a prime number is a positive integer p with $p > 1$ such that p is not divisible by any positive integer n with $1 < n < p$. The prime numbers are $2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$

Fix a prime number p.

Every non-zero rational number $x \in \mathbb{Q}$ can be written

$$x = \frac{a}{b} p^n$$

where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are not divisible by p. n is the valuation of x and is denoted $\mathrm{ord}(x)$. On the rational numbers $\mathbb{Q}$ define a norm by

$$|x|_p = p^{-\mathrm{ord}(x)} \qquad x \neq 0$$

$$|0|_p = 0$$

This norm satisfies

$$|x+y|_p \leqq \max\{|x|_p, |y|_p\}$$

If $x, y \in \mathbb{Q}$, the distance from x to y is defined to be $|x-y|_p$. This is a

③ LICZBY p-ADYCZNE I ZESPOLONE

metric on $\mathbb{Q}$, and $\mathbb{Q}$ can now be completed in this metric. The completion is done using Cauchy sequences just as in the usual case of completing $\mathbb{Q}$ to obtain the real numbers $\mathbb{R}$.

The completion of $\mathbb{Q}$ using $| \, |_p$ is a topological field which is locally compact and totally disconnected. This field is known as $\boxed{\text{the p-adic numbers}}$ and is denoted $\mathbb{Q}_p$.

A p-adic number can be viewed as a possibly infinite sum

$$\boxed{\sum_{n \in \mathbb{Z}} a_n p^n \qquad a_n \in \{0, 1, \dots, p-1\}}$$

Only a finite number of a_n with $n<0$ are allowed to be non-zero. With $n>0$, infinitely many a_n may be non-zero. Thus each p-adic number can have only a finite pole.

Przykład (suma odwrotności liczb pierwszych)

Chcemy pokazać że suma odwrotności wszystkich kolejnych liczb pierwszych jest nieskończona: $\sum_{i=0}^{\infty} \frac{1}{p_i} = \infty$, gdzie $(p_i)_{i \in \mathbb{N}}$ to ciąg kolejnych liczb pierwszych.

Zauważmy że każda $n \in \mathbb{Z}_+$ daje się zapisać jednoznacznie jako $n = p_{i_1} p_{i_2} \ldots p_{i_m} k^2$, gdzie $k \leq n$, zaś $\{p_{i_1}, \ldots, p_{i_m}\}$ jest zbiorem (być może pustym) niepowtarzających się liczb pierwszych $\leq n$. Zatem

$$\sum_{j=1}^{n} \frac{1}{j} \leq \prod_{p_i \leq n} \left(1 + \frac{1}{p_i}\right) \sum_{k=1}^{n} \frac{1}{k^2}.$$

Z drugiej strony, $\sum_{k=1}^{n} \frac{1}{k^2} \leq M$ oraz $\forall x > 0: \ 1 + x < e^x$. Stąd

$$\sum_{j=1}^{n} \frac{1}{j} \leq M \prod_{p_i \leq n} e^{p_i^{-1}} = M e^{\sum_{p_i \leq n} p_i^{-1}}.$$

To implikuje

$$\ln \sum_{j=1}^{n} \frac{1}{j} - \ln M \leq \sum_{p_i \leq n} \frac{1}{p_i}.$$

To kończy dowód gdyż $\ln \sum_{j=1}^{n} \frac{1}{j} \xrightarrow[n \to \infty]{} \infty$.

Third proof

Here is another proof that actually gives a lower estimate for the partial sums; in particular, it shows that these sums grow at least as fast as $\ln(\ln(n))$. The proof is an adaptation of the product expansion idea of Euler. In the following, a sum or product taken over p always represents a sum or product taken over a specified set of primes.

The proof rests upon the following four inequalities:

- Every positive integer i can be uniquely expressed as the product of a square-free integer and a square. This gives the inequality

$$\sum_{i=1}^{n} \frac{1}{i} \le \prod_{p \le n} \left(1 + \frac{1}{p}\right) \sum_{k=1}^{n} \frac{1}{k^2},$$

  where for every i between 1 and n the (expanded) product contains to the square-free part of i and the sum contains to the square part of i (see fundamental theorem of arithmetic).

