THE SPECTRUM AND THE SPECTRAL RADIUS

Let A be a unital algebra. <u>The spectrum of $a \in A$</u> is the set

$$\boxed{\operatorname{spec}_A(a) := \{\lambda \in \mathbb{C} \mid \nexists (a-\lambda)^{-1}\}}.$$

Note that a is invertible $\iff 0 \notin \operatorname{spec}_A(a)$.

<u>Elementary examples</u>:

(1) For $A = M_n(\mathbb{C})$, $\operatorname{spec}_A(a) = \{\lambda \in \mathbb{C} \mid \det(a-\lambda)=0\}$.

(2) For $A = C(X)$, where X is a compact Hausdorff space, $\operatorname{spec}_A(a) = a(X) = \operatorname{Im} a$.

<u>Theorem</u>: For any unital algebra A,

$$\boxed{\forall a,b \in A: \ \operatorname{spec}_A(ab)\setminus\{0\} = \operatorname{spec}_A(ba)\setminus\{0\}}.$$

<u>Proof</u>: We need to show that

$$(\mathbb{C}\setminus \operatorname{spec}_A(ab)) \cup \{0\} = (\mathbb{C}\setminus \operatorname{spec}_A(ba)) \cup \{0\}.$$

It suffices to prove that

$$\mathbb{C}\setminus \operatorname{spec}_A(ab) \subseteq \mathbb{C}\setminus \operatorname{spec}(ba) \cup \{0\},$$

i.e. $\exists (ab-\lambda)^{-1} \Rightarrow (\lambda = 0 \vee \exists (ba-\lambda)^{-1})$.

This is logically equivalent to

$(\exists (ab-\lambda)^{-1} \wedge \lambda \neq 0) \Rightarrow \exists (ba-\lambda)^{-1}$.

Indeed, $(p \Rightarrow (q \vee r)) \Leftrightarrow \sim(p \wedge \sim q \wedge \sim r)$

$\Leftrightarrow ((p \wedge \sim q) \Rightarrow r)$. Without the loss of generality, we can take $\lambda = 1$.

Finally, note that $c(ab-1) = 1 \Rightarrow$

$bc(ab-1)a = ba \Leftrightarrow bcaba - bca = ba$

$\Leftrightarrow bca(ba-1) = ba \Leftrightarrow bca(ba-1) - 1 = ba-1$

$\Leftrightarrow (bca-1)(ba-1) = 1$. Likewise,

$(ab-1)c = 1 \Rightarrow b(ab-1)ca = ba \Leftrightarrow$

$babca - bca = ba \Leftrightarrow (ba-1)bca = ba$

$\Leftrightarrow (ba-1)(bca-1) = 1$. ∎

Theorem: Let A be a unital algebra and p a polynomial function. Then

$$\boxed{\mathrm{spec}_A(a) \neq \emptyset \Rightarrow \mathrm{spec}_A(p(a)) = p(\mathrm{spec}_A(a))}.$$

Proof: The claim is obvious when $p = \text{const.}$

$\forall p \neq \text{const.}$ and $\mu \in \mathbb{C}$:

$$p(a) - \mu = \lambda_0 (a - \lambda_1) \dots (a - \lambda_n), \quad \lambda_0 \neq 0.$$

On the other hand, in any commutative ring R, if $r = r_1 \cdots r_n$, then r is invertible $\iff \forall i \in \{1, \dots, n\}$: r_i is invertible.

Hence $\mu \in \mathrm{spec}_A(p(a)) \Rightarrow \lambda_i \in \mathrm{spec}_A(a)$ for some $i \in \{1, \dots, n\}$, so that $\mu = p(\lambda_i) \in p(\mathrm{spec}_A(a))$. Furthermore, if $\lambda \in \mathrm{spec}_A(a)$, then $p(a) - p(\lambda) = \hat{p}(a)(a - \lambda)$ implies that $p(\lambda) \in \mathrm{spec}_A(p(a))$. ∎

Theorem: Let A be a unital Banach algebra and $a \in A$ such that $\|a\| < 1$. Then $\boxed{(1-a)^{-1} = \sum_{n=0}^{\infty} a^n}$ (Neumann series).

