テープリッツ環 (Toeplitz algebra)

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1 Hardy space, Toeplitz operators

Unit circle: $\mathbb{S}^1$, closed unit disc: $\mathbb{D}$, its interior: $\mathring{\mathbb{D}}(= \mathbb{D} \setminus \mathbb{S}^1)$. Integers: $\mathbb{Z}$, non-negative integers: $\mathbb{N} = \{0, 1, \dots\}$, strictly positive integers: $\mathbb{N}_{>0} = \{1, 2, \dots\}$

Definition 1. We consider the Hilbert space $L^2(\mathbb{S}^1)$ with respect to the rotation invariant Lebesgue measure on $\mathbb{S}^1$. The *Hardy space* H^2 is the subspace of $L^2(\mathbb{S}^1)$ consisting of the holomorphic L^2-functions.

The vectors $e_n = e^{2\pi int}$ for $n \in \mathbb{Z}$ is an orthonormal basis of $L^2(\mathbb{S}^1)$; any vector can be represented by $\mathbb{Z}$-indexed square summable sequence (Fourier decomposition). The elements of H^2 are precisely the ones with support $\mathbb{N}$.

Remark. When $f \in H^2$ and $\lambda \in \mathring{\mathbb{D}}$, the *evaluation of f at λ* makes sense: if $f = \sum_{n=0}^{\infty} \alpha_n e^{2\pi int}$, the value of f at $\lambda = re^{2\pi is}$ is given by the absolutely convergent series $\sum_n \alpha_n r^n e^{2\pi ins}$.

Bounded continuous (or measurable) functions on $\mathbb{S}^1$ act as bounded operators on $L^2(\mathbb{S}^1)$.

Definition 2. Let P be the orthogonal projection $L^2(\mathbb{S}^1) \to H^2$. When f is a continuous function on $\mathbb{S}^1$, the associated *Toeplitz operator* T_f is defined as PfP.

Caution

- The composition of two Toeplitz operators is usually not Toeplitz
- Two Toeplitz operators almost never commute

Facts

- If f is a continuous function on $\mathbb{S}^1$, the convolution by the Poisson kernel defines a (unique) extension ϕ_f of f as a harmonic function on $\mathbb{D}$.

- If f is holomorphic, anti-holomorphic, or real, the spectrum of T_f is equal to the range of ϕ_f. In general, $\sigma(T_f)$ is contained in the convex hull of $f(\mathbb{S}^1)$. (We will come back to this later)

Observation

- (by Gelfand-Naimark) $\mathbb{S}^1$ can be recovered as the object behind the pointwise multiplication operators $C(\mathbb{S}^1) \subset B(L^2(\mathbb{S}^1))$
- does the Toeplitz algebra represent something similar, but similar to the closed disk?

1.1 Unilateral shift operator

Let's look at the most important function $z \in C(\mathbb{S}^1)$, the complex coordinate function w.r.t. the embedding $\mathbb{S}^1 \subset \mathbb{C}$ as the unit circle. The associated Toeplitz operator is $e^{2\pi n t} \to e^{2\pi(n+1)t}$.

Definition 3. The *unilateral shift operator* S is the isometry on $\ell^2\mathbb{N}$ given by $e_n \mapsto e_{n+1}$.

Under the identification $H^2 = \ell^2\mathbb{N}$, the unilateral shift operator is equal to T_z we don't distinguish these two from now on.

Exercise. Check that $S^n = T_{z^n}$ and $(S^*)^n = T_{\bar{z}^n}$ for $n \in \mathbb{N}$. Describe these operators in terms of the basis $(e_n)_{n\in\mathbb{N}}$.

Proposition 1. The spectrum of S is equal to $\mathbb{D}$.

Proof. Since $\|S\|$ is equal to 1, $\sigma(S)$ has to be contained in $\mathbb{D}$. Moreover, since $T \mapsto T^*$ is conjugate linear and reverses the order of multiplication, $\sigma(S) = \overline{\sigma(S^*)}$. If we show that $\mathring{\mathbb{D}}$ is contained in $\sigma(S^*)$, the compactness of spectrum implies the assertion.

Suppose $\|\lambda\| < 1$. Then, $\xi = \sum_{n=0}^{\infty} \lambda^n e_n$ is in $\ell^2\mathbb{N}$. We have $S^*\xi = \lambda\xi$ (use the exercise), and this shows $\lambda \in \sigma(S^*)$. □

2 Toeplitz algebra

Definition 4 (First definition of the Toeplitz algebra)**.** The *Toeplitz algebra T* is the C*-algebra on $\ell^2\mathbb{N}$ generated by S (i.e. the operator norm closure of sums and products of S and S^*).

Why do we add S^*?

