Fredholm determinant

From Wikipedia, the free encyclopedia

In mathematics, the **Fredholm determinant** is a complex-valued function which generalizes the determinant of a matrix. It is defined for bounded operators on a Hilbert space which differ from the identity operator by a trace-class operator. The function is named after the mathematician Erik Ivar Fredholm.

Fredholm determinants have had many applications in mathematical physics, the most celebrated example being Gábor Szegő's limit formula, proved in response to a question raised by Lars Onsager and C. N. Yang on the spontaneous magnetization of the Ising model.

Contents

Definition

Let H be a Hilbert space and G the set of bounded invertible operators on H of the form $I + T$, where T is a trace-class operator. G is a group because

$$(I+T)^{-1} - I = -T(I+T)^{-1}.$$

It has a natural metric given by $d(X, Y) = \|X - Y\|_1$, where $\|\cdot\|_1$ is the trace-class norm.

If H is a Hilbert space with inner product $(\cdot,\cdot)$, then so too is the kth exterior power $\Lambda^k H$ with inner product

$$(v_1 \wedge v_2 \wedge \cdots \wedge v_k, w_1 \wedge w_2 \wedge \cdots \wedge w_k) = \det(v_i, w_j).$$

In particular

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}, \qquad (i_1 < i_2 < \cdots < i_k)$$

gives an orthonormal basis of $\Lambda^k H$ if (e_i) is an orthonormal basis of H. If A is a bounded operator on H, then A functorially defines a bounded operator $\Lambda^k(A)$ on $\Lambda^k H$ by

$$\Lambda^k(A) v_1 \wedge v_2 \wedge \cdots \wedge v_k = Av_1 \wedge Av_2 \wedge \cdots \wedge Av_k.$$

If A is trace-class, then $\Lambda^k(A)$ is also trace-class with

$$\|\Lambda^k(A)\|_1 \le \|A\|_1^k / k!.$$

This shows that the definition of the **Fredholm determinant** given by

$$\det(I+A)=\sum_{k=0}^{\infty}\mathrm{Tr}\Lambda^k(A)$$

makes sense.

Properties

- If A is a trace-class operator.

  $$\det(I+zA)=\sum_{k=0}^{\infty}z^k\mathrm{Tr}\Lambda^k(A)$$

  defines an entire function such that

  $$|\det(I+zA)|\leq\exp(|z|\cdot\|A\|_1).$$

- The function det(I + A) is continuous on trace-class operators, with

  $$|\det(I+A)-\det(I+B)|\leq\|A-B\|_1\exp(\|A\|_1+\|B\|_1+1).$$

One can improve this inequality slightly to the following, as noted in Chapter 5 of Simon:

$$|\det(I+A)-\det(I+B)|\leq\|A-B\|_1\exp(\max(\|A\|_1,\|B\|_1)+1).$$

- If A and B are trace-class then

  $$\det(I+A)\cdot\det(I+B)=\det(I+A)(I+B).$$

- The function *det* defines a homomorphism of G into the multiplicative group **C*** of non-zero complex numbers.
- If T is in G and X is invertible,

  $$\det XTX^{-1}=\det T.$$

- If A is trace-class, then

  $$\det e^A=\exp\ \mathrm{Tr}(A).$$

  $$\log\det(I+zA)=\mathrm{Tr}(\log(I+zA))=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\mathrm{Tr}A^k}{k}z^k$$

Fredholm determinants of commutators

A function $F(t)$ from (a, b) into G is said to be *differentiable* if $F(t)$ -I is differentiable as a map into the trace-class operators, i.e. if the limit

$$\dot{F}(t)=\lim_{h\to0}\frac{F(t+h)-F(t)}{h}$$

exists in trace-class norm.

If $g(t)$ is a differentiable function with values in trace-class operators, then so too is exp $g(t)$ and

$$F^{-1}\dot{F} = \frac{\mathrm{id} - \exp -\mathrm{ad}g(t)}{\mathrm{ad}g(t)} \cdot \dot{g}(t),$$

where

$$\mathrm{ad}(X) \cdot Y = XY - YX.$$

Israel Gohberg and Mark Krein proved that if F is a differentiable function into G, then $f = \det F$ is a differentiable map into $\mathbf{C}^*$ with

$$f^{-1}\dot{f} = \mathrm{Tr} F^{-1}\dot{F}.$$

This result was used by Joel Pincus, William Helton and Roger Howe to prove that if A and B are bounded operators with trace-class commutator $AB - BA$, then

$$\det e^A e^B e^{-A} e^{-B} = \exp \mathrm{Tr}(AB - BA).$$

Szegő limit formula

See also: Szegő limit theorems

Let $H = L^2(S^1)$ and let P be the orthogonal projection onto the Hardy space $H^2(S^1)$.