Proof: Note first that $\forall k \in \mathbb{N}$: $\left\| \sum_{n=0}^{k} a^n \right\| \leq \sum_{n=0}^{k} \|a\|^n \leq \sum_{n=0}^{\infty} \|a\|^n = (1 - \|a\|)^{-1}$.

Consequently, $\sum_{n=0}^{k} a^n$ is a Cauchy sequence because $\forall k \in \mathbb{N}: \| \sum_{n=k}^{l} a^n \| \leq \|a\|^k \| \sum_{n=0}^{l-k} a^n \|$

$\leq \|a\|^k (1-\|a\|)^{-1} \xrightarrow{k\to\infty} 0$. By the completeness of the Banach algebra, $\sum_{n=0}^{\infty} a^n \in A$. Finally, by the continuity of the multiplication, $\left(\lim_{k\to\infty} \sum_{n=0}^{k} a^n\right)(1-a)$

$$= \lim_{k\to\infty} \left(\left(\sum_{n=0}^{k} a^n\right)(1-a)\right) = \lim_{k\to\infty} (1-a^{k+1}) = 1.$$

In the same way, $(1-a)\sum_{n=0}^{\infty} a^n = 1$. ∎

<u>Theorem</u>: Let A be a unital Banach algebra. $\forall a \in A$: $\mathrm{spec}_A(a)$ is a closed subset of $\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|a\|\}$.

<u>Proof</u>: $\exists a^{-1} \in A$ and $\|a-b\| < \|a^{-1}\|^{-1}$

$\Rightarrow \|1 - ba^{-1}\| \leq \|a-b\| \|a^{-1}\| < 1$. Hence, using the Neumann series, $1-(1-ba^{-1}) = ba^{-1}$

is invertible. Therefore $b = (ba^{-1})a$ is invertible, and we can conclude that $\mathrm{Inv}(A) := \{a \in A \mid \exists a^{-1} \in A\}$ is an open subset of A. Hence $\mathrm{spec}_A(a)$ is a closed subset of $\mathbb{C}$ because $\mathbb{C} \setminus \mathrm{spec}_A(a)$ is open:

$\lambda \notin \mathrm{spec}_A(a)$ and $|\lambda' - \lambda| = \| a - \lambda - (a - \lambda') \|$
$< \| (a-\lambda)^{-1} \|^{-1} \implies \lambda' \notin \mathrm{spec}_A(a)$.

Finally, if $|\lambda| > \|a\|$, then $\| \frac{a}{\lambda} \| < 1$, so that $\exists (1 - \frac{a}{\lambda})^{-1}$. It follows that $(a-\lambda)^{-1} = -\lambda^{-1}(1 - \frac{a}{\lambda})^{-1}$, whence $\lambda \notin \mathrm{spec}_A(a)$. ∎

<u>Theorem (Gelfand):</u> Let A be a unital Banach algebra. Then

$$\boxed{\forall a \in A : \ \mathrm{spec}_A(a) \neq \emptyset}.$$

If a is an element of a unital Banach algebra A, then its <u>spectral radius</u> is

$$\boxed{r(a) := \sup_{\lambda \in \mathrm{spec}_A(a)} |\lambda|}.$$

It exists because $\lambda \in \mathrm{spec}_A(a) \implies |\lambda| \leq \|a\|$.

Theorem (Beurling): Let A be a unital Banach algebra. Then $\boxed{\forall a \in A: r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}}$.

Corollary: If A is a unital C^*-algebra, $a = a^* \in A$, then $r(a) = \|a\|$.

Proof: Since $\|a^2\| = \|a^* a\| = \|a\|^2$, by induction $\|a^{2^n}\| = \|a\|^{2^n}$. Consequently,

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \|a\|.$$

∎

Corollary: There is at most one norm on a unital $*$-algebra making it a unital C^*-algebra.

Proof: $\|a\|_1^2 = \|a^* a\|_1 = r(a^* a) = \|a^* a\|_2 = \|a\|_2^2$.

∎

Theorem: Let A be a unital C^*-algebra.

$\boxed{\text{If } a = a^* \in A, \text{ then } \operatorname{spec}_A(a) \subset \mathbb{R}.}$

Theorem: Let B be a unital C^*-subalgebra of a unital C^*-algebra A. Then

$$\boxed{\forall b \in B: \operatorname{spec}_B(b) = \operatorname{spec}_A(a)}.$$