EXERCISES 5

① Let V be a normed space. Prove that the norm completion of V admits a unique Banach space structure such that V is a normed subspace of the completion.

② Let A be a unital C^*-algebra. Show that, if $u \in A$ satisfies $\|u\| = 1 = \|u^{-1}\|$, then u is a unitary.

③ Show that the universal C^*-algebra of the algebra $\mathbb{C}[u, u^{-1}]$ with the $*$-structure given by $u^* = u^{-1}$ is isomorphic with $C(S^1)$.

④ Let A be a Banach with an involution satisfying $\forall x \in A : \|x\|^2 \leq \|x^* x\|$.

Prove that A is a C^*-algebra.

⑤ Let u be a unitary in a unital C^*-algebra. Show that $\|1 - u\| < 2 \Rightarrow \mathrm{spec}(u) \neq U(1)$.