Duality for Grassmann bundles and applications (1

HI 10. 11. 16 with L. Darondeau

X alg. variety over a field

$$\begin{array}{c} E \\ \downarrow \\ X \end{array} \quad \text{vector bundle of rk } n$$

flag: $E_\bullet = E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n = E \qquad \mathrm{rk}\, E_i = i$

$G_d(E)$ Grassmann bundle of d-subspaces in the fibres of E

$$\begin{array}{c} G_d(E) \\ \downarrow \pi \\ X \end{array} \qquad 0 \to \mathcal{U}^{d} \to E \to Q^{n-d} \to 0$$

univ. seq.

partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_d) \qquad \lambda_1 \leq n-d$

$$\square = (n-d)^d = (n-d, \dots, n-d) \qquad \lambda \subset \square$$

technical numbers: $n_i = n-d+i-\lambda_i$, $1 < n_1 < n_2 < \dots < n_d \leq n$

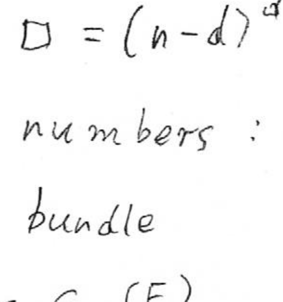
Schubert bundle

$$\begin{array}{c} \Omega_\lambda(E_\bullet) \subset G_d(E) \\ \downarrow \\ X \ni x \end{array} \qquad \Omega_\lambda(E)_x = \{ V \in G_d(E_x) : \dim(V \cap E_{n_i, x}) \geq i,\ i = 1, \dots, d \}$$

$$\mathrm{codim}\, \Omega_\lambda(E_\bullet) = |\lambda|$$

Thm (duality) $X = x$, $\Omega_\lambda \subset G_d(E)$

$\lambda \subset \square$



$$\lambda^c = (n-d-\lambda_d, \dots, n-d-\lambda_1)$$

$$[\Omega_\lambda] \cdot [\Omega_\mu] = \begin{cases} 1 & \mu = \lambda^c \\ 0 & \text{else} \end{cases}$$

$|\lambda| + |\mu| = d(n-d)$

Basis thm $A^\bullet G_d(E) = \bigoplus_{\lambda \subset \square} \mathbb{Z}[\Omega_\lambda]$.

duality $\Rightarrow$

$W \subset G_d(E_x)$ subvariety, $[W] = \sum m_\lambda [\Omega_\lambda]$, $m_\lambda = [W] \cdot [\Omega_{\lambda^c}]$.

Schur functions

$t = (t_1, \dots, t_d)$ variables of deg. 1

$$s_\lambda(t) = \frac{\det\left(t_j^{\lambda_i + d - i}\right)}{\prod_{i<j}(t_i - t_j)} = \det\left(h_{\lambda_i - i + j}(t)\right)$$

$\uparrow$ complete homog. symm. fts

Let $x_1, \dots, x_n$ be the Chern roots of E.

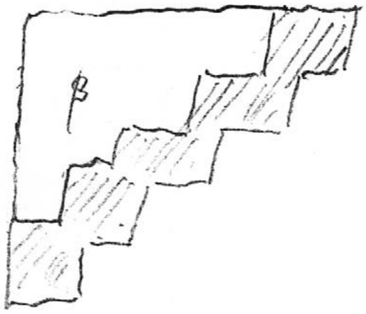
$s_\lambda(E) := s_\lambda(-x_1, \dots, -x_n)$

Giambelli formula $\quad X = x \quad [\Omega_\lambda] = s_\lambda(\mathcal{U})$.

