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Gysin maps, duality, and Schubert classes

(with L. Darondeau)

X alg var. / $k = \bar{k}$

$A^{\cdot}X$ Chow ring, X smooth

E vector bundle of rk n
$\downarrow$
X flag $E_\bullet : E_1 \subset E_2 \subset \dots \subset E_{n-1} \subset E_n = E$ rk $E_i = i$

$G_d(E)$ Grassmann bundle of d-subspaces in E

$\downarrow \pi$ $\qquad P(E) = G_1(E)$

X

$\Omega_\lambda(E_\bullet)$ Schubert bundle for $\lambda = (\lambda_1 \geq \dots \geq \lambda_d)$

$\downarrow \omega_\lambda$ $\qquad \lambda_1 \leq n-d$

X $\qquad \square = (n-d)^d$, $\lambda \subseteq \square$

$x \in X$ $\Omega_\lambda(E_\bullet)_x = \{ V \in G_d(E_x) : \dim(V \cap E_{n-d-\lambda_i+i,x}) \geq i \ \forall i \}$

$\alpha = (\alpha_1, \dots, \alpha_d)$, $\beta = (\beta_1, \dots, \beta_d)$ partitions $\alpha \pm \beta := (\alpha_1 \pm \beta_1, \dots, \alpha_d \pm \beta_d)$

$\alpha^{\leftarrow} := (\alpha_d, \alpha_{d-1}, \dots, \alpha_1)$

Let $\nu = (\nu_1, \dots, \nu_d) = (\dots, n-d-\lambda_i+i, \dots)^{\leftarrow}$ Have

$\nu = \lambda^c + \rho$ where $\boxed{\lambda \, \lambda^c}$ d and $\rho = (d, d-1, \dots, 1)$ ν strict, $\ell(\nu) = d$

$n-d$

$$\Omega_\lambda(E_\bullet) = \{ V \in G_d(E) : \dim(V \cap E_{\nu_i}) \geq d+1-i \} \supset \mathring{\Omega}_\lambda(E_\bullet)$$

open dense

Let $\mu \subseteq (n)^d$ be a strict partition with d parts.

$F_\mu(E_\bullet)$ Kempf-Laksov flag bundle parametrizing

$\downarrow \vartheta_\mu$ flags of subbundles $V_1 \subset \dots \subset V_d$ with rk $V_i = i$

X such that $V_i \subset E_{\mu_{d+1-i}}$.

Prop. 1 The forgetful map $F(1, \dots, d)(E) \to G_d(E)$ establishes a desingularization $\varphi : F_\nu(E_\bullet) \to \Omega_\lambda(E_\bullet)$. (On $\mathring{\Omega}_\lambda(E_\bullet)$ the inverse map is $V \mapsto (V \cap E_{\nu_d}, \dots, V \cap E_{\nu_1})$.)

$\mathcal{U}_1 \subset \dots \subset \mathcal{U}_d = \mathcal{U}$ univ. flag

<u>Lemma 1</u> The forgetful map:

$$F(1,\dots,d-e+1)(E) \to F(1,\dots,d-e)(E)$$

$e=1,\dots,d-1$

induces a map $F_{(\mu_e,\dots,\mu_d)}(E) \to F_{(\mu_{e+1},\dots,\mu_d)}(E)$ isomorphic to $P(E_{\mu_e}/\mathcal{U}_{d-e})$. Lastly $F_{(\mu_d)}(E_\cdot) = P(E_{\mu_d})$.

Get:

$$F(\mu_1,\dots,\mu_d)(E_\cdot) \to F_{(\mu_2,\dots,\mu_d)}(E_\cdot) \to \dots \to F_{(\mu_{d-1},\mu_d)}(E_\cdot) \to F_{\mu_d}(E_\cdot) \to X$$

$$P(E_{\mu_1}/\mathcal{U}_{d-1}) \to P(E_{\mu_2}/\mathcal{U}_{d-2}) \to \dots \to P(E_{\mu_{d-1}}/\mathcal{U}_1) \to P(E_{\mu_d}) \to X.$$