If f is a smooth function on the circle, let $m(f)$ denote the corresponding multiplication operator on H.

The commutator

$Pm(f) - m(f)P$

is trace-class.

Let $T(f)$ be the Toeplitz operator on $H^2(S^1)$ defined by

$$T(f) = Pm(f)P,$$

then the additive commutator

$$T(f)T(g) - T(g)T(f)$$

is trace-class if f and g are smooth.

Berger and Shaw proved that

$$\mathrm{tr}(T(f)T(g) - T(g)T(f)) = \frac{1}{2\pi i} \int_0^{2\pi} f dg.$$

If f and g are smooth, then

$$T(e^{f+g})T(e^{-f})T(e^{-g})$$

is in G.

Harold Widom used the result of Pincus-Helton-Howe to prove that

$$\det T(e^f)T(e^{-f}) = \exp \sum_{n>0} n a_n a_{-n},$$

where

$$f(z) = \sum a_n z^n.$$

He used this to give a new proof of Gábor Szegő's celebrated limit formula:

$$\lim_{N\to\infty} \det P_N m(e^f) P_N = \exp \sum_{n>0} n a_n a_{-n},$$

where P_N is the projection onto the subspace of H spanned by 1, z, ..., z^N and $a_0 = 0$.

Szegő's limit formula was proved in 1951 in response to a question raised by the work Lars Onsager and C. N. Yang on the calculation of the spontaneous magnetization for the Ising model. The formula of Widom, which leads quite quickly to Szegő's limit formula, is also equivalent to the duality between bosons and fermions in conformal field theory. A singular version of Szegő's limit formula for functions supported on an arc of the circle was proved by Widom; it has been applied to establish probabilistic results on the eigenvalue distribution of random unitary matrices.

Informal presentation

The section below provides an informal definition for the Fredholm determinant. A proper definition requires a presentation showing that each of the manipulations are well-defined, convergent, and so on, for the given situation for which the Fredholm determinant is contemplated. Since the kernel K may be defined on a large variety of Hilbert spaces and Banach spaces, this is a non-trivial exercise.

The Fredholm determinant may be defined as

$$\det(I - \lambda K) = \left[\sum_{n=0}^{\infty}(-\lambda)^n \operatorname{Tr} K^n\right] = \exp\left(\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\operatorname{Tr} A^n}{n} z^n\right)$$

where K is an integral operator. The trace of the operator is given by

$$\operatorname{Tr} K = \int K(x,x)\, dx$$

and

$$\operatorname{Tr} \Lambda^2(K) = \frac{1}{2!}\iint K(x,x)K(y,y) - K(x,y)K(y,x)\, dxdy$$

and in generall :$\operatorname{Tr} K^n = \frac{1}{n!}\int \cdots \int \det K(x_i, x_j)|_{1\le i,j\le n}\, dx_1 \cdots dx_n$. The trace is well-defined for these kernels, since these are trace-class or nuclear operators.

Applications

The Fredholm determinant was used by physicist John A. Wheeler (1937, Phys. Rev. 52:1107) to help provide mathematical description of the wavefunction for a composite nucleus composed of antisymmetrized combination of partial wavefunctions by the method of Resonating Group Structure. This method corresponds to the various possible ways of distributing the energy of neutrons and protons into fundamental boson and fermion nucleon cluster groups or building blocks such as the alpha-particle, helium-3, deuterium, triton, di-neutron, etc. When applied to the method of Resonating Group Structure for beta and alpha stable isotopes, use of the Fredholm determinant: (1) determines the energy values of the composite system, and (2) determines scattering and disintegration cross sections. The method of Resonating Group Structure of Wheeler provides the theoretical bases for all subsequent Nucleon Cluster Models and associated cluster energy dynamics for all light and heavy mass isotopes (see review of Cluster Models in physics in N.D. Cook, 2006).

References

- Simon, Barry (2005), *Trace Ideals and Their Applications*, Mathematical Surveys and Monographs, **120**, American Mathematical Society, ISBN 0-8218-3581-5
- Wheeler, John A. (1937), *On the Mathematical Description of Light Nuclei by the Method of Resonating Group Structure*, Physical Review, **52**, p. 1107
- Cook, Norman D. (2006), *Models of the Atomic Nucleus*, Springer
- Bornemann, Folkmar (2010), "On the numerical evaluation of Fredholm determinants", *Math. Comp.* (Springer) **79**: 871-915

Categories: Fredholm theory | Determinants

Navigation menu