WHY HILBERT SPACES AND C*-ALGEBRAS?

① $\bigoplus_{\mathbb{N}} \mathbb{C} \quad \subset \quad \ell^2(\mathbb{N}) \quad \subset \quad \prod_{\mathbb{N}} \mathbb{C}$

finite $\subset$ square summable $\subset$ all

The left is too small, the right is too large, and the middle is a Hilbert space.

② In quantum mechanics, the wave functions are elements of a Hilbert space.

③ Results of measurements are given as eigenvalues of operators on a Hilbert space.

④ The Uncertaintity Principle is a theorem following from a simple commutation relation:

$$\left[\frac{d}{dx}, x\right] = I,$$

$$\left[\frac{d}{dx}, x\right](f) = \frac{d}{dx}(x f(x)) - x\left(\frac{d}{dx} f(x)\right) = f(x).$$

⑤ There plenty of interesting norm-closed $*$-subalgebras in the algebra $B(H)$ of all bounded operators on a Hilbert space H.

⑥ The Gelfand-Naimark theorems ensure that all C^*-algebras are operator algebras, and all commutative C^*-algebras are algebras of all continuous $\mathbb{C}$-valued vanishing at infinity functions on a locally compact Hausdorff space.

⑦ Plenty of topological aspects of locally compact Hausdorff spaces can be expressed in C^*-terms in a way that makes sense for arbitrary C^*-algebras.

(8) Spectacular success of noncommutative topology: K-theory (KK-theory) (locally) compact quantum groups.

(9) Possibility to use Dirac operators to introduce geometry on C^*-algebras.

(10) Spectacular success of cyclic homology as a replacement of de Rham cohomology.

(11) Usefullness of the noncommutative geometry point of view in solving classical problems.

(12) Differential geometry and topology is the language of General Relativity. Operators on Hilbert spaces are the language of Quantum Mechanics. There is a hope that noncommutative geometry and topology will be the language of the long awaited quantum gravity.

Standing assumption: Unless otherwise stated, our ground field is always assumed to be the field of complex numbers (with its standard Euclidean topology).

A normed vector space V is a vector space V equipped with a norm $\| \ \|$, i.e. a map $\| \ \| : V \to [0, +\infty[\subset \mathbb{R}$ satisfying:

i) $\|v\| = 0 \Rightarrow v = 0$,

ii) $\forall v \in V, \lambda \in \mathbb{C} : \|\lambda v\| = |\lambda| \, \|v\|$,

iii) $\forall v, w \in V : \|v + w\| \leq \|v\| + \|w\|$

A Banach space V is a normed vector space V whose all Cauchy sequences are convergent with respect to the norm-induced topology. In other words, a normed space is Banach if it is (sequencially) complete.

A Hilbert space H is a Banach space H equipped with a norm-compatible scalar product, that is a map $H \times H \xrightarrow{\langle | \rangle} \mathbb{C}$ satisfying:

i) $\forall v_1, v_2, v_3, v_4 \in H$: $\langle v_1 + v_2 | v_3 + v_4 \rangle = \langle v_1 | v_3 \rangle + \langle v_1 | v_4 \rangle + \langle v_2 | v_3 \rangle + \langle v_2 | v_4 \rangle$,

ii) $\forall v, w \in H, \lambda \in \mathbb{C}$: $\langle w | \lambda v \rangle = \lambda \langle w | v \rangle$,

iii) $\forall v, w \in H$: $\langle w, v \rangle = \overline{\langle v | w \rangle}$,

iv) $\forall v \in H$: $\langle v | v \rangle = \|v\|^2$.

An algebra A is a vector space A equipped with an associative bilinear multiplication $A \times A \longrightarrow A$.

A normed algebra A is an algebra A equipped with a norm $A \xrightarrow{\| \|} \mathbb{R}$ making it a normed vector space and satisfying

$$\forall a, b \in A : \|ab\| \leq \|a\| \, \|b\| .$$

A **Banach algebra** A is a normed algebra A complete with respect to the norm-induced metric.

A **C^*-algebra** A is a Banach algebra A equipped with $A \xrightarrow{*} A$ satisfying

i) $\forall a, b \in A: (a+b)^* = a^* + b^*$

ii) $\forall a \in A, \lambda \in \mathbb{C}: (\lambda a)^* = \bar{\lambda} a^*$

iii) $\forall a, b \in A: (ab)^* = b^* a^*$

iv) $\forall a \in A: (a^*)^* = a$,

v) $\forall a \in A: \|aa^*\| = \|a\|^2$.

An endomorphism T of a normed space $(V, \| \ \|)$ is called **bounded** $\Leftrightarrow$ its operator norm $\boxed{\|T\| := \sup_{\substack{v \in V \\ \|v\|=1}} \|Tv\| < \infty}$.

The algebra of all bounded endomorphisms of a ① normed space, ② Banach space, ③ Hilbert space, is a ① normed algebra, ② Banach algebra, ③ C^*-algebra, respectively.