BOUNDED LINEAR MAPS

Let V and W be normed vector spaces. A linear map $V \xrightarrow{T} W$ is called <u>bounded</u> $\Leftrightarrow$

$$\|T\| := \sup_{\substack{v \in V \\ \|v\|_V = 1}} \|Tv\|_W < \infty.$$

<u>Theorem</u>: Let $T: V \to W$ be a linear map between two normed vector spaces. Then the following are equivalent:

① T is bounded,

② T is continuous,

③ T is continuous at one point,

④ T is continuous at 0.

<u>Proof</u>: ① $\Rightarrow$ ② Let $\varepsilon > 0$, $x, v \in V$, $x \neq v$. Then $\|x - v\|_V < \frac{\varepsilon}{\|T\|}$ (if $\|T\| = 0$, then $T = 0$, whence continuous) implies that $\|Tx - Tv\|_W =$

$$= \|T(x-v)\|_W = \left\|T \frac{x-v}{\|x-v\|_V}\right\|_W \|x-v\|_V \leq$$

$\|T\| \, \|x - v\| < \varepsilon$. ② $\Rightarrow$ ③ obvious.

③ $\Rightarrow$ ④ Assume that T is continuous at $v \in V$. Choose $\varepsilon > 0$ and $x \neq 0$. Then $x + v \neq v$, and $\exists\, \delta > 0 : \|x\|_V = \|x + v - v\|_V < \delta$

$$\Rightarrow \|Tx\|_W = \|Tx + Tv - Tv\|_W = \|T(x+v) - Tv\|_W < \varepsilon.$$

④ $\Rightarrow$ ① Take $\varepsilon = 1$ and choose $\delta > 0$ s.t. $\|v\|_V < \delta \Rightarrow \|Tv\|_W < 1$. Then

$$\forall x \neq 0 : \left\|T\frac{x}{\|x\|}\right\|_W = \frac{2}{\delta}\left\|T\frac{\delta x}{2\|x\|}\right\|_W < \frac{2}{\delta} < \infty. \blacksquare$$

Theorem: If V is a normed space and Y is a Banach space, then the set $B(V, Y)$ of all bounded linear maps from V to Y is a Banach space with respect to the operator norm.

Corollary: If V is a normed space, then $V^{\#} := B(V, \mathbb{C}) = \{f \in V^* \mid f \text{ is continuous}\}$ is a Banach space.

The Hahn-Banach Theorem: If V_0 is a subspace of a normed space V and $g_0 \in V_0^\#$, then $\exists\, g \in V^\#$ s.t.

① $\|g_0\|_{V_0^\#} = \|g\|_{V^\#}$,

② $\forall v \in V_0 : g_0(v) = g(v)$.

The Riesz Theorem: Let H be a Hilbert space. Then $\forall y \in H$:

$\langle y| \in H^\#$ and $\| \langle y| \|_{H^\#} = \|y\|_H$.

Moreover, $\boxed{\forall \varphi \in H^\# \ \exists!\, y_\varphi \in H : \varphi = \langle y_\varphi|}$.

Proof: $\| \langle y| \| = \sup\limits_{x \in H \setminus \{0\}} \left| \langle y | \frac{x}{\|x\|} \rangle \right| \leq \left\{ \begin{array}{l} \text{by the} \\ \text{BCS} \\ \text{inequality} \end{array} \right\}$

$\leq \sup\limits_{x \in H \setminus \{0\}} \frac{\|y\| \, \|x\|}{\|x\|} = \|y\|$. On the other hand $\| \langle y| \| \geq \left| \langle y | \frac{y}{\|y\|} \rangle \right| =$

$= \frac{\|y\|^2}{\|y\|} = \|y\|$. Hence $\| \langle y| \| = \|y\|$.

The remaining key part will be proven in steps.

Step 1 Let V be a closed vector subspace of H. Then $\boxed{\forall x_0 \in H \; \exists y_0 \in V : \|x_0 - y_0\| = \inf_{v \in V} \|x_0 - v\|}$.

Indeed, let $\{y_n\}_n$ be a sequence s.t.

$\lim_{n\to\infty} \|x_0 - y_n\| = \inf_{v \in V} \|x_0 - y_n\|$. We want to prove that it is a Cauchy sequence.

By the parallelogram law, $\forall m, n \in \mathbb{N}$:

$$2\|x_0 - y_m\|^2 + 2\|x_0 - y_n\|^2 = 4\left\|x_0 - \frac{y_m + y_n}{2}\right\|^2 + \|y_m - y_n\|^2$$

so that $\|y_m - y_n\|^2 = 2\|x_0 - y_m\|^2 + 2\|x_0 - y_n\|^2 - 4\left\|x_0 - \frac{y_m + y_n}{2}\right\|^2$

$$\leq 2\|x_0 - y_m\|^2 + 2\|x_0 - y_n\|^2 - 4\left(\inf_{v \in V}\|x_0 - v\|\right)^2 \xrightarrow[\min(m,n)\to\infty]{} 0.$$

Consequently, $\{y_n\}_n$ is a Cauchy sequence, as needed. By the completeness of H, $\exists\, y_0 := \lim_{n\to\infty} y_n \in H$.

