Spectrum (functional analysis)

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In functional analysis, the concept of the **spectrum** of a bounded operator is a generalisation of the concept of eigenvalues for matrices. Specifically, a complex number λ is said to be in the spectrum of a bounded linear operator T if $\lambda I - T$ is not invertible, where I is the identity operator. The study of spectra and related properties is known as spectral theory, which has numerous applications, most notably the mathematical formulation of quantum mechanics.

The spectrum of an operator on a finite-dimensional vector space is precisely the set of eigenvalues. However an operator on an infinite-dimensional space may have additional elements in its spectrum, and may have no eigenvalues. For example, consider the right shift operator R on the Hilbert space ℓ^2,

$$(x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots).$$

This has no eigenvalues, since if $Rx=\lambda x$ then by expanding this expression we see that $x_1=0$, $x_2=0$, etc. On the other hand 0 is in the spectrum because the operator $R - 0$ (i.e. R itself) is not invertible: it is not surjective since any vector with non-zero first component is not in its range. In fact *every* bounded linear operator on a complex Banach space must have a non-empty spectrum.

The notion of spectrum extends to densely-defined unbounded operators. In this case a complex number λ is said to be in the spectrum of such an operator $T:D\to X$ (where D is dense in X) if there is no bounded inverse $(\lambda I - T)^{-1}:X\to D$. If T is a closed operator (which includes the case that T is a bounded operator), boundedness of such inverses follow automatically if the inverse exists at all.

The space of bounded linear operators $B(X)$ on a Banach space X is an example of a unital Banach algebra. Since the definition of the spectrum does not mention any properties of $B(X)$ except those that any such algebra has, the notion of a spectrum may be generalised to this context by using the same definition verbatim.

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Definition

Let T be a a bounded linear operator acting on a Banach space $\mathbb{X}$ over the scalar field $\mathbb{K}$, and I be the identity operator on $\mathbb{X}$. The **spectrum** of T is the set of all $\lambda \in \mathbb{K}$ for which the operator $\lambda I - T$ does not have an inverse that is a bounded linear operator.

Since $\lambda I - T$ is a linear operator, the inverse is linear if it exists; and, by the bounded inverse theorem, it is bounded. Therefore the spectrum consists precisely of those scalars λ for which $\lambda I - T$ is not bijective.

The spectrum of a given operator T is often denoted $\sigma(T)$, and its complement, the resolvent set, is denoted $\rho(T) = \mathbb{K} \setminus \sigma(T)$.

Spectrum and eigenvalues

If λ is an eigenvalue of T, then the operator $T - \lambda I$ is not one-to-one, and therefore its inverse $(T - \lambda I)^{-1}$ is not defined. However, the converse statement is not true: the operator $T - \lambda I$ may not have an inverse, even if λ is not an eigenvalue. Thus the spectrum of an operator always contains all its eigenvalues, but is not limited to them.

For example,consider the Hilbert space $\ell^2(\mathbb{Z})$, that consists of all bi-infinite sequences of real numbers

$$v = (\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots)$$

that have a finite sum of squares $\sum_{i=-\infty}^{+\infty} v_i^2$. The bilateral shift operator T simply displaces every element of the sequence by one position; namely if $u = T(v)$ then $u_i = v_{i-1}$ for every integer i. The eigenvalue equation $T(v) = \lambda v$ has no solution in this space, since it implies that all the values v_i have the same absolute value (if $\lambda = 1$) or are a geometric progression (if $\lambda \neq 1$); either way, the sum of their squares would not be finite. However, the operator $T - \lambda I$ is not invertible if $|\lambda| = 1$. For example, the sequence u such that $u_i = 1/(|i| + 1)$ is in $\ell^2(\mathbb{Z})$; but there is no sequence v in $\ell^2(\mathbb{Z})$ such that $(T - I)v = u$ (that is, $v_{i-1} = u_i + v_i$ for all i).

Basic properties

The spectrum of a bounded operator *T* is always a closed, bounded and non-empty subset of the complex plane.

If the spectrum were empty, then the *resolvent function*

$$R(\lambda) = (\lambda I - T)^{-1}$$

would be defined everywhere on the complex plane and bounded. But it can be shown that the resolvent function *R* is holomorphic on its domain. By the vector-valued version of Liouville's theorem, this function is constant, thus everywhere zero as it is zero at infinity. This would be a contradiction.

