EXERCISES 3

Let $A \xrightarrow{\tau} \mathbb{C}$ be a positive linear functional on a C^*-algebra A, i.e. a linear functional s.t. $\forall a \in A$: $\tau(a^*a) \geq 0$.

① Show that $\forall a, b \in A$:

Ⓐ $\tau(a^*b) = \overline{\tau(ba^*)}$.

Ⓑ $|\tau(a^*b)|^2 \leq \tau(a^*a)\,\tau(b^*b)$

② Prove that $N_\tau := \{a \in A \mid \tau(a^*a) = 0\}$ is a left ideal using ⓘⒷ.

③ Assume that A is unital.
Using the fact that $\|f\| = f(1)$ for <u>any</u> positive linear functional on A, show that $\tau(b^*a^*ab) \leq \|a\|^2 \tau(b^*b)$, $\forall a, b \in A$.

④ Taking advantage of the commutative Gelfand-Naimark theorem, prove that $f(1) = \|f\|$ for any positive linear functional on a unital C^*-algebra.