SCALAR PRODUCTS ON BANACH SPACES

<u>Proposition</u>: Let $(V, \| \ \|)$ be a normed vector space with a norm-compatible scalar product $\langle \ | \ \rangle$. Then $\forall x, y \in V$:

① $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$
(the parallelogram law),

② $4\langle y|x\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$.

<u>Corollary</u>: There exists at most one norm-compatible scalar product.

<u>Lemma</u>: Norm is a continuous function on any normed vector space with the norm-induced topology.

Theorem: A normed vector space $(V, \| \ \|)$ admits a norm-compatible scalar product

$$\Longleftrightarrow \forall x, y \in V: \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Proof: We need to show that the formula

$$\langle y | x \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

defines a norm-compatible scalar product whenever the parallelogram law holds.

Norm compatibility: $\forall x \in V$:

$$\langle x | x \rangle = \frac{1}{4}\left(\|2x\|^2 + i\,\|(1+i)x\|^2 - i\,\|(1-i)x\|^2\right)$$

$$= \frac{1}{4}\|x\|^2\left(4 + i|1+i|^2 - i|1-i|^2\right) = \|x\|^2.$$

Conjugation property: $\forall x, y \in V$:

$$\overline{\langle y | x \rangle} = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2\right) = \frac{1}{4}\left(\|y+x\|^2 - \|y-x\|^2 + i\|-i(y+ix)\|^2 - i\|i(y-ix)\|^2\right) = \langle x | y \rangle.$$

i-linearity of $\langle y|$: $\forall x, y \in V$:

$$\langle y|ix\rangle = \frac{1}{4}\big(\|ix+y\|^2 - \|ix-y\|^2 + i\|ix+iy\|^2$$

$$-i\|ix-iy\|^2 = \frac{i}{4}\big(-i\|i(x-iy)\|^2 +$$

$$i\|i(x+iy)\|^2 + \|x+y\|^2 - \|x-y\|^2\big) = i\langle y|x\rangle$$

+-linearity of $\langle y|$: It follows from the parallelogram law that $\forall x_1, x_2, y \in V$:

$$\|x_1+y\|^2 + \|x_2+y\|^2 = \frac{1}{2}\|x_1+x_2+2y\|^2 +$$

$$+\frac{1}{2}\|x_1-x_2\|^2 = 2\|\tfrac{1}{2}(x_1+x_2)+y\|^2 + \frac{1}{2}\|x_1-x_2\|^2.$$

We obtain 3 analogous expressions replacing y by $-y$, iy, and $-iy$ respectively. Hence

$$\langle y|x_1\rangle + \langle y|x_2\rangle = \frac{1}{4}\big(\|x_1+y\|^2 + \|x_2+y\|^2$$

$$-\|x_1-y\|^2 - \|x_2-y\|^2 + i\|x_1+iy\|^2 + i\|x_2+iy\|^2$$

$$-i\|x_1-iy\|^2 - i\|x_2-iy\|^2\big) =$$

$$\frac{1}{4}\Big(2\left\|\tfrac{1}{2}(x_1+x_2)+y\right\|^2+\tfrac{1}{2}\|x_1-x_2\|^2 -$$

$$-2\left\|\tfrac{1}{2}(x_1+x_2)-y\right\|^2-\tfrac{1}{2}\|x_1-x_2\|^2+$$

$$+i\Big(2\left\|\tfrac{1}{2}(x_1+x_2)+iy\right\|^2+\tfrac{1}{2}\|x_1-x_2\|^2\Big)$$

$$-i\Big(2\left\|\tfrac{1}{2}(x_1+x_2)-iy\right\|^2+\tfrac{1}{2}\|x_1-x_2\|^2\Big)\Big)$$

$$=\frac{2}{4}\Big(\left\|\tfrac{1}{2}(x_1+x_2)+y\right\|^2-\left\|\tfrac{1}{2}(x_1+x_2)-y\right\|^2$$

$$+i\left\|\tfrac{1}{2}(x_1+x_2)+iy\right\|^2-i\left\|\tfrac{1}{2}(x_1+x_2)-iy\right\|^2\Big)$$

$= 2\langle y \mid \tfrac{1}{2}(x_1+x_2)\rangle$. Thus we obtained

$$\boxed{\langle y|x_1\rangle+\langle y|x_2\rangle=2\langle y|\tfrac{1}{2}(x_1+x_2)\rangle}.$$

Now, putting $x_1=x$ and $x_2=0$ we get

$\langle y|x\rangle = 2\langle y|\frac{x}{2}\rangle$. Indeed, $\langle y|0\rangle$

$= \frac{1}{4}\left(\|y\|^2-\|-y\|^2+i\|iy\|^2-i\|-iy\|^2\right)=0.$

Consequently, $\langle y|x_1\rangle+\langle y|x_2\rangle=\langle y|x_1+x_2\rangle$. $\square$

$\mathbb{Z}$-linearity of $\langle y|$: We already know that $\forall x, y \in V$, $n \in \mathbb{N}$: $\langle y | n x \rangle = n \langle y | x \rangle$. Next, observe that $\forall x, y \in V$: $\langle y | -x \rangle =$

$$= \frac{1}{4}\left(\|-x+y\|^2 - \|-x-y\|^2 + i\|-x+iy\|^2 - i\|-x-iy\|^2 \right) = -\frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 \right) = -\langle y | x \rangle.$$

$\mathbb{Q}$-linearity of $\langle y|$: $\forall x, y \in V$, $n \in \mathbb{Z}\setminus\{0\}$:

$$\langle y | \tfrac{x}{n} \rangle = \langle y | n \tfrac{x}{n^2} \rangle = n \langle y | \tfrac{1}{n} \tfrac{x}{n} \rangle.$$

Since x is arbitrary, so is $x' := \frac{x}{n}$. Therefore, $\forall x' \in V$: $\langle y | \frac{1}{n} x' \rangle = \frac{1}{n} \langle y | x' \rangle$.

$\mathbb{C}$-linearity of $\langle y|$: It follows from i-linearity and $\mathbb{Q}$-linearity that $\langle y|$ is $(\mathbb{Q} + i\mathbb{Q})$-linear. To prove the $\mathbb{C}$-linearity of $\langle y|$, we define $\mathbb{F} := \{a \in \mathbb{C} \mid \forall x \in V : \langle y|ax\rangle = a\langle y|x\rangle\}$.

Since the norm and algebraic operations are norm-continuous for any normed vector space, we conclude that $\langle y| : V \to \mathbb{C}$ is norm-continuous for any $y \in V$. Consequently, the function $\mathbb{C} \ni a \longmapsto f_{yx}(a) := \langle y|ax\rangle - a\langle y|x\rangle \in \mathbb{C}$ is continuous $\forall x, y \in V$, so that $f_{yx}^{-1}(\{0\})$ is a closed set. It follows that $\mathbb{F} = \bigcap_{x \in V} f_{yx}^{-1}\{0\}$ is closed. On the other hand, $\mathbb{Q} + i\mathbb{Q} \subseteq \mathbb{F}$. Hence $\mathbb{F} = \mathbb{C}$, as needed. ∎

<u>Corollary</u>: A Banach space is a Hilbert space $\Leftrightarrow$ its norm satisfies the parallelogram law.

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