

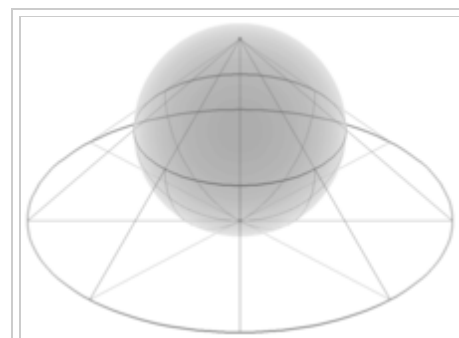
Riemann sphere

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In mathematics, the **Riemann sphere**, named after the 19th century mathematician Bernhard Riemann, is a model of the **extended complex plane**, the complex plane plus a point at infinity. This extended plane represents the **extended complex numbers**, that is, the complex numbers plus a value ∞ for infinity. With the Riemann model, the point " ∞ " is near to very large numbers, just as the point " 0 " is near to very small numbers.

The extended complex numbers are useful in complex analysis because they allow for division by zero in some circumstances, in a way that makes expressions such as $1/0 = \infty$ well-behaved. For example, any rational function on the complex plane can be extended to a continuous function on the Riemann sphere, with the poles of the rational function mapping to infinity. More generally, any meromorphic function can be thought of as a continuous function whose codomain is the Riemann sphere.

In geometry, the Riemann sphere is the prototypical example of a Riemann surface, and is one of the simplest complex manifolds. In projective geometry, the sphere can be thought of as the **complex projective line** $\mathbf{P}^1(\mathbf{C})$, the projective space of all complex lines in \mathbf{C}^2 . As with any compact Riemann surface, the sphere may also be viewed as a projective algebraic curve, making it a fundamental example in algebraic geometry. It also finds utility in other disciplines that depend on analysis and geometry, such as quantum mechanics and other branches of physics.



The Riemann sphere can be visualized as the complex number plane wrapped around a sphere (by some form of stereographic projection – details are given below).

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Extended complex numbers

The **extended complex numbers** consist of the complex numbers \mathbf{C} together with ∞ . The extended complex numbers may be written as $\mathbf{C} \cup \{\infty\}$, and are often denoted by adding some decoration to the letter \mathbf{C} , such as

$$\hat{\mathbf{C}}, \quad \bar{\mathbf{C}}, \quad \text{or} \quad \mathbf{C}_{\infty}.$$

Geometrically, the set of extended complex numbers is referred to as the **Riemann sphere** (or **extended complex plane**).

Arithmetic operations

Addition of complex numbers may be extended by defining, for $z \in \mathbf{C}$,

$$z + \infty = \infty$$

for any complex number z , and multiplication may be defined by

$$z \cdot \infty = \infty$$

for all nonzero complex numbers z , with $\infty \cdot \infty = \infty$. Note that $\infty + \infty$, $\infty - \infty$ and $0 \cdot \infty$ are left undefined. Unlike the complex numbers, the extended complex numbers do not form a field, since ∞ does not have a multiplicative inverse. Nonetheless, it is customary to define division on $\mathbf{C} \cup \{\infty\}$ by

$$z/0 = \infty \quad \text{and} \quad z/\infty = 0$$

for all nonzero complex numbers z , with $\infty/0 = \infty$ and $0/\infty = 0$.

Rational functions

Any rational function $f(z) = g(z)/h(z)$ can be extended to a continuous function on the Riemann sphere. Specifically, if z_0 is a complex number such that the denominator $h(z_0)$ is zero but the numerator $g(z_0)$ is nonzero, then $f(z_0)$ can be defined as ∞ . (If both the numerator and denominator are zero, then they share a common factor, and the fraction should first be reduced to lowest terms.) Moreover, $f(\infty)$ can be defined as the limit of $f(z)$ as $z \rightarrow \infty$, which may be finite or infinite.

For example, given the function

$$f(z) = \frac{6z + 1}{2z - 10}$$

we may define $f(5) = \infty$ since the denominator is zero at $z = 5$, and $f(\infty) = 3$ since $f(z) \rightarrow 3$ as $z \rightarrow \infty$. Using these definitions, f becomes a continuous function from the Riemann sphere to itself.

When viewed as a complex manifold, these rational functions are in fact holomorphic functions from the Riemann sphere to itself.

As a complex manifold

As a one-dimensional complex manifold, the Riemann sphere can be described by two charts, both with domain equal to the complex number plane \mathbf{C} . Let ζ and ξ be complex coordinates on \mathbf{C} . Identify the nonzero complex numbers ζ with the nonzero complex numbers ξ using the transition maps

$$\zeta = 1/\xi,$$

$$\xi = 1/\zeta.$$

Since the transition maps are holomorphic, they define a complex manifold, called the **Riemann sphere**.

Intuitively, the transition maps indicate how to glue two planes together to form the Riemann sphere. The planes are glued in an "inside-out" manner, so that they overlap almost everywhere, with each plane

contributing just one point (its origin) missing from the other plane. In other words, (almost) every point in the Riemann sphere has both a ζ value and a ξ value, and the two values are related by $\zeta = 1/\xi$. The point where $\xi = 0$ should then have ζ -value "1/0"; in this sense, the origin of the ξ -chart plays the role of " ∞ " in the ζ -chart. Symmetrically, the origin of the ζ -chart plays the role of ∞ in the ξ -chart.