The boundedness of the spectrum follows from the Neumann series expansion in λ; the spectrum $\sigma(T)$ is bounded by $||T||$. A similar result shows the closedness of the spectrum.

The bound $||T||$ on the spectrum can be refined somewhat. The *spectral radius*, $r(T)$, of T is the radius of the smallest circle in the complex plane which is centered at the origin and contains the spectrum $\sigma(T)$ inside of it, i.e.

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

The **spectral radius formula** says[1] that for any element T of a Banach algebra,

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

Classification of points in the spectrum of an operator

Further information: Decomposition of spectrum (functional analysis)

A bounded operator *T* on a Banach space is invertible, i.e. has a bounded inverse, if and only if *T* is bounded below and has dense range. Accordingly, the spectrum of *T* can be divided into the following parts:

1. $\lambda \in \sigma(T)$, if $\lambda - T$ is not bounded below. In particular, this is the case, if $\lambda - T$ is not injective, that is, λ is an eigenvalue. The set of eigenvalues is called the **point spectrum** of *T* and denoted by $\boldsymbol{\sigma_p(T)}$. Alternatively, $\lambda - T$ could be one-to-one but still not be bounded below. Such λ is not an eigenvalue but still an *approximate eigenvalue* of *T* (eigenvalues themselves are also approximate eigenvalues). The set of approximate eigenvalues (which includes the point spectrum) is called the **approximate point spectrum** of *T*, denoted by $\boldsymbol{\sigma_{ap}(T)}$.
2. $\lambda \in \sigma(T)$, if $\lambda - T$ does not have dense range. No notation is used to describe the set of all λ, which satisfy this condition, but for a subset: If $\lambda - T$ does not have dense range but is injective, λ is said to be in the **residual spectrum** of *T*, denoted by $\boldsymbol{\sigma_r(T)}$.

Note that the approximate point spectrum and residual spectrum are not necessarily disjoint (however, the point spectrum and the residual spectrum are).

The following subsections provide more details on the three parts of $\sigma(T)$ sketched above.

Point spectrum

If an operator is not injective (so there is some nonzero *x* with $T(x) = 0$), then it is clearly not invertible. So if λ is an eigenvalue of *T*, one necessarily has $\lambda \in \sigma(T)$. The set of eigenvalues of *T* is also called the **point spectrum** of *T*, denoted by $\boldsymbol{\sigma_p(T)}$.

Approximate point spectrum

More generally, *T* is not invertible if it is not bounded below; that is, if there is no $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in X$. So the spectrum includes the set of **approximate eigenvalues**, which are those λ such that $T - \lambda I$ is not bounded below; equivalently, it is the set of λ for which there is a sequence of unit vectors x_1, x_2, ... for which

$$\lim_{n\to\infty} \|Tx_n - \lambda x_n\| = 0.$$

The set of approximate eigenvalues is known as the **approximate point spectrum**, denoted by $\boldsymbol{\sigma_{ap}(T)}$.

It is easy to see that the eigenvalues lie in the approximate point spectrum.

Example Consider the bilateral shift *T* on $l^2(\mathbf{Z})$ defined by

$$T(\cdots, a_{-1}, \hat{a}_0, a_1, \cdots) = (\cdots, \hat{a}_{-1}, a_0, a_1, \cdots)$$

where the ˆ denotes the zero-th position. Direct calculation shows T has no eigenvalues, but every λ with |λ| = 1 is an approximate eigenvalue; letting x_n be the vector

$$\frac{1}{\sqrt{n}}(\ldots, 0, 1, \lambda^{-1}, \lambda^{-2}, \ldots, \lambda^{1-n}, 0, \ldots)$$

then $\|x_n\| = 1$ for all n, but

$$\|Tx_n - \lambda x_n\| = \sqrt{\frac{2}{n}} \to 0.$$

Since T is a unitary operator, its spectrum lie on the unit circle. Therefore the approximate point spectrum of T is its entire spectrum. This is true for a more general class of operators.

A unitary operator is normal. By spectral theorem, a bounded operator on a Hilbert space is normal if and only if it is a multiplication operator. It can be shown that, in general, the approximate point spectrum of a bounded multiplication operator is its spectrum.