- The upper estimate for the natural logarithm

$$\ln(n+1) = \int_1^{n+1} \frac{dx}{x} = \sum_{i=1}^{n} \underbrace{\int_i^{i+1} \frac{dx}{x}}_{<1/i} < \sum_{i=1}^{n} \frac{1}{i}.$$

- The lower estimate $1 + x < \exp(x)$ for the exponential function, which holds for all $x > 0$.

- Let $n \ge 2$. The upper bound (using a telescoping sum) for the partial sums (convergence is all we really need)

$$\sum_{k=1}^{n} \frac{1}{k^2} < 1 + \sum_{k=2}^{n} \underbrace{\left(\frac{1}{k-1/2} - \frac{1}{k+1/2}\right)}_{=1/(k^2-1/4)>1/k^2} = 1 + \frac{2}{3} - \frac{1}{n+1/2} < \frac{5}{3}.$$

Combining all these inequalities, we see that

$$\ln(n+1) < \sum_{i=1}^{n} \frac{1}{i} \le \prod_{p \le n} \left(1 + \frac{1}{p}\right) \sum_{k=1}^{n} \frac{1}{k^2} < \frac{5}{3} \prod_{p \le n} \exp\left(\frac{1}{p}\right) = \frac{5}{3} \exp\left(\sum_{p \le n} \frac{1}{p}\right).$$

Dividing through by 5/3 and taking the natural logarithm of both sides gives

$$\ln \ln(n+1) - \ln \frac{5}{3} < \sum_{p \le n} \frac{1}{p}$$

as desired. ∎

Using

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

(see Basel problem), the above constant $\ln(5/3) = 0.51082...$ can be improved to $\ln(\pi^2/6) = 0.4977...$; in fact it turns out that

Liczby zespolone $\mathbb{C} := \mathbb{R} \times \mathbb{R}$. Wkładamy $\mathbb{R} \subseteq \mathbb{C}$ poprzez $r \mapsto (r,0)$. Rozszerzamy z $\mathbb{R}$ operacje

i) $(r,s) + (r',s') = (r+r', s+s')$

ii) $(r,s)(r',s') = (rr'-ss', rs'+r's)$

iii) $|(r,s)| = \sqrt{r^2+s^2}$

Tracimy porządek, zyskujemy *sprzężenie zespolone*: $\overline{(r,s)} := (r,-s)$. Mamy:

i) $\overline{(r,s)+(r',s')} = \overline{(r,s)} + \overline{(r',s')}$

ii) $\overline{(r,s)\cdot(r',s')} = \overline{(r,s)} \cdot \overline{(r',s')}$

iii) $(r,s)\,\overline{(r,s)} = |(r,s)|^2$

W $\mathbb{C}$ możemy rozwiązać równanie $x^2 = -1$, bo $(0,1)^2 = -(1,0)$. Tradycyjnie $(0,1) =: i$, $i^2 = -1$. Liczby urojone to $(0,r)$, $r \in \mathbb{R}$. Zapisujemy $(r,s) = r + is$.

Fundamentalne Twierdzenie Algebry:

Dla każdego wielomianu $p(z)$ nad $\mathbb{C}$ istnieje $z_0 \in \mathbb{C}$ takie że $p(z_0) = 0$.

Wniosek: $\forall p \, \exists a, z_1, \ldots, z_n \in \mathbb{C} : p(z) = a(z-z_1)\ldots(z-z_n)$, gdzie n to stopień wielomianu p.

Zadanie 3: Udowodnić że ciąg liczb wymiernych $\mathbb{N} \ni n \longmapsto e_n := \sum_{k=0}^{n} \frac{1}{k!} \in \mathbb{Q}$ jest ciągiem Cauchy'ego. Oznaczmy wyznaczoną przezeń liczbę rzeczywistą przez $e := \sum_{k=0}^{\infty} \frac{1}{k!}$. Pokazać że $e \notin \mathbb{Q}$.

Zadanie 4:

Korzystając ze wzoru $e^{i\varphi} = \cos\varphi + i\sin\varphi$ udowodnić że

$$\sum_{k=1}^{n} \cos kx = \frac{\sin\frac{nx}{2}\cos\frac{(n+1)x}{2}}{\sin\frac{x}{2}},$$

$$\sum_{k=1}^{n} \sin kx = \frac{\sin\frac{nx}{2}\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}},$$

$$\forall\, x \in \mathbb{R} \setminus \{2\pi m\}_{m\in\mathbb{Z}}.$$