5 Normed Spaces Vector Spaces

All vector spaces over $\mathbb{R}$ or $\mathbb{C} =: \mathbb{K}$.

Def^n: A norm on a vector space V is a map
$$\|\cdot\| : V \to [0,\infty) \subset \mathbb{R}$$
s.t. $\forall v,w \in V$ & $\lambda \in \mathbb{K}$

1) $\|v\| \geq 0$ & $\|v\| = 0$ iff $v = 0$ — positivity
2) $\|\lambda v\| = |\lambda|\,\|v\|$ — homogeneity
3) $\|v+w\| \leq \|v\| + \|w\|$ — triangle inequality

Eg

$\mathbb{R}^n$ $\quad \|\sum \alpha_i e_i\|_2 = \sqrt{\sum |\alpha_i|^2}$
$\mathbb{C}^n$ $\quad \|\sum \alpha_i e_i\|_2 = \sqrt{\sum |\alpha_i|^2}$ $\Big\}$ Euclidean norm

$\mathbb{C}^n$ $\quad \|\sum \alpha_i e_i\|_p = \left(\sum |\alpha_i|^p\right)^{1/p} \longleftrightarrow$ p-norm $1 \leq p < \infty$

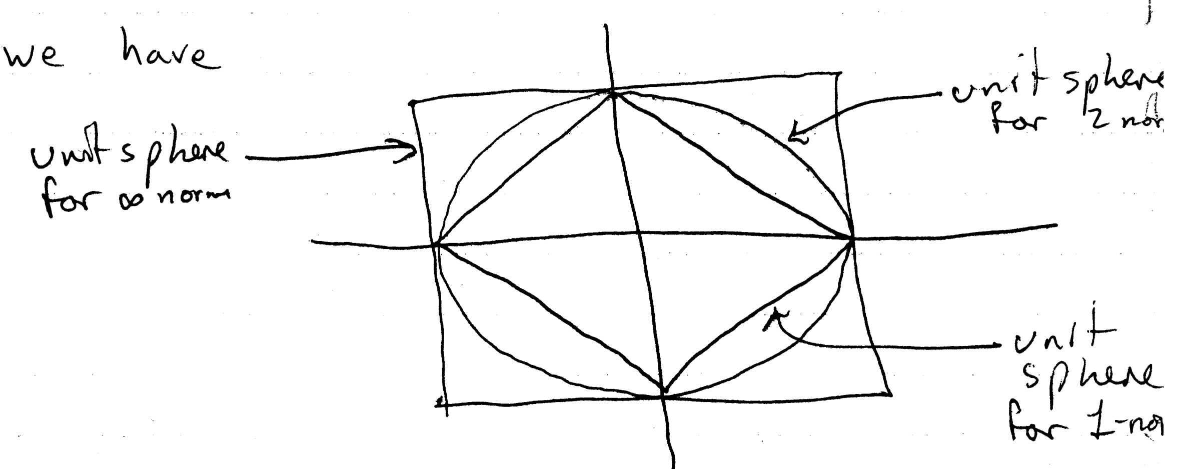
$\mathbb{C}^n$ $\quad \|\sum \alpha_i e_i\|_\infty = \max |\alpha_i| \leftarrow$ sup or ∞ norm

$$\ell^p = \{(\alpha_i)_{i=1}^\infty : \alpha_i \in \mathbb{C},\ \sum^\infty |\alpha_i|^p < \infty\}$$
&
$$\|(\alpha_i)_{i=1}^\infty\|_p = \left(\sum^\infty |\alpha_i|^p\right)^{1/p}$$

$$\ell^\infty = \{(\alpha_i)_{i=1}^\infty : \alpha_i \in \mathbb{C},\ \sup|\alpha_i| < \infty\}$$
&
$$\|(\alpha_i)_{i=1}^\infty\|_\infty := \sup|\alpha_i|$$

That the p-norms are norms – in particular the triangle inequality – is hard. $p=1$ & $p=\infty$ easy

In $\mathbb{R}^2$ we have



<u>Prop^n</u> Every normed vector space is a metric space & so a topological vector space (over $\mathbb{R}$ or $\mathbb{C}$)

<u>Pf:</u> Define $d(v,w) := \|v-w\|$ $v,w \in V$. This is a metric

(1) $d(v,w) \geq 0$ because $\|v-w\| \geq 0$
& $d(v,w) = 0$ iff $\|v-w\| = 0$ iff $v = w$.