Residual spectrum

An operator may be injective, even bounded below, but not invertible. The unilateral shift on $l^2(\mathbf{N})$ is such an example. This shift operator is an isometry, therefore bounded below by 1. But it is not invertible as it is not surjective. The set of λ for which $\lambda I - T$ is injective but does not have dense range is known as the **residual spectrum** or **compression spectrum** of T and is denoted by $\boldsymbol{\sigma_r(T)}$.

Continuous spectrum

The set of all λ for which $\lambda I - T$ is injective and has dense range, but is not surjective, is called the **continuous spectrum** of T, denoted by $\boldsymbol{\sigma_c(T)}$. The continuous spectrum therefore consists of those approximate eigenvalues which are not eigenvalues and do not lie in the residual spectrum. That is,

$$\sigma_c(T) = \sigma_{ap}(T) \setminus (\sigma_r(T) \cup \sigma_p(T)).$$

Peripheral spectrum

The peripheral spectrum of an operator is defined as the set of points in its spectrum which have modulus equal to its spectral radius.

Example

The hydrogen atom provides an example of this decomposition. The eigenfunctions of the hydrogen atom Hamiltonian are called **eigenstates** and are grouped into two categories. The bound states of the hydrogen atom correspond to the discrete part of the spectrum (they have a discrete set of eigenvalues that can be computed by Rydberg formula) while the ionization processes are described by the continuous part (the energy of the collision/ionization is not quantized).

Further results

If T is a compact operator, then it can be shown that any nonzero λ in the spectrum is an eigenvalue. In other words, the spectrum of such an operator, which was defined as a generalization of the concept of eigenvalues, consists in this case only of the usual eigenvalues, and possibly 0.

If X is a Hilbert space and T is a normal operator, then a remarkable result known as the spectral theorem gives an analogue of the diagonalisation theorem for normal finite-dimensional operators (Hermitian matrices, for example).

Spectrum of an unbounded operator

One can extend the definition of spectrum for unbounded operators on a Banach space X, operators which are no longer elements in the Banach algebra $B(X)$. One proceeds in a manner similar to the bounded case. A complex number λ is said to be in the **resolvent set**, that is, the complement of the spectrum of a linear operator

$$T : D \subset X \to X$$

if the operator

$$T - \lambda I : D \to X$$

has a bounded inverse, i.e. if there exists a bounded operator

$$S : X \to D$$

such that

$$S(T - I\lambda) = I_D,\ (T - I\lambda)S = I_X.$$

A complex number λ is then in the **spectrum** if this property fails to hold. One can classify the spectrum in exactly the same way as in the bounded case.

The spectrum of an unbounded operator is in general a closed, possibly empty, subset of the complex plane.

For λ to be in the resolvent (i.e. not in the spectrum), as in the bounded case $\lambda I - T$ must be bijective, since it must have a two-sided inverse. As before if an inverse exists then its linearity is immediate, but in general it may not be bounded, so this condition must be checked separately.

However, boundedness of the inverse *does* follow directly from its existence if one introduces the additional assumption that T is closed; this follows from the closed graph theorem. Therefore, as in the bounded case, a complex number λ lies in the spectrum of a closed operator T if and only if $\lambda I - T$ is not bijective. Note that the class of closed operators includes all bounded operators.

Via its spectral measures, one can define a decomposition of the spectrum of any self adjoint operator, bounded or otherwise into absolutely continuous, pure point, and singular parts.

Spectrum of a unital Banach algebra

Let B be a complex Banach algebra containing a unit e. Then we define the spectrum $\sigma(x)$ (or more explicitly $\sigma_B(x)$) of an element x of B to be the set of those complex numbers λ for which $\lambda e - x$ is not invertible in B. This extends the definition for bounded linear operators $B(X)$ on a Banach space X, since $B(X)$ is a Banach algebra.

See also

- Essential spectrum
- Self-adjoint operator

- Pseudospectrum

References

1. ^ Theorem 3.3.3 of Kadison & Ringrose, 1983, *Fundamentals of the Theory of Operator Algebras, Vol. I: Elementary Theory,* New York: Academic Press, Inc.

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Categories: Spectral theory
