UNIVERSAL C*-ALGEBRAS

Removing the axioms concerning the norm from the definition of a C^*-algebra yields the definition of a $*$-algebra.

The <u>universal C^*-algebra</u> A_u of a (unital) $*$-algebra $\mathcal{A}$, if it exists, it is constructed as follows:

① Check that there exists at least one <u>bounded</u> representation of $\mathcal{A}$, i.e. a unital $*$-homomorphism $\mathcal{A} \to B(H)$ for some Hilbert space H. Recall that the relation $[a,b]=1$ admits <u>no</u> bounded representations.

② Check that there exists the supremum $\sup_{\rho \in \mathrm{Rep}(\mathcal{A})} \|\rho(x)\| < \infty$ for any $x \in \mathcal{A}$. Here $\mathrm{Rep}(\mathcal{A})$ is the set of <u>all</u> bounded representations of $\mathcal{A}$.

Since $\|\sum_{\text{finite}} \alpha_{in} x_i^n\| \leq \sum_{\text{finite}} |\alpha_{in}| \|x_i\|^n$,

it suffices to verify that for all generators $x_i \in \mathcal{A}$: $\sup_{\rho \in \mathrm{Rep}(\mathcal{A})} \|x_i\| < \infty$.

For instance, for $\mathcal{A} = \mathbb{C}[x]$ with the $*$-structure given by $x^* = x$ such a supremum does not exist. Indeed, $\forall n \in \mathbb{N} \setminus \{0\}$: $\rho_n : \mathbb{C}[x] \xrightarrow{\wedge} C([-n,n]) \xhookrightarrow{GN} B(H_n)$, where $\forall t \in [-n,n] : \hat{x}(t) = t$ and GN is the faithful (isometric) Gelfand-Naimark representation, is a bounded representation and $\|\rho_n(x)\| = \|\hat{x}\|$. Since $\hat{x} = \hat{x}^*$, $\|\hat{x}\| = r(\hat{x}) = \sup_{\lambda \in \mathrm{spec}(\hat{x}) = [-n,n]} |\lambda| = n$.

Therefore $\sup_{\rho \in \mathrm{Rep}(\mathbb{C}[x])} \|\rho(x)\| = \infty$.

3 Turn $\mathcal{A}$ into the normed algebra

$$\boxed{\tilde{\mathcal{A}} := \mathcal{A} \Big/ \bigcap_{\rho \in \mathrm{Rep}(\mathcal{A})} \mathrm{Ker}\,\rho, \quad \|[x]\|_u := \sup_{\rho \in \mathrm{Rep}(\mathcal{A})} \|\rho(x)\|}$$

$\tilde{\mathcal{A}}$ is evidently a unital $*$-algebra.
It is also a normed algebra because

(A) $\|[x]\|_u = 0 \Rightarrow \forall \rho \in \mathrm{Rep}(\mathcal{A}) : \rho(x) = 0$

$\Leftrightarrow x \in \bigcap_{\rho \in \mathrm{Rep}(\mathcal{A})} \mathrm{Ker}\,\rho \Leftrightarrow [x] = 0.$

(B) $\|\lambda[x]\|_u = \sup_{\rho \in \mathrm{Rep}(\mathcal{A})} |\lambda| \, \|\rho(x)\| = |\lambda| \sup_{\rho \in \mathrm{Rep}(\mathcal{A})} \|\rho(x)\|$

$= |\lambda| \, \|[x]\|_u .$

(C) $\|[x]+[y]\|_u \leq \sup_{\rho \in \mathrm{Rep}(\mathcal{A})} (\|\rho(x)\| + \|\rho(y)\|) \leq$

$\sup_{\rho \in \mathrm{Rep}(\mathcal{A})} \|\rho(x)\| + \sup_{\rho \in \mathrm{Rep}(\mathcal{A})} \|\rho(y)\| = \|[x]\|_u + \|[y]\|_u .$

(D) $\|[x][y]\|_u \leq \|[x]\|_u \, \|[y]\|_u$ much in the same way.

