

The above differs from the case of monomorphisms where it is more frequently true that monomorphisms are precisely those whose underlying functions are injective.

As to examples of epimorphisms in non-concrete categories:

- If a monoid or ring is considered as a category with a single object (composition of morphisms given by multiplication), then the epimorphisms are precisely the right-cancellable elements.
- If a directed graph is considered as a category (objects are the vertices, morphisms are the paths, composition of morphisms is the concatenation of paths), then *every* morphism is an epimorphism.

Properties

Every isomorphism is an epimorphism; indeed only a right-sided inverse is needed: if there exists a morphism $j : Y \to X$ such that $fj = \mathrm{id}_Y$, then f is easily seen to be an epimorphism. A map with such a right-sided inverse is called a **split epi**. In a topos, a map that is both a monic morphism and an epimorphism is an isomorphism.

The composition of two epimorphisms is again an epimorphism. If the composition fg of two morphisms is an epimorphism, then f must be an epimorphism.

As some of the above examples show, the property of being an epimorphism is not determined by the morphism alone, but also by the category of context. If D is a subcategory of C, then every morphism in D

which is an epimorphism when considered as a morphism in C is also an epimorphism in D; the converse, however, need not hold; the smaller category can (and often will) have more epimorphisms.

As for most concepts in category theory, epimorphisms are preserved under equivalences of categories: given an equivalence $F : C \to D$, then a morphism f is an epimorphism in the category C if and only if $F(f)$ is an epimorphism in D. A duality between two categories turns epimorphisms into monomorphisms, and vice versa.

The definition of epimorphism may be reformulated to state that $f : X \to Y$ is an epimorphism if and only if the induced maps

$$\begin{array}{ccc} \operatorname{Hom}(Y,Z) & \rightarrow & \operatorname{Hom}(X,Z) \\ g & \mapsto & gf \end{array}$$

are injective for every choice of Z. This in turn is equivalent to the induced natural transformation

$$\operatorname{Hom}(Y,-) \;\rightarrow\; \operatorname{Hom}(X,-)$$

being a monomorphism in the functor category $\mathbf{Set}^C$.

Every coequalizer is an epimorphism, a consequence of the uniqueness requirement in the definition of coequalizers. It follows in particular that every cokernel is an epimorphism. The converse, namely that every epimorphism be a coequalizer, is not true in all categories.

In many categories it is possible to write every morphism as the composition of a monomorphism followed by an epimorphism. For instance, given a group homomorphism $f : G \to H$, we can define the group $K = \operatorname{im}(f) = f(G)$ and then write f as the composition of the surjective homomorphism $G \to K$ which is defined like f, followed by the injective homomorphism $K \to H$ which sends each element to itself. Such a factorization of an arbitrary morphism into an epimorphism followed by a monomorphism can be carried out in all abelian categories and also in all the concrete categories mentioned above in the Examples section (though not in all concrete categories).

Related concepts

Among other useful concepts are *regular epimorphism*, *extremal epimorphism*, *strong epimorphism*, and *split epimorphism*. A regular epimorphism coequalizes some parallel pair of morphisms. An extremal epimorphism is an epimorphism that has no monomorphism as a second factor, unless that monomorphism is an isomorphism. A strong epimorphism satisfies a certain lifting property with respect to commutative squares involving a monomorphism. A split epimorphism is a morphism which has a right-sided inverse.

A morphism that is both a monomorphism and an epimorphism is called a bimorphism. Every isomorphism is a bimorphism but the converse is not true in general. For example, the map from the half-open interval [0,1) to the unit circle S^1 (thought of as a subspace of the complex plane) which sends x to $\exp(2\pi ix)$ (see Euler's formula) is continuous and bijective but not a homeomorphism since the inverse map is not continuous at 1, so it is an instance of a bimorphism that is not an isomorphism in the category **Top**. Another example is the embedding $\mathbf{Q} \to \mathbf{R}$ in the category **Haus**; as noted above, it is a bimorphism, but it is not bijective and therefore not an isomorphism. Similarly, in the category of rings, the maps $\mathbf{Z} \to \mathbf{Q}$ and $\mathbf{Q} \to \mathbf{R}$ are bimorphisms but not isomorphisms.

Epimorphisms are used to define abstract quotient objects in general categories: two epimorphisms $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ are said to be *equivalent* if there exists an isomorphism $j : Y_1 \to Y_2$ with $j\, f_1 = f_2$. This is an equivalence relation, and the equivalence classes are defined to be the quotient objects of X.

Terminology

The companion terms *epimorphism* and *monomorphism* were first introduced by Bourbaki. Bourbaki uses *epimorphism* as shorthand for a surjective function. Early category theorists believed that epimorphisms were the correct analogue of surjections in an arbitrary category, similar to how monomorphisms are very nearly an exact analogue of injections. Unfortunately this is incorrect; strong or regular epimorphisms behave much more closely to surjections than ordinary epimorphisms. Saunders Mac Lane attempted to create a distinction between *epimorphisms*, which were maps in a concrete category whose underlying set maps were surjective, and *epic morphisms*, which are epimorphisms in the modern sense. However, this distinction never caught on.

It is a common mistake to believe that epimorphisms are either identical to surjections or that they are a better concept. Unfortunately this is rarely the case; epimorphisms can be very mysterious and have unexpected behavior. It is very difficult, for example, to classify all the epimorphisms of rings. In general, epimorphisms are their own unique concept, related to surjections but fundamentally different.

See also

- List of category theory topics

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