Topologically, the resulting space is the one-point compactification of a plane into the sphere. However, the Riemann sphere is not merely a topological sphere. It is a sphere with a well-defined complex structure, so that around every point on the sphere there is a neighborhood that can be biholomorphically identified with \mathbf{C} .

On the other hand, the uniformization theorem, a central result in the classification of Riemann surfaces, states that the only simply-connected one-dimensional complex manifolds are the complex plane, the hyperbolic plane, and the Riemann sphere. Of these, the Riemann sphere is the only one that is a closed surface (a compact surface without boundary). Hence the two-dimensional sphere admits a unique complex structure turning it into a one-dimensional complex manifold.

As the complex projective line

The Riemann sphere can also be defined as the **complex projective line**. This is the subset of \mathbf{C}^2 consisting of all pairs (α, β) of complex numbers, not both zero, modulo the equivalence relation

$$(\alpha, \beta) = (\lambda\alpha, \lambda\beta)$$

for all nonzero complex numbers λ . The complex plane \mathbf{C} , with coordinate ζ , can be mapped into the complex projective line by

$$(\alpha, \beta) = (\zeta, 1).$$

Another copy of \mathbf{C} with coordinate ξ can be mapped in by

$$(\alpha, \beta) = (1, \xi).$$

These two complex charts cover the projective line. For nonzero ξ the identifications

$$(1, \xi) = (1/\xi, 1) = (\zeta, 1)$$

demonstrate that the transition maps are $\zeta = 1/\xi$ and $\xi = 1/\zeta$, as above.

This treatment of the Riemann sphere connects most readily to projective geometry. For example, any line (or smooth conic) in the complex projective plane is biholomorphic to the complex projective line. It is also convenient for studying the sphere's automorphisms, later in this article.

As a sphere

The Riemann sphere can be visualized as the unit sphere $x^2 + y^2 + z^2 = 1$ in the three-dimensional real space \mathbf{R}^3 . To this end, consider the stereographic projection from the unit sphere minus the point $(0, 0, 1)$ onto the plane $z = 0$, which we identify with the complex plane by $\zeta = x + iy$. In Cartesian coordinates (x, y, z) and spherical coordinates (ϕ, θ) on the sphere (with ϕ the zenith and θ the azimuth), the projection is

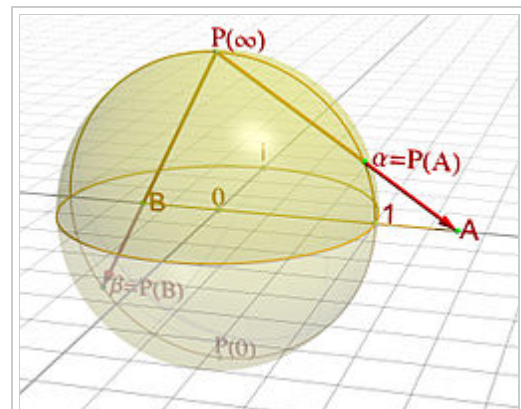
$$\zeta = \frac{x + iy}{1 - z} = \cot\left(\frac{1}{2}\phi\right) e^{i\theta}.$$

Similarly, stereographic projection from $(0, 0, -1)$ onto the plane $z = 0$, identified with another copy of the

complex plane by $\xi = x - iy$, is written

$$\xi = \frac{x - iy}{1 + z} = \tan\left(\frac{1}{2}\phi\right) e^{-i\theta}.$$

In order to cover the unit sphere, one needs the two stereographic projections: the first will cover the whole sphere except the point $(0,0,1)$ and the second except the point $(0,0,-1)$. Hence, one needs two complex planes, one for each projection, which can be intuitively seen as glued back-to-back at $z=0$. Note that the two complex planes are identified differently with the plane $z = 0$. An orientation-reversal is necessary to maintain consistent orientation on the sphere, and in particular complex conjugation causes the transition maps to be holomorphic.



Stereographic projection of a complex number A onto a point α of the Riemann sphere

The transition maps between ζ -coordinates and ξ -coordinates are obtained by composing one projection with the inverse of the other. They turn out to be $\zeta = 1/\xi$ and $\xi = 1/\zeta$, as described above. Thus the unit sphere is diffeomorphic to the Riemann sphere.

Under this diffeomorphism, the unit circle in the ζ -chart, the unit circle in the ξ -chart, and the equator of the unit sphere are all identified. The unit disk $|\zeta| < 1$ is identified with the southern hemisphere $z < 0$, while the unit disk $|\xi| < 1$ is identified with the northern hemisphere $z > 0$.

Metric

A Riemann surface does not come equipped with any particular Riemannian metric. However, the complex structure of the Riemann surface does uniquely determine a metric up to conformal equivalence. (Two metrics are said to be conformally equivalent if they differ by multiplication by a positive smooth function.) Conversely, any metric on an oriented surface uniquely determines a complex structure, which depends on the metric only up to conformal equivalence. Complex structures on an oriented surface are therefore in one-to-one correspondence with conformal classes of metrics on that surface.

