Normal operator

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In mathematics, especially functional analysis, a **normal operator** on a complex Hilbert space H is a continuous linear operator $N : H \to H$ that commutes with its hermitian adjoint N^*, that is: $NN^* = N^*N$.

Normal operators are important because the spectral theorem holds for them. Today, the class of normal operators is well-understood. Examples of normal operators are

- unitary operators: $N^* = N^{-1}$
- Hermitian operators (i.e., selfadjoint operators): $N^* = N$; (also, anti-selfadjoint operators: $N^* = -N$)
- positive operators: $N = MM^*$
- normal matrices can be seen as normal operators if one takes the Hilbert space to be $\mathbf{C}^n$.

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Properties

Normal operators are characterized by the spectral theorem. A compact normal operator (in particular, a normal operator on a finite-dimensional linear space) is unitarily diagonalizable.

Let T be a bounded operator. The following are equivalent.

- T is normal.
- T^* is normal.
- $\|Tx\| = \|T^*x\|$ for all x (use $\|Tx\|^2 = \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle = \|T^*x\|^2$).
- The selfadjoint and anti-selfadjoint parts of T (i.e., $T \equiv T_1 + iT_2$, with $T_1 := \frac{T+T^*}{2}$ rsp. $i\,T_2 := \frac{T-T^*}{2}$), commute.[1]

If N is a normal operator, then N and N^* have the same kernel and range. Consequently, the range of N is dense if and only if N is injective. Put in another way, the kernel of a normal operator is the orthogonal complement of its range; thus, the kernel of the operator N^k coincides with that of N for any k. Every generalized eigenvalue of a normal operator is thus genuine. λ is an eigenvalue of a normal operator N if and only if its complex conjugate $\overline{\lambda}$ is an eigenvalue of N^*. Eigenvectors of a normal operator corresponding to different eigenvalues are orthogonal, and it stabilizes orthogonal complements to its eigenspaces.[2] This implies the usual spectral theorem: every normal operator on a finite-dimensional space is diagonalizable by a unitary operator. There is also an infinite-dimensional generalization in terms of projection-valued measures. Residual spectrum of a normal operator is empty.[2]

The product of normal operators that commute is again normal; this is nontrivial and follows from Fuglede's theorem, which states (in a form generalized by Putnam):

If N_1 and N_2 are normal operators and if *A* is a bounded linear operator such that $N_1 A = A N_2$, then $N_1^* A = A N_2^*$.

The operator norm of a normal operator equals its numerical radius and spectral radius.

A normal operator coincides with its Aluthge transform.

Properties in finite-dimensional case

If a normal operator *T* on a *finite-dimensional* real or complex Hilbert space (inner product space) *H* stabilizes a subspace *V*, then it also stabilizes its orthogonal complement $V^\perp$. (This statement is trivial in the case where *T* is self-adjoint)

Proof. Let P_V be the orthogonal projection onto *V*. Then the orthogonal projection onto $V^\perp$ is $\mathbf{1}_H - P_V$. The fact that *T* stabilizes *V* can be expressed as $(\mathbf{1}_H - P_V) T P_V = 0$, or $T P_V = P_V T P_V$. The goal is to show that $X := P_V T(\mathbf{1}_H - P_V) = 0$. Since $(A, B) \mapsto \mathrm{tr}(AB^*)$ is an inner product on the space of endomorphisms of *H*, it is enough to show that $\mathrm{tr}(XX^*) = 0$. But first we express XX^* in terms of orthogonal projections:

$$XX^* = P_V T(\mathbf{1}_H - P_V)^2 T^* P_V = P_V T(\mathbf{1}_H - P_V) T^* P_V = P_V T T^* P_V - P_V T P_V T^* P_V$$
,

Now using properties of the trace and of orthogonal projections we have:

$$\begin{aligned}
\mathrm{tr}(XX^*) &= \mathrm{tr}\left(P_V T T^* P_V - P_V T P_V T^* P_V\right) \\
&= \mathrm{tr}(P_V T T^* P_V) - \mathrm{tr}(P_V T P_V T^* P_V) \\
&= \mathrm{tr}(P_V^2 T T^*) - \mathrm{tr}(P_V^2 T P_V T^*) \\
&= \mathrm{tr}(P_V T T^*) - \mathrm{tr}(P_V T P_V T^*) \\
&= \mathrm{tr}(P_V T T^*) - \mathrm{tr}(T P_V T^*) \\
&= \mathrm{tr}(P_V T T^*) - \mathrm{tr}(P_V T^* T) \\
&= \mathrm{tr}(P_V (T T^* - T^* T)) \\
&= 0.
\end{aligned}$$

The same argument goes through for compact normal operators in infinite dimensional Hilbert spaces, where one make use of the Hilbert-Schmidt inner product.[3] However, for bounded normal operators orthogonal complement to a stable subspace may not be stable.[4] It follows that such subspaces cannot be spanned by eigenvectors. Consider, for example, the bilateral shift, which has no eigenvalues. The invariant subspaces of the bilateral shift is characterized by Beurling's theorem.

Normal elements

The notion of normal operators generalizes to an involutive algebra; namely, an element *x* of an involutive algebra is said to be normal if $xx^* = x^*x$. The most important case is when such an algebra is a C*-algebra. A positive element is an example of a normal element.

Unbounded normal operators

The definition of normal operators naturally generalizes to some class of unbounded operators. Explicitly, a closed operator *N* is said to be normal if

$$N^*N = NN^*$$

Here, the existence of the adjoint *N** implies that the domain of *N* is dense, and the equality implies that the domain of *N**N* equals that of *NN**, which is not necessarily the case in general.

The spectral theorem still holds for unbounded normal operators, but usually requires a different proof.

Generalization

The success of the theory of normal operators led to several attempts for generalization by weakening the commutativity requirement. Classes of operators that include normal operators are (in order of inclusion)

- Quasinormal operators
- Subnormal operators
- Hyponormal operators
- Paranormal operators
- Normaloids

Notes

1. ^ In contrast, for the important class of Creation and annihilation operators of, e.g., quantum field theory, they don't commute
2. ^ ***a*** ***b*** Naylor, Arch W.; Sell George R. (1982). *Linear Operator Theory in Engineering and Sciences* (http://books.google.com/books?id=t3SXs4-KrE0C&dq=naylor+sell+linear). New York: Springer. ISBN 978-0-387-95001-3.
3. ^ Andô, Tsuyoshi (1963). "Note on invariant subspaces of a compact normal operator". *Archiv der Mathematik* **14**: 337–340. doi:10.1007/BF01234964 (http://dx.doi.org/10.1007%2FBF01234964).
4. ^ Garrett, Paul (2005). "Operators on Hilbert spaces" (http://www.math.umn.edu/~garrett/m/fun/Notes/04a_ops_hsp.pdf).

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