<u>Gysin formula</u>

$f: X \to Y$ proper $\rightsquigarrow f_*: A^\cdot X \to A^\cdot Y$ additive map induced by push-forward of cycles

$\xi_i = -c_1(\mathcal{U}_{d+1-i}/\mathcal{U}_{d-i})$ – hyperplane classes

$t = t_1,\dots,t_d$ – variables ; $f(t) \in A^\cdot X[t]$

Fundamental formula for $\pi: P(E) \to X$; $\xi = c_1 \mathcal{O}_{P(E)}(1)$

$(*)\ \pi_* \xi^i = s_{i-n+1}(E)$ Segre class

Segre polynomial $s_x(E) = 1 + x s_1(E) + x^2 s_2(E) + \dots$

$s_{i-n+1}(E) = [x^{i-n+1}](s_x(E)) = [x^{-n+1}](x^{-i} s_x(E))$

$x = 1/t,\ s_{i-n+1}(E) = [t^{n-1}](t^i s_{1/t}(E))$

$(*)\ \pi_* \xi^i = [t^{n-1}](t^i s_{1/t}(E))$

$f(t) \in A^\cdot X[t] \Rightarrow \pi_* f(\xi) = [t^{n-1}](f(t)\, s_{1/t}(E))$

ff

3

Thm 1 (Parondeau – P) $(\nu_\mu)_* f(\xi_1,\dots,\xi_d) = \left[\prod_{i=1}^d t_i^{\mu_i-1}\right]\left(f(t)\cdot \prod_{i<j}(t_i-t_j)\prod s_{1/t_i}(E_{\mu_i})\right)$.

$d=3$

$$P(E_{\mu_1}/U_2) \xrightarrow[p_1]{\xi_1} P(E_{\mu_2}/U_1) \xrightarrow[p_2]{\xi_2} P(E_{\mu_3}) \xrightarrow[p_3]{\xi_3} X$$

$$s_x(E+F) = s_x(E)\, s_x(F) \qquad s_x(E/F) = s_x(E)\, c_x(F)$$

$$p_{1*} f(\xi_1,\xi_2,\xi_3) \overset{\text{ff}}{=} [t_1^{\mu_1-3}]\left(f(t_1,\xi_2,\xi_3)\ \boxed{s_{1/t_1}(E_{\mu_1}/U_2)}\right)$$

$$\boxed{s_{1/t_1}(E_{\mu_1}/U_2)} = s_{1/t_1}(E_{\mu_1})\left(1-\frac{\xi_2}{t_1}\right)\left(1-\frac{\xi_3}{t_1}\right)$$

$$= s_{1/t_1}(E_{\mu_1})\frac{(t_1-\xi_2)(t_1-\xi_3)}{t_1^2}$$

$$\underline{= [t_1^{\mu_1-1}]\left(f(t_1,\xi_2,\xi_3)\, s_{1/t_1}(E_{\mu_1})(t_1-\xi_2)(t_1-\xi_3)\right)}$$

$$p_{2*}p_{1*} f(\xi_1,\xi_2,\xi_3) \overset{\text{ff}}{=}$$

$$[t_1^{\mu_1-1} t_2^{\mu_2-2}]\left(f(t_1,t_2,\xi_3)\, s_{1/t_1}(E_{\mu_1})(t_1-t_2)(t_1-\xi_3)\ \boxed{s_{1/t_2}(E_{\mu_2}/U_1)}\right)$$

$$s_{1/t_2}(E_{\mu_2}/U_1) = s_{1/t_2}(E_{\mu_2})\left(1-\frac{\xi_3}{t_2}\right) = s_{1/t_2}(E_{\mu_2})\frac{t_2-\xi_3}{t_2}$$

$$= [t_1^{\mu_1-1} t_2^{\mu_2-1}]\left(f(t_1,t_2,\xi_3)\, s_{1/t_1}(E_{\mu_1})\, s_{1/t_2}(E_{\mu_2})(t_1-t_2)(t_1-\xi_3)(t_2-\xi_3)\right)$$

