EXERCISES 4

① Let A be a unital Banach algebra. Show that $\sum_{n=0}^{\infty} |\alpha_n| \, \|a\|^n < \infty \Rightarrow \exists \sum_{n=0}^{\infty} \alpha_n a^n \in A$.

② Show that, if A is a unital Banach algebra in which every non-zero element is invertible, then $A \cong \mathbb{C}$.

③ Let $A := M_2(\mathbb{C})$, $a := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Compute the norm $\|a\|$ and the spectral radius $r(a)$.

④ Let A be a unital Banach algebra. Show that $\forall a, b \in A : r(ab) = r(ba)$. Prove also that there are <u>no bounded</u> operators a and b on a Banach space such that $[a, b] := ab - ba = 1$.

⑤ Let A be a unital C^*-algebra. We say that $u \in A$ is <u>unitary</u> $\Leftrightarrow$ $uu^* = 1 = u^*u$. Prove that $\operatorname{spec}(u) \subseteq U(1) := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ for any unitary $u \in A$.

(6) Let A be a unital C^*-algebra. We say that $r \in A$ is <u>self-adjoint</u> $\Leftrightarrow$ $r^* = r$. Prove that $\mathrm{spec}(r) \subset \mathbb{R}$ for any self-adjoint $r \in A$.