THE GNS-REPRESENTATION

Corollary: Let H be a Hilbert space. The algebra $B(H) = B(H,H)$ of all bounded linear maps from H to H is a C^*-algebra with respect to the operator norm and the involution defined by $\boxed{\langle T^* x | y \rangle = \langle x | Ty \rangle}$, $\forall x, y \in H$ (the Hermitian adjoint).

Proof: Follows from:

① For any normed space V, the set of all bounded linear maps $B(V)$ is a normed algebra with respect to the operator norm.

② For any Banach space V, the normed algebra $B(V)$ is a Banach space. Consequently, $B(V)$ is a Banach algebra.

③ The Riesz Theorem makes the Hermitian adjoint well defined on $B(H)$.

Remark: If $\dim H = n$, then $B(H) \cong M_n(\mathbb{C})$.

Now our goal is to prove that not only $B(H)$ is a C^*-algebra, but also every C^*-algebra is a subalgebra of $B(H)$ for some Hilbert space H. Therefore, we will need to construct a Hilbert space out of a C^*-algebra.

Let A be a C^* algebra and H a Hilbert space. A *-algebra homomorphism $\boxed{\varphi : A \to B(H)}$ is called a representation of A on H. A representation φ is called faithful $\Longleftrightarrow \operatorname{Ker}\varphi = 0$.

The Gelfand-Naimark Noncommutative Thm.:

Every C^*-algebra admits a faithful representation.

This theorem is a major advantage of C^*-algebras over Banach algebras.

Proof strategy:

1. Use norm-one positive linear functionals on A (states).
2. Construct a pre-Hilbert space (normed space with the norm compatible scalar product) out of a positive linear functional. (The GNS-construction.)
3. Complete the thus obtained pre-Hilbert space to a Hilbert space.
4. Define a (natural) represtation of A on this Hilbert space. (The GNS-representation.)
5. Take the direct sum of all such representations given by states to conclude faithfulness. (The Gelfand-Naimark Noncommutative Theorem.)

States on a C^*-algebra: A linear functional for a C^*-algebra A is called positive $\iff$ $\forall a \in A: f(a^* a) \geq 0$.

Examples of positive linear functionals:

① A $*$-algebra homomorphism $A \longrightarrow \mathbb{C}$.

② Trace $M_n(\mathbb{C}) \ni a \mapsto \mathrm{Tr}(a) := \sum_{i=1}^{n} a_{ii} \in \mathbb{C}$.

Indeed, $\mathrm{Tr}(a^*a) = \sum_{i=1}^{n} \sum_{j=1}^{n} (a^*)_{ij} a_{ji} =$

$$= \sum_{i,j=1}^{n} \overline{a_{ji}}\, a_{ji} \geq 0.$$

A positive linear functional f is called a <u>state</u> $\Leftrightarrow \|f\| = 1$.

<u>Lemma 1</u>: A bounded linear functional f on a unital C^*-algebra A is positive $\Leftrightarrow f(1) = \|f\|$.

An element a of a C^*-algebra is called <u>normal</u> $\Leftrightarrow [a^*, a] = 0$ (it commutes with its adjoint.

<u>Lemma 2</u>: If a is a normal element of a non-zero C^*-algebra A, then there exists a state f on A such that $|f(a)| = \|a\|$.

Lemma 3: Let f be a positive linear functional on a C^*-algebra A. Then:

① $\forall a \in A: \ f(a^*a) = 0 \Longleftrightarrow (\forall b \in A: f(ba) = 0)$.

② $\forall a, b \in A: \ f(b^*a^*ab) \leq \|a^*a\| f(b^*b)$.

③ $N_f := \{ a \in A \mid f(a^*a) = 0 \}$ is a closed left ideal of A.

④ The formula $\boxed{\langle [a]_f \mid [b]_f \rangle := f(a^*b)}$

defines a scalar product on A/N_f making it a pre-Hilbert space.

⑤ The formula $\boxed{\varrho_f(a)[b]_f := [ab]_f}$

defines a linear map from A/N_f to A/N_f with the operator norm satisfying $\forall a \in A: \|\varrho_f(a)\| \leq \|a\|$.

Lemma 4: Let X be a metric space. Any continuous function $X \longrightarrow \mathbb{C}$ extends uniquely to a continuous map on the completion of X.

Denote by H_f the completion of A/N_f. Since norm is continuous, it extends uniquely to H_f.

Lemma 5: Let H be the completion of a pre-Hilbert space. Then it is a Hilbert space for the unique scalar product extending the scalar product on the pre-Hilbert space.

Lemma 6: If T is a bounded linear map from a normed space V to a Banach space W, then it extends uniquely to a bounded linear map from the completion of V to W with the same operator norm.

The GNS-Representation Theorem: For any positive linear functional f on a C^*-algebra A, the assignment $A \ni a \mapsto \rho_f(a)^{ext} \in B(H_f)$, where $\rho_f(a)^{ext}$ is the extension of $\rho_f(a) \in B(A/N_f)$, defines a $*$-algebra homomorphism.