(2) $d(w,v) = \|w-v\| = \|(-1)(v-w)\|$
$= |(-1)| \, \|v-w\| = d(v,w)$

(3) $d(v,w) + d(w,z) = \|v-w\| + \|w-z\|$
$\geq \|v-w+w-z\| = \|v-z\| = d(v,z)$

Continuity of scalar mult follows from homogeneity & continuity of addition follows from triangle inequality □

<u>Def^n:</u> A Banach space is a normed linear space which is complete in the metric defined above.

Eg All my previous examples are Banach spaces. Here is another:

$$L^2(\mathbb{R}^n) := \{ f : \mathbb{R}^n \to \mathbb{C} : f \text{ measurable}, \int |f|^2 d^n x < \infty$$

$$\& \quad \|f\|_2 = \sqrt{\int |f|^2 d^n x}$$

[OK, there is a small lie here about sets of measure zero

<u>Claim:</u> All norms on finite dimensional vector spaces (over $\mathbb{R}$ or $\mathbb{C}$) are equivalent, meaning

$\exists c_1, c_2 > 0 \; \forall v \in V$ $\quad c_1 \|v\|_a \leq \|v\|_b \leq c_2 \|v\|_a$

for any two norms $\|\cdot\|_a$, $\|\cdot\|_b$ [c_1, c_2 depend on the norms]

[fin dim = finite dimensional]

To prove this we need

1. All fin dim vector spaces are isomorphic to $\mathbb{R}^n$ some n (Exercise using bases)
2. All fin dim vector spaces are Banach spaces (Exercise using triangle inequality)

Strategy of proof: show all norms are equivalent to the 1-norm.

Lemma: Let X be an n-dimensional normed space (over $\mathbb{R}$ say) with basis $x_1, \dots, x_n$. Then $\exists\, c > 0$ s.t. $\forall \alpha_1, \dots, \alpha_n \in \mathbb{R}$

$$\|\alpha_1 x_1 + \dots + \alpha_n x_n\| \geq c\left(|\alpha_1| + \dots + |\alpha_n|\right)$$

Pf: Let $S = \sum^n |\alpha_i|$ & $\beta_i = \frac{\alpha_i}{S}$. Then we want

$$\|\beta_1 x_1 + \dots + \beta_n x_n\| \geq c$$

for all β_i s.t. $\sum^n |\beta_i| = 1$.

Suppose, for a contradiction, that no such c exists. Then we can find a sequence

$$y_m = \sum \beta_i^m x_i, \quad \sum^n |\beta_i^m| = 1 \ \forall m$$

bdd = bounded

s.t. $\|y_m\| \to 0$ (and so $y_m \to 0$).

For each i, $\{\beta_i^m\}_{m=1}^{\infty}$ is a bdd sequence, & so has a convergent subsequence by Bolzano-Weierstraß.

Pick such a subsequence for (β_1^m) with limit β_1, & let $y_{1,m}$ be the resulting subsequence of y_m.

Repeat for $\beta_2^m, \dots, \beta_n^m$ ending up with

$$y_{n,m} = \sum \beta_i^m x_i \quad , \quad \sum^n |\beta_i^m| = 1 \quad \forall m$$

& $\beta_i^m \to \beta_i$ for each i. Let $y = \sum \beta_i x_i$.

Now

$$\|y_{n,m} - y\| = \|\sum (\beta_i^m - \beta_i) x_i\|$$

$$\leq \sum |\beta_i^m - \beta_i| \, \|x_i\| \quad \text{homog \& triangle}$$

$$\to 0$$

So $y_{n,m} \xrightarrow{m\to\infty} y \neq 0$. Contradiction. Hence there is such a $c > 0$. $\square$

Theorem All norms on a finite dimensional vector space are equivalent.

Pf: Let $\|\cdot\|_1$ be the 1-norm & $\|\cdot\|$ any other norm. Then for $y = \sum \alpha_i x_i$ (x_i basis) & with $c > 0$ as in the Lemma

$$c\|y\|_1 = c \sum_{i=1}^n |\alpha_i| \leq \|y\| \leq \sum_{i=1}^n |\alpha_i| \, \|x_i\| \leq m \sum_{i=1}^n |\alpha_i|$$

$$= m\|y\|_1$$

where $m = \max \|x_i\|$.