$$\xi_3 \to t_3$$

$p_{3*}p_{2*}p_{1*}$ $\qquad t_3^{\mu_3-1}$ $\qquad s_{1/t_3}(E_{\mu_3})$

Schur functions

$\alpha = (\alpha_1, \dots, \alpha_d)$ partition

$$s_\alpha(t_1, \dots, t_d) = \frac{\det(t_j^{\alpha_i + d - i})}{\prod_{i<j}(t_i - t_j)} = \det(h_{\alpha_i - i + j})$$

complete symm. fts

$x_1, \dots, x_m$ – Chern roots of E

$s_\lambda(E) := s_\lambda(-x_1, \dots, -x_m)$

Giambelli formule $\quad X = x \quad [\Omega_\lambda] = s_\lambda(\mathcal{U})$, $\mathcal{U}$ – universal bundle on $G_d(E)$.

Prop. 2 $\quad \vartheta_{\nu *} s_\alpha(\mathcal{U}) = \det(s_{\alpha_i - i + d + 1 - \nu_j}(E_{\nu_j}))_{1 \le i,j \le d}$

Pf By the theorem,

$$\vartheta_{\nu *} s_\alpha(\mathcal{U}) = [\prod_j t_j^{\nu_j - 1}]\left(\det(t_j^{\alpha_i + d - i} s_{1/t_j}(E_{\nu_j}))\right) =$$

Lemma 2 If $f_{ij} \in A^\circ X[t_j]$ $\forall i$, then

$$[\prod_j t_j^{e_j}] \det(f_{ij}) = \det([t_j^{e_j}] f_{ij}).$$

$$= \det\left([t_j^{\nu_j - 1}]\left(t_j^{\alpha_i + d - i} s_{1/t}(E_{\nu_j})\right)\right)$$

$$= \det(s_{\alpha_i - i + d + 1 - \nu_j}(E_{\nu_j}))_{1 \le i,j \le d}. \quad \text{Q.E.D.}$$

Prop. 3 $\quad \omega_{\lambda *} s_\alpha(\mathcal{U}) = \det(s_{\alpha_i - i + j - \lambda_j^c}(E_{\nu_j}))_{1 \le i,j \le d}$.

$$\begin{array}{ccc} F_\nu(E_\bullet) & \xrightarrow[\text{bir.}]{\varphi} & \Omega_\lambda(E_\bullet) \\ & \vartheta_\nu \searrow \quad \swarrow \omega_\lambda & \\ & X & \end{array}$$

This follows from $\omega_{\lambda *} s_\alpha(\mathcal{U}) = \vartheta_{\nu *} s_\alpha(\varphi^* \mathcal{U})$ and

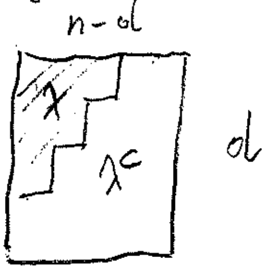
$\nu_j = \lambda_j^c + d + 1 - j$.

(skew Schur class)

Duality

$X = x$; absolute duality :

$\lambda \subseteq \square$



$$\lambda^c = (n-d-\lambda_d, \dots, n-d-\lambda_1)$$

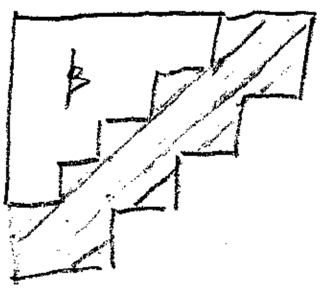
$$[\Omega_\lambda]\cdot[\Omega_\mu] = \begin{cases} 1 & \mu = \lambda^c \\ 0 & \text{else} \end{cases}$$

$$|\lambda| + |\mu| = d(n-d)$$

Skew Schur functions

$\beta \subseteq \alpha$ $\qquad \alpha/\beta$

$$s_{\alpha/\beta} = \det\left(h_{\alpha_i - i + j - \beta_j}\right)$$



Lemma 3 Let $\ell(\alpha), \ell(\beta) \leq d$. Then ~~for~~ any $\beta \subset (\ell)^d$,

$$s_\alpha \cdot s_\beta = s_{(\ell)^d + \alpha / (\ell)^d - \beta^{\leftarrow}}.$$