The $\| \ \|_u$ and $*$ are compatible:

$\|[x]^*[x]\|_u = \sup_{\rho \in \mathrm{Rep}(\mathcal{A})} \|\rho(x)^*\rho(x)\| = \sup_{\rho \in \mathrm{Rep}(\mathcal{A})} (\|\rho(x)\|^2)$

$= \left(\sup_{\rho \in \mathrm{Rep}(\mathcal{A})} \|\rho(x)\| \right)^2 = \|[x]\|_u^2 .$

Note that $\bigcap_{\varrho \in \mathrm{Rep}(\mathcal{A})} \operatorname{Ker} \varrho$ might be non-zero.

Indeed, if $x^2 = 0$ and $x^* = x$, then $\forall \varrho \in \mathrm{Rep}(\mathcal{A})$: $\|\varrho(x)\|^2 = \|\varrho(x)^* \varrho(x)\|$ $= \|\varrho(x^2)\| = 0$, so that $x \in \bigcap_{\varrho \in \mathrm{Rep}(\mathcal{A})} \operatorname{Ker} \varrho$.

④ Norm complete $\tilde{\mathcal{A}}$ to a unital C^*-algebra A_4.
The norm completion of a normed space is a Banach space including the ~~normed space we started with~~ as a normed subspace (non-trivial result!). A_4 is a Banach algebra because $\|\lim x_n \lim y_n\|_4$ $= \|\lim (x_n y_n)\|_4 = \lim \|x_n y_n\|_4 \leq \lim (\|x_n\|_4 \|y_n\|_4)$ $= \|x\|_4 \|y\|_4$. On the other hand, it is a $*$-algebra because $*$ is $\|\ \|_4$-isometric. Finally, the C^*-condition holds because $\|(\lim x_n)^* \lim x_n\|_4 = \lim \|x_n^* x_n\|_4 =$ $\lim \|x_n\|_4^2 = \|\lim x_n\|_4^2$. We call the thus constructed A_4 the <u>universal (enveloping) C^*-algebra</u> of the unital $*$-algebra $\mathcal{A}$.

Claim: Let A_u be the universal C^*-algebra of a unital $*$-algebra $\mathcal{A}$. Then for any unital C^*-algebra B and any unital $*$-algebra homomorphism $\mathcal{A} \xrightarrow{f} B$, there exists a unique unital $*$-algebra homomorphism $A_u \xrightarrow{f_u} B$ rendering the following diagram commutative:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f"] \arrow[d] & B \\ \hat{\mathcal{A}} \arrow[r, hook] & A_u \arrow[u, "f_u"'] \end{tikzcd}$$

Here $\mathcal{A} \twoheadrightarrow \hat{\mathcal{A}}$ and $\hat{\mathcal{A}} \hookrightarrow A_u$ are the canonical quotient map and inclusion respectively.

Proof: Composing f with the GN-representation of B we obtain a bounded representation of $\mathcal{A}$: $\rho_f : \mathcal{A} \xrightarrow{f} B \xhookrightarrow{GN} B(H_B)$. Hence $\rho_f\left(\bigcap_{\rho \in Rep(\mathcal{A})} Ker\, \rho\right) = 0$, so that $f\left(\bigcap_{\rho \in Rep(\mathcal{A})} Ker\, \rho\right) = 0$ by the injectivity of GN. This implies that f factors through $\hat{\mathcal{A}}$. On the other hand, the induced map $\tilde{f} : \hat{\mathcal{A}} \to B$ satisfies the norm inequality $\forall [x] \in \hat{\mathcal{A}}$:

$$\|\tilde{f}([x])\| = \|f(x)\| = \|\varrho_f(x)\| \leq \|x\|_u .$$

Hence $\tilde{f}$ is continuous and can be uniquely extended to a (continuous) unital $*$-algebra homomorphism $A_u \xrightarrow{f_u} B$. Thus our square diagram consists of two commutative triangle diagrams

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "f"] \arrow[d, two heads] & B \\
\hat{\mathcal{A}} \arrow[ur, "\tilde{f}"] \arrow[r, hook] & A_u \arrow[u, "f_u"']
\end{tikzcd},$$

so that the square diagram commutes as well. Finally, assume that $\exists A_u \xrightarrow{f_u'} B$ making the square diagram commutative. It follows from the surjectivity of the down arrow that f_u and f_u' agree on a dense subset. Therefore, by the continuity, $f_u = f_u'$. $\blacksquare$