Gysin homomorphism : $f : X \longrightarrow Y$ proper $\rightsquigarrow f_* : A_* X \to A_* Y$
additive map induced by pushforward of cycles

Skew Schur function $\quad \beta \subset \alpha$

α / β



$$s_{\alpha/\beta} = \det\left(h_{\alpha_i - \beta_j - i + j}\right)$$

Lemma: Let $\ell(\alpha), \ell(\beta) \le d$. Then for any $(\ell)^d \supset \beta$

$$s_\alpha(t)\, s_\beta(t) = s_{(\ell)^d + \alpha / (\ell)^d - \beta^{\leftarrow}}$$

$\beta^{\leftarrow} = (\beta_d, \dots, \beta_1)$
$\alpha \pm \beta = (\alpha_1 \pm \beta_1, \dots, \alpha_d \pm \beta_d)$

– proof by bijection

Jacobi 1840
Naegelbash 1871
Porteous
Lascoux

$$s_\alpha \cdot s_\beta = \det\left(h_{\alpha_i + \beta_{d+1-j} - i + j}\right)$$

(Darondeau, P)

Thm $\ell(\alpha), \ell(\beta) \leq d$. For $(\ell)^d \supseteq \beta$ (3

$$\pi_*\left(s_\alpha(\mathcal{U})\cdot s_\beta(\mathcal{U})\right) = s_{(\ell-n+d)^d + \alpha/(\ell)^d - \beta^{\leftarrow}}(E)\,.$$

In particular, if $\beta \subset (n-d)^d$

$$\pi_*\left(s_\alpha(\mathcal{U})\cdot s_\beta(\mathcal{U})\right) = s_{\alpha/\beta^c}(E)\,.$$

We need Littlewood-Richardson numbers $c^\gamma_{\alpha\beta}$

$$s_\alpha s_\beta = \sum_\gamma c^\gamma_{\alpha\beta}\, s_\gamma \Rightarrow c^\gamma_{\alpha\beta} = c^\gamma_{\beta\alpha}$$

$$s_{\alpha/\beta} = \sum c^\alpha_{\beta\gamma}\, s_\gamma \Rightarrow c^\alpha_{\beta\gamma} \text{ depends only on } \alpha/\beta \text{ and } \gamma.$$

$\Rightarrow c^\alpha_{\beta\gamma} = c^\alpha_{\gamma\beta}$ depends only on α/γ and β.

$$\Rightarrow c^\alpha_{\beta\gamma} = c^{\alpha+\square}_{\beta,\gamma+\square} \qquad \left(\alpha+\square/\gamma+\square = \alpha/\gamma\right).$$

$$\pi_* s_\alpha(\mathcal{U}) = s_{\alpha-\square}(E) \qquad \text{(Józefiak, Lascoux, P 1978)}$$

Thus
$$\pi_*\left(s_\alpha(\mathcal{U})\, s_\beta(\mathcal{U})\right) = \sum_\gamma c^\gamma_{\alpha\beta}\, s_{\gamma-\square}(E)$$
$$= \sum c^{\gamma+\square}_{\alpha\beta}\, s_\gamma(E)$$

For $\beta \subset \square$, $\ell(\alpha) \leq d$,

$$s_\alpha(t)\cdot s_\beta(t) = s_{\square+\alpha/\beta^c}(t)\,.$$

Cor $c^\gamma_{\alpha\beta} = c^{\alpha+\square}_{\beta^c\gamma}$

To prove thm, we need

$$LHS \rightsquigarrow c^{\gamma+\square}_{\alpha\beta} = c^{\alpha}_{\beta^c\gamma} \leftsquigarrow RHS$$

But $c^{\gamma+\square}_{\alpha\beta} \underset{\text{Cor.}}{=} c^{\alpha+\cancel{\square}}_{\beta^c(\gamma+\cancel{\square})} = c^{\alpha}_{\beta^c\gamma}$. $\square$

<u>Applications</u>

$$\begin{tikzcd} W \arrow[r, phantom, "\subset"] \arrow[dr, "\eta"'] & G_d(E) \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

i

$A^\cdot G_d(E)$ has an $A^\cdot X$-basis $s_\beta \mathcal{U}$, where $\beta \subset \square$.

$$[W] = \sum m_\beta s_\beta \mathcal{U}$$

1) Determine η_* $\qquad i_* s_\alpha \mathcal{U} = s_\alpha \mathcal{U} \cdot [W]$