Proposition 2. If $f \in C(\mathbb{S}^1)$, there are sequences of polynomials $(P_n(x))_n$ and $(Q_n(x))_n$ such that $\|P_n(S) + Q_n(S^*) - T_f\| \to 0$ (in operator norm).

Proof. By the Stone-Weierstrass theorem, we can find sequences of polynomials $(P_n)_n$, $(Q_n)_n$ such that $P_n(z) + Q_n(\bar{z}) \to f$ uniformly. If we represent these polynomials on $L^2(\mathbb{S}^1)$ and restrict them to H^2, $P_n(z)$ (resp. $Q_n(\bar{z})$) becomes $P_n(S)$ (resp. $Q_n(S^*)$). Since the operation $T \mapsto PTP$ only decreases the operator norm, we get the assertion. □

Definition 5 (Second definition of T)**.** The Toeplitz algebra T is the C*-algebra on H^2 generated by T_f for $f \in C(\mathbb{S}^1)$ (but still need to consider their products!)

What do we get from the products of S, S^*, etc.? First of all, from the definition, $S^*S = I$. But $SS^* \neq I$, and....

Exercises.

1. Show that SS^* is the orthogonal projection onto $\ell^2\mathbb{N}_{>0} = \ell^2\mathbb{N} \ominus \mathbb{C}.e_0$
2. Show that $SS^* - S^2(S^*)^2 = S(I - SS^*)S^*$ is the orthogonal projection onto e_1.
3. Show that, when $m, n \in \mathbb{N}$, $S^n(I - SS^*)(S^*)^m$ is the rank 1 operator sending e_m to e_n.

Proposition 3. For $f, g \in C(\mathbb{S}^1)$, the difference $T_f T_g - T_{fg}$ is a compact operator. The algebra T contains the ideal K of all the compact operators on $\ell^2\mathbb{N}$.

Proof. (First assertion) When f and g are polynomials in S and S^*, we can reduce it to the case of monomials, then use the above exercises and prove the statement by the induction on the monomial degree (actually, $T_f T_g - T_{fg}$ is going to be finite rank in this case). The general case follows from the operator norm approximation of T_f and T_g by polynomials in S and S^*.

(Second assertion) Again by the last exercise, any matrix unit '$e_m \mapsto e_n$' is contained in T. Since we take the operator norm closure, the whole K is contained in T. □

Definition 6 (Third definition of T)**.** The Toeplitz algebra is (the norm closure of) the linear span of $(T_f)_{f \in C(\mathbb{S}^1)}$ and K.

Remark. The compact operators K form an closed bilateral ideal of T (it is already so in $B(\ell^2\mathbb{N})$.)

2.1 Noncommutative disks

Let $0 < q < 1$, and consider the operator $Z_q \colon e_n \mapsto \sqrt{1 - q^n} e_{n+1}$ on $\ell^2\mathbb{N}$.

1. We have $I - Z_q^* Z_q = q(I - Z_q Z_q^*)$, which has the 'classical limit' at $q = 1$
2. The C*-algebra generated by Z_q is equal to T ($Z_q - S$ is a compact operator)

2.2 Gauge action

Consider the 1-parameter unitary $U_s = e^{2\pi ns} e_n$ on $\ell^2\mathbb{N}$.

- This has period 1.
- It is induced by the rotation action of $\mathbb{S}^1$ on $L^2(\mathbb{S}^1)$.

Exercise. Show that $U_s S^m = e^{2\pi ms} S^m U_s$.

Definition 7. The *gauge action* γ_s is the transformation $x \mapsto U_s x U_s^*$ on T.

3 Toeplitz extension

The Toeplitz operators represent a 'noncommutative disk' $\mathbb{D}_q$, (roughly) through the boundary value problem of harmonic functions. So, what is the boundary of $\mathbb{D}_q$? It should be an analogue of the algebra homomorphism $C(\mathbb{D}) \to C(\mathbb{S}^1)$, sending each function on $\mathbb{D}$ to its restriction on $\mathbb{S}^1$.

Proposition 4. The quotient of T by K is isomorphic to $C(\mathbb{S}^1)$, in a way $[T_f] = f$ for $f \in C(\mathbb{S}^1)$.

Proof. Let s be the image of S in the quotient by K. Recall that $S^*S - SS^*$ is a compact operator (a rank 1 projection). Thus, in the quotient, we have $s^*s = ss^* = 1$.