Within a given conformal class, one can use conformal symmetry to find a representative metric with convenient properties. In particular, there is always a complete metric with constant curvature in any given conformal class.

In the case of the Riemann sphere, the Gauss-Bonnet theorem implies that a constant-curvature metric must have positive curvature K . It follows that the metric must be isometric to the sphere of radius $1/\sqrt{K}$ in \mathbf{R}^3 via stereographic projection. In the ζ -chart on the Riemann sphere, the metric with $K = 1$ is given by

$$ds^2 = \left(\frac{2}{1 + |\zeta|^2}\right)^2 |d\zeta|^2 = \frac{4}{(1 + \zeta\bar{\zeta})^2} d\zeta d\bar{\zeta}.$$

In real coordinates $\zeta = u + iv$, the formula is

$$ds^2 = \frac{4}{(1 + u^2 + v^2)^2} (du^2 + dv^2).$$

Up to a constant factor, this metric agrees with the standard Fubini–Study metric on complex projective space (of which the Riemann sphere is an example).

Conversely, let S denote the sphere (as an abstract smooth or topological manifold). By the uniformization

theorem there exists a unique complex structure on S . It follows that any metric on S is conformally equivalent to the round metric. All such metrics determine the same conformal geometry. The round metric is therefore not intrinsic to the Riemann sphere, since "roundness" is not an invariant of conformal geometry. The Riemann sphere is only a conformal manifold, not a Riemannian manifold. However, if one needs to do Riemannian geometry on the Riemann sphere, the round metric is a natural choice.

Automorphisms

Main article: Möbius transformation

The study of any mathematical object is aided by an understanding of its group of automorphisms, meaning the maps from the object to itself that preserve the essential structure of the object. In the case of the Riemann sphere, an automorphism is an invertible biholomorphic map from the Riemann sphere to itself. It turns out that the only such maps are the Möbius transformations. These are functions of the form

$$f(\zeta) = \frac{a\zeta + b}{c\zeta + d},$$

where a , b , c , and d are complex numbers such that $ad - bc \neq 0$.

Examples of Möbius transformations include dilations, rotations, translations, and complex inversion. In fact, any Möbius transformation can be written as a composition of these.

The Möbius transformations are profitably viewed as transformations on the complex projective line. In projective coordinates, the transformation f' can be written

$$f(\alpha, \beta) = (a\alpha + b\beta, c\alpha + d\beta) = \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

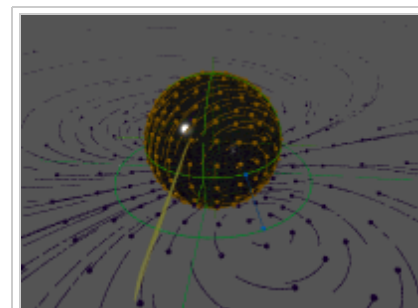
Thus the Möbius transformations can be described as 2×2 complex matrices with nonzero determinant; two matrices yield the same Möbius transformation if and only if they differ by a nonzero factor. Thus the Möbius transformations exactly correspond to the projective linear transformations $\text{PGL}(2, \mathbf{C})$.

If one endows the Riemann sphere with the Fubini–Study metric, then not all Möbius transformations are isometries; for example, the dilations and translations are not. The isometries form a proper subgroup of $\text{PGL}(2, \mathbf{C})$, namely $\text{PSU}(2)$. This subgroup is isomorphic to the rotation group $\text{SO}(3)$, which is the group of symmetries of the unit sphere in \mathbf{R}^3 (which, when restricted to the sphere, become the isometries of the sphere).

Applications

In complex analysis, a meromorphic function on the complex plane (or on any Riemann surface, for that matter) is a ratio f/g of two holomorphic functions f and g . As a map to the complex numbers, it is undefined wherever g is zero. However, it induces a holomorphic map (f, g) to the complex projective line that is well-defined even where $g = 0$. This construction is helpful in the study of holomorphic and meromorphic functions. For example, on a compact Riemann surface there are no non-constant holomorphic maps to the complex numbers, but holomorphic maps to the complex projective line are abundant.

The Riemann sphere has many uses in physics. In quantum mechanics, points on the complex projective line are natural values for photon polarization states, spin states of massive particles of spin 1/2, and 2-state particles in general. The Riemann sphere has been suggested as a relativistic model for the celestial sphere. In



A Möbius transformation acting on the sphere, and on the plane by stereographic projection

string theory, the worldsheets of strings are Riemann surfaces, and the Riemann sphere, being the simplest Riemann surface, plays a significant role. It is also important in twistor theory.

See also

- Conformal geometry
- Cross-ratio
- Hopf bundle
- Dessin d'enfant
- Directed infinity

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External links

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- Moebius Transformations Revealed (<http://www.ima.umn.edu/~arnold/moebius/>) , by Douglas N. Arnold and Jonathan Rogness (a video by two University of Minnesota professors explaining and illustrating Möbius transformations using stereographic projection from a sphere)

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