2) Determine m_β

1.) $W = LG(E) \subset G_d(E)$ Lagrangian Grassmann bundle

$$\begin{tikzcd} W \arrow[dr, "\eta"'] & & G_d(E) \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

$\rho(k) := (k, k-1, \dots, 1)$

$[LG(E)] = c_{top}\, \Lambda^2 \mathcal{U}^* = s_{\rho(n-1)}(\mathcal{U})$.

$$\eta_*(s_\alpha \mathcal{U}) = \pi_* i_* s_\alpha \mathcal{U} = \pi_*(s_\alpha \mathcal{U} \cdot s_{\rho(n-1)} \mathcal{U}) \underset{\text{duality}}{=}$$

$$= s_{\alpha/\rho(n-1)^c}(E) = s_{\alpha/\rho(n)}(E).$$

2) $[\Omega_\lambda(E_\cdot)]$. Desing. $\theta: F \to X$ of $\Omega_\lambda(E_\cdot)$. (Kempf - Laksov 1974)

$x \in X \quad F_x = \{V_1 \subset \dots \subset V_d : \dim V_i = i,\ V_i \subset E_{n_i, x}\}$

F as a chain of projective bundles:

$$F = P(E_{n_d}/\mathcal{U}_{d-1}) \longrightarrow \dots \longrightarrow P(E_{n_2}/\mathcal{U}_1) \to P(E_{n_1}) \to X$$

$\pi : P(E) = G_1(E) \longrightarrow X \qquad \mathcal{O}_{P(E)}(-1)$ univ. line bundle

$\xi = c_1 \mathcal{O}_{P(E)}(1)$

(*) $\pi_* \xi^i = s_{i-n+1}(E)$ Segre class $(-1)^i h_i$ (Chern roots of E).

Segre polynomial $s_x(E) = 1 + x s_1(E) + x^2 s_2(E) + \dots$

$x = \frac{1}{t}$, rewrite (*)

$\pi_* \xi^{i} = [t^{n-1}]\left(t^{i} s_{1/t}(E)\right)$

$f(t) \in A^\bullet X[t]$, $\pi_* f(\xi) = [t^{n-1}]\left(f(t)\, s_{1/t}(E)\right)$

"fundamental formula" - good to iterate.

Write $\nu = (n_d, n_{d-1}, \dots, n_1)$ $\quad \nu_i = n_{d+1-i}$

$\xi_i = -c_1(\mathcal{U}_{d+1-i}/\mathcal{U}_i)$.

Have a new type of a Gysin formula:

<u>Thm (D-P)</u> $\theta_* f(\xi_1, \dots, \xi_d) =$

$$[t_1^{\nu_1-1} \cdots t_d^{\nu_d-1}]\left(f(t) \prod_{i<j}(t_i - t_j) \prod s_{1/t_i}(E_{\nu_i})\right).$$

$$\begin{tikzcd}
F \arrow[r, "\text{desing.}"] \arrow[dr, "\theta"'] & \Omega_\lambda(E_\cdot) \arrow[r, hook] \arrow[d] & G_d(E) \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

push generators of $A_\cdot \Omega_\lambda(E_\cdot)$ in two ways and get a system of equations

$\forall_\alpha \qquad \theta_*(s_\alpha \mathcal{U}) = \pi_*\left(\sum m_\beta\, s_\alpha \mathcal{U} \cdot s_\beta \mathcal{U}\right)$

$(\alpha) \qquad \det\left(s_{\alpha_i - i + d + 1 - \nu_j}\right)(E_{\nu_j}) = \sum \underline{m_\beta}\, s_{\alpha/\beta^c}(E)$

Solving in m_β: (K-L, 1974)

$[\Omega_\lambda(E_\cdot)] = \sum m_\beta s_\beta \mathcal{U} = \det\left(c_{\lambda_i - i + j}(Q - E_{n_i})\right)_{1 \le i,j \le d}$

as long as $\alpha \nsubseteq \beta^c$, $s_{\alpha/\beta^c}(E) = 0$,
if $\alpha = \beta^c$, $s_{\alpha/\beta^c}(E) = 1$; get invertible triangular system in m_β