By continuous functional calculus, we can identify T/K with $C(\sigma(s))$. Since s is unitary, $\sigma(s)$ is a subset of $\mathbb{S}^1$, and s itself gives the inclusion map $\sigma(s) \to \mathbb{S}^1$. We need to show that this is surjective. The gauge action induces an action of $\mathbb{S}^1$ on $\sigma(s)$, and the map s is equivariant. Since the rotation action of $\mathbb{S}^1$ on $\mathbb{S}^1$ is transitive, this map has to be surjective. □

So, we get an extension of the form

$$0 \to K \to T \to C(\mathbb{S}^1) \to 0,$$

called the *Toeplitz extension*. This is the analogue of

$$0 \to C_0(\mathring{\mathbb{D}}) \to C(\mathbb{D}) \to C(\mathbb{S}^1) \to 0.$$

Remark. We get the essential spectrum of the Toeplitz operators: $\sigma_e(T_f) = f(\mathbb{S}^1)$.

Fact. Any operator of the form T_f for $f \in C(\mathbb{S}^1)$ has either trivial kernel or cokernel. Check this for holomorphic functions as an exercise. (Hint: if f is holomorphic, T_f on H^2 is simply the pointwise multiplication. What can be said about the evaluation on $\mathring{\mathbb{D}}$?)

Remark. We can use the above proposition to identify the spectrum of T_f. From the Fredholm theory, an operator is invertible only if it is Fredholm of index 0 and has the trivial kernel (or equivalently, trivial cokernel). In our setting, T_f is Fredholm precisely when f is invertible in $C(\mathbb{S}^1)$, and its Fredholm index is the winding number of f (which classifies f among the nowhere vanishing complex functions on $\mathbb{S}^1$ up to homotopy). The following conditions are equivalent

1. $\lambda \notin \sigma(T_f)$, i.e. $T_f - \lambda = T_{f-\lambda}$ is invertible
2. λ is not in the range of f, and the winding number of $f - \lambda$ is 0.

4 Universality of Toeplitz algebra

Proposition 5. Let V be an isometry on a Hilbert space H. Then, H decomposes as a direct sum $H_0 \oplus H_1$ of V-invariant subspaces, such that

1. the restriction of V on H_0 is unitarily equivalent to $(\ell^2\mathbb{N} \otimes \ker V^*, S \otimes I_{\ker V^*})$,
2. the restriction of V on H_1 is unitary.

Proof. Let $(f_i)_{i\in I}$ be an orthonormal basis of $\ker V^*$. For each $n \in \mathbb{N}$, we consider the vector $f_i^{(n)} = V^n f_i$. Thus, when $n \neq 0$, the vector $f_i^{(n)}$ is orthogonal to f_j, for any $i, j \in I$. Since V is isometry, $(f_i^{(n)})_{n,i}$ is a mutually orthonormal vectors in H.

We define H_0 to be the closed linear span of $\{f_i^{(n)} \mid n \in \mathbb{N}, i \in I\}$. Then, the map $e_n \otimes f_i \mapsto f_i^{(n)}$ for $n \in \mathbb{N}$ and $i \in I$ extends to a unitary map U from $\ell^2\mathbb{N} \otimes \ker V^*$ to H_0. We have

$$U(S \otimes I_{\ker V^*})(e_n \otimes f_i) = U(e_{n+1} \otimes f_i) = f_i^{(n+1)} = VU(e_n \otimes f_i).$$

We remark that H_0 is also V^*-invariant (exercise!).

Next, define H_1 to be $H \ominus H_0$. Since H_0 was invariant under both V and V^*, H_1 has the same property. By definition of H_0, the restriction of V^* to H_1 has trivial kernel. This means, $V^*|_{H_1}$ is also an isometry, and $V|_{H_1}$ is an unitary. $\square$

Recall that, if U is a unitary operator on H, there is a continuous functional calculus map $C(\mathbb{S}^1) \to B(H)$ sending z to U.

Theorem 1 (Coburn)**.** Let V be an isometry on a Hilbert space H. Then, there exists a *-homomorphism $\phi\colon T \to B(H)$ satisfying $\phi(S) = V$.

Proof. Take the decomposition $H = H_0 \oplus H_1$ as in the proposition. Then, there is a *-homomorphism $T \to B(H_0)$ which sends S to $V|_{H_0}$. By the remark above, we also get $C(\mathbb{S}^1) \to B(H_1)$ which sends z to $V|_{H_1}$. Thus, we get a *-homomorphism $\mathcal{T} \oplus C(\mathbb{S}^1) \to B(H)$ which sends $S \oplus z$ to $V|_{H_0} \oplus V|_{H_1} = V$.

From the Toeplitz extension, we know that there is a homomorphism $\mathcal{T} \to \mathcal{T} \oplus C(\mathbb{S}^1)$ which sends S to $S \oplus z$. Combining these, we get the desired homomorphism. $\square$

In other words, $\mathcal{T}$ is a universal C*-algebra generated by an element S subject to the relation $S^*S = 1$.