EXERCISES 6

1. Let J be the (unital) C^*-subalgebra of $B(\ell^2(\mathbb{N}))$ generated by the unilateral shift s given by

$$\forall i \in \mathbb{N} : \; s e_i := e_{i+1},$$

where $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $\ell^2(\mathbb{N})$. Let J_q be the (unital) C^*-subalgebra of $B(\ell^2(\mathbb{N}))$ generated by the weighted unilateral shift x given by

$$\forall i \in \mathbb{N} : \; x e_i := \sqrt{1-q^{i+1}}\, e_{i+1},$$

where $0 \leq q < 1$. Show that

$$s^* s = 1 \;, \quad x^* x - q x x^* = 1 - q \;,$$

and prove that the C^*-algebras J and J_q are isomorphic.

(2) Let $C(\mathbb{T}^2_\theta)$ be the universal C^*-algebra generated by unitaries U and V satisfying the commutation relation

$$UV = e^{2\pi i \theta} VU, \quad \theta \in \mathbb{R} \setminus \mathbb{Q}.$$

Find a representation of this C^*-algebra on $\ell^2(\mathbb{Z})$.

(3) The <u>unitisation</u> $\tilde{A}$ of a C^*-algebra A is the vector space $A \oplus \mathbb{C}$ endowed with the $*$-algebra structure given by the formulae

$$(a,\lambda)(b,\mu) := (ab + \lambda b + \mu a, \lambda\mu),$$

$$(a,\lambda)^* := (a^*, \bar{\lambda}).$$

Prove that there exists a norm making $\tilde{A}$ a C^*-algebra. Show that A is a C^* subalgebra of $\tilde{A}$. (Murphy Theorem 2.1.6.) Check that every $*$-homomorphism $\varphi: A \to B$ extends uniquely to a unital $*$-homomorphism $\tilde{\varphi}: \tilde{A} \to \tilde{B}$.

(4) Show that a $*$-homomorphism $\varphi : A \to B$ between C^*-algebras is necessarily preserving or decreasing the norm. (Murphy Theorem 2.1.7.)

(5) Prove that a non-zero algebra homomorphism from any C^*-algebra A to $\mathbb{C}$ is automatically $*$-preserving. (Murphy Theorem 2.1.9.)

(6) Prove that any closed ideal I in any C^*-algebra are selfadjoint, i.e. $x \in I \Rightarrow x^* \in I$. (Murphy Theorem 3.1.3)

(7) Let I be a closed ideal for a C^*-algebra A. Prove that the formula $\|[a]\| := \inf_{j \in I} \|a + j\|$ defines a norm on A/I making it a C^*-algebra. (Murphy Theorem 3.1.4.)

(8) Prove that an injective $*$-homomorphism between C^*-algebras is necessarily isometric. (Murphy Theorem 3.1.5.)

(9) Let $\varphi : A \to B$ be a $*$-homomorphism between C^*-algebras. Prove that $\varphi(A)$ is closed in B. (Murphy Theorem 3.1.6.)

(10) Let I and J be closed ideals in A.

Prove that: (Murphy Theorem 3.1.7)

(A) $I+J$ is a closed ideal,

(B) $I \cap J = IJ$. (Murphy p. 82.)