Furthermore, since V is closed, $y_0 \in V$.

Finally, from the continuity of the norm, we infer that

$$\|x_0 - y_0\| = \|x_0 - \lim_{n\to\infty} y_n\| = \lim_{n\to\infty}\|x_0 - y_n\| = \inf_{v \in V}\|x_0 - v\|.$$

Step 2 : $\boxed{\forall y \in V : \mathrm{Re}\langle x_0 - y_0 | y_0 \rangle \geq \mathrm{Re}\langle x_0 - y_0 | y \rangle.}$

Indeed, $\forall t > 0 : \|x_0 - y_0\|^2 = \left(\inf_{v \in V} \|x_0 - v\|\right)^2 \leq \|x_0 - \underbrace{(y_0 + t(y - y_0))}_{\in V}\|^2$

$= \|x_0 - y_0\|^2 + t^2 \|y - y_0\|^2 - 2t\,\mathrm{Re}\langle x_0 - y_0 | y - y_0 \rangle.$

Hence $\forall t > 0 : 2\,\mathrm{Re}\langle x_0 - y_0 | y - y_0 \rangle \leq t \|y - y_0\|^2$,

so that $\mathrm{Re}\langle x_0 - y_0 | y - y_0 \rangle \leq 0$, i.e.

$$\mathrm{Re}\langle x_0 - y_0 | y_0 \rangle \geq \mathrm{Re}\langle x_0 - y_0 | y \rangle.$$

Step 3 : $\boxed{\forall y \in V : \langle x_0 - y_0 | y \rangle = 0.}$

Indeed, by step 2, $\forall \substack{a \in \mathbb{C} \\ y \in V} : \mathrm{Re}\langle x_0 - y_0 | a y \rangle$

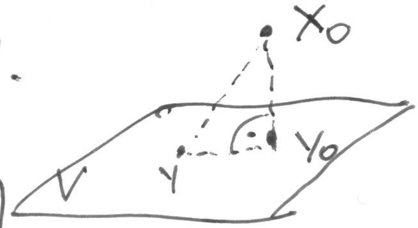
$\leq \alpha \in \mathbb{R}$. In particular, $\forall a \in \mathbb{R}$:

$a\,\mathrm{Re}\langle x_0 - y_0 | y \rangle \leq \alpha \in \mathbb{R}$, so that

$\mathrm{Re}\langle x_0 - y_0 | y \rangle = 0$. Similarly, $\forall a \in \mathbb{R}$:

$\mathrm{Re}\langle x_0 - y_0 | i a y \rangle = -a\,\mathrm{Im}\langle x_0 - y_0 | y \rangle \leq \alpha \in \mathbb{R}$,

so that $\mathrm{Im}\langle x_0 - y_0 | y \rangle = 0$.



Hence $x_0 - y_0 \in V^{\perp} := \{x \in \mathcal{H} \mid \forall y \in V : \langle x | y \rangle = 0\}$.

Step 4: $\boxed{H = V \oplus V^\perp}$. Indeed, $y \in V \cap V^\perp \Rightarrow$

$\langle y | y \rangle = 0 \Leftrightarrow \|y\|^2 = 0 \Leftrightarrow y = 0$.

On the other hand, $\forall x_0 \in H : x_0 = y_0 + x_0 - y_0$,

$y_0 \in V$ and $x_0 - y_0 \in V^\perp$ by Step 3.

Step 5: $\boxed{V^{\perp\perp} = V.}$ The inclusion $V \subseteq V^{\perp\perp}$

is obvious. $\forall x_0 \in V^{\perp\perp} : x_0 = y_0 + x_0 - y_0$.

Since $y_0 \in V \subseteq V^{\perp\perp}$, $x_0 - y_0 \in V^{\perp\perp}$.

However, by Step 3, $x_0 - y_0 \in V^\perp$. Therefore,

$x_0 - y_0 \in V^\perp \cap V^{\perp\perp}$. By Step 4, $x_0 - y_0 = 0$,

so that $x_0 = y_0 \in V$.

Step 6: $\boxed{V \neq H \Rightarrow V^\perp \neq 0} \Leftrightarrow (V^\perp = 0 \Rightarrow V = H)$,

which is true by Step 5: $V = V^{\perp\perp} = 0^\perp = H$.

Step 7: $\varphi \neq 0 \Rightarrow \operatorname{Ker}\varphi \neq H$. By the continuity of φ, $\operatorname{Ker}\varphi$ is a closed subspace. By Step 6,

$\exists u \in (\operatorname{Ker}\varphi)^\perp : \|u\| = 1$. $\forall x \in H : \varphi(\varphi(u)x - \varphi(x)u) = 0$,

i.e. $\varphi(u)x - \varphi(x)u \in \operatorname{Ker}\varphi$. As $u \in (\operatorname{Ker}\varphi)^\perp$, $0 = \langle u | \varphi(u)x - \varphi(x)u \rangle$

$= \varphi(u)\langle u | x \rangle - \varphi(x)$. Hence $\varphi(x) = \langle \varphi(u) u | x \rangle$, $\forall x \in H$.

The case $\varphi = 0$ is trivial. Uniqueness follows from $H^\perp = 0^{\perp\perp} = 0$ by Step 5. ∎