EXERCISES 1

① Let A be a C^*-algebra. Show that $\forall a \in A: \|a^*\| = \|a\|$. If A is unital, show also that $1^* = 1$ and $\|1\| = 1$.

② Let A be a unital C^*-algebra, and $q \in \mathbb{C}$. Find all values of q for which the following implication holds: $(a \in A \text{ and } a^*a - q\, a a^* = 1 - q)$

$$\Rightarrow \|a\| \leq 1.$$

③ Let $(V, \| \|)$ be a normed vector space. Prove that $V \xrightarrow{\| \|} \mathbb{R}$ is continuous with respect to the norm-induced topology.

④ Let $(V, \| \|)$ be a normed vector space with a norm-compatible scalar product. Show that $\forall x, y \in V$:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Explain why it is called the "parallelogram law".

⑤ With the assumptions as in the preceeding exercise, show that $\forall x, y \in V$:

$$4\langle y|x\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2.$$

⑥ Prove that the formula

$$\|(x_0, x_1, \ldots, x_n, 0, \ldots)\| := \sqrt{\sum_{i=0}^{n} |x_i|^2}$$ defines a norm on $\bigoplus_{n\in\mathbb{N}} \mathbb{C}$ making it a normed vector space. Show that $\bigoplus_{n\in\mathbb{N}} \mathbb{C}$ is <u>not</u> a Banach space with respect to this norm.