So $\|\cdot\|$ is equivalent to $\|\cdot\|_1$. Now let $\|\cdot\|_a$, $\|\cdot\|_b$ be any two norms on our vector space. Then $\exists$ constants $C_1, C_2, C_3, C_4 > 0$ s.t. $\forall y$

$$\|y\|_a \leq C_1 \|y\|_1 \leq C_2 \|y\|_b \leq C_3 \|y\|_1 \leq C_4 \|y\|_a$$

gives equivalence. $\square$

Inner Product & Hilbert Spaces

Def^n: Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. An inner product on V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}$ s.t.

1) $\langle v, v \rangle \geq 0$ & $\langle v, v \rangle = 0$ iff $v = 0$
2) $\langle v, w \rangle = \overline{\langle w, v \rangle}$
3) $\langle \lambda v, w \rangle = \lambda \langle v, w \rangle$

The pair $(V, \langle \cdot, \cdot \rangle)$ is an inner product space.

Lemma (Cauchy-Schwarz) If $(V, \langle \cdot, \cdot \rangle)$ is an inner product space, then for all $v, w \in V$

$$|\langle v, w \rangle| \leq \sqrt{\langle v, v \rangle \langle w, w \rangle}$$

Pf: Easy if $w = 0$, so assume $w \neq 0$ & $m \in \mathbb{K}$. Then

$$0 \leq \langle v - mw, v - mw \rangle = \langle v, v \rangle - \bar{m} \langle v, w \rangle - m \langle w, v \rangle + |m|^2 \langle w, w \rangle$$

$$2) \& 3) \mapsto \left(\langle v, mw \rangle = \overline{\langle mw, v \rangle} = \overline{m \langle w, v \rangle} = \bar{m} \langle v, w \rangle \right)$$

Now choose $m = \dfrac{\langle v, w \rangle}{\langle w, w \rangle}$, so that

$$0 \leq \langle v, v \rangle - \frac{|\langle v, w \rangle|^2}{\langle w, w \rangle} \quad \text{or} \quad |\langle v, w \rangle|^2 \leq \langle v, v \rangle \langle w, w \rangle$$

$\square$

Lemma: If $(V, \langle \cdot, \cdot \rangle)$ is an inner product space then

$$\|v\|_2 := \sqrt{\langle v, v \rangle} \qquad v \in V$$

defines a norm on V.

Pf: Everything is easy except the triangle inequality (a universal truth!!). For this...

$$\begin{aligned}
\|v+w\|_2^2 &= \langle v+w, v+w\rangle \\
&= \|v\|^2 + \|w\|^2 + \langle v, w\rangle + \langle w, v\rangle \\
&= \|v\|^2 + \|w\|^2 + 2\,\mathrm{Re}\langle v, w\rangle \\
&\leq \|v\|^2 + \|w\|^2 + 2|\langle v, w\rangle| \\
&\leq \|v\|^2 + \|w\|^2 + 2\|v\|\,\|w\| \qquad \text{Cauchy-Schwarz} \\
&= (\|v\| + \|w\|)^2 . \qquad \square
\end{aligned}$$

Def^n: A Hilbert space is an inner product space which is complete in the metric coming from the norm coming from the inner product

Eg $\mathbb{C}^n$, $\langle \sum \alpha_i e_i, \sum \beta_i e_i\rangle = \sum^n \alpha_i \overline{\beta_i}$

$\ell^2(\mathbb{N})$, $\langle (\alpha_i)_{i=1}^\infty, (\beta_i)_{i=1}^\infty\rangle = \sum^\infty \alpha_i \overline{\beta_i}$

$L^2(\mathbb{R}^n)$, $\langle f, g\rangle = \int f \overline{g}\, d^n x$

Riesz Theorem: Let H be a Hilbert space and $H^* := \{\varphi \in \mathrm{Hom}(H, \mathbb{C}) \mid \varphi \text{ is continuous}\}$ the dual space of continuous linear functionals. Then

$$\boxed{\forall \varphi \in H^*\ \exists!\, v_\varphi \in H\ \forall v \in H: \ \varphi(v) = \langle v_\varphi | v\rangle}$$

In addition, $\|v_\varphi\| = \|\varphi\|$.