$\beta^{\leftarrow} = (\beta_d, \dots, \beta_1)$

$\alpha \pm \beta = (\alpha_1 \pm \beta_1, \dots)$

- proof by bijection

Jacobi 1840
Naegelbash 1871
Porteous
Lascoux

$$s_\alpha \cdot s_\beta = \det\left(h_{\alpha_i + \beta_{d+1-j} - i + j}\right)$$

Thm 2 (Darondeau-P) If $\ell(\alpha) \leq d$ and $\beta \subseteq \square$,

$$\pi_*\left(s_\alpha(\mathcal{U}) \cdot s_\beta(\mathcal{U})\right) = s_{\alpha/\beta^c}(E).$$

$X = x$, get absolute duality.

Littlewood–Richardson numbers $c^{\gamma}_{\alpha\beta}$

$$s_\alpha s_\beta = \sum_\gamma c^\gamma_{\alpha\beta} s_\gamma \Rightarrow c^\gamma_{\alpha\beta} = c^\gamma_{\beta\alpha}.$$

$$s_{\alpha/\beta} = \sum c^\alpha_{\beta\gamma} s_\gamma \Rightarrow c^\alpha_{\beta\gamma} \text{ depends only on } \alpha/\beta \text{ and } \gamma.$$

$$\Rightarrow c^\alpha_{\beta\gamma} = c^\alpha_{\gamma\beta} \text{ depends only on } \alpha/\gamma \text{ and } \beta.$$

$$\Rightarrow c^\alpha_{\beta\gamma} = c^{\alpha+\square}_{\beta,\gamma+\square} \qquad (\alpha+\square/\gamma+\square = \alpha/\gamma)$$

<u>Prop. 3</u> (Józefiak, Lascoux, P 1978)

$$\pi_* s_\alpha(\mathcal{U}) = s_{\alpha-\square}(E).$$

$$\Rightarrow \pi_*\big(s_\alpha(\mathcal{U})\, s_\beta(\mathcal{U})\big) = \sum_\gamma c^\gamma_{\alpha\beta} s_{\gamma-\square}(E)$$

$$= \sum c^{\gamma+\square}_{\alpha\beta} s_\gamma(E).$$

We already know: for $\beta \subseteq \square$, $\ell(\alpha) \le d$,

$$s_\alpha \cdot s_\beta = s_{\square+\alpha/\beta^c}$$

<u>Cor</u> $\quad c^\gamma_{\alpha\beta} = c^{\alpha+\square}_{\beta^c\gamma}$

To prove thm, we need LHS $c^{\gamma+\square}_{\alpha\beta} = c^{\alpha}_{\beta^c\gamma}$ RHS $\quad \downarrow s_{\alpha/\beta^c}$

But $c^{\gamma+\square}_{\alpha\beta} \underset{\text{Cor.}}{=} c^{\alpha+\cancel{\square}}_{\beta^c(\gamma+\cancel{\square})} = c^\alpha_{\beta^c\gamma}$. Q.E.D.

We pass to applications to Schubert classes.

Schubert classes

Def. (Flagged skew Schur classes) For partitions λ, γ with at most d parts and partial flags $A_\bullet, B_\bullet$ with d members define $s_{\lambda/\gamma}(A_\bullet, B_\bullet) = \det\left(s_{\lambda_i - i + j - \gamma_j}(A_i + B_j)\right)_{1 \le i,j \le d}$

Also $s_{\lambda/\gamma}(A_\bullet) = s_{\lambda/\gamma}(A_\bullet, 0)$ and $s^*_{\lambda/\gamma}(A_\bullet) = s_{\lambda/\gamma}(-A_\bullet, 0)$.

We need a variation of

$$s_{\lambda/\mu}(A+B) = \sum_{\mu \le \nu \le \lambda} s_{\lambda/\nu}(A)\, s_{\nu/\mu}(B)$$

Lemma 4 For partitions λ, γ with at most d parts and partial flags $A_\bullet, B_\bullet$ with d members

$$s_{\lambda/\mu}(A_\bullet, B_\bullet) = \sum_{\mu \le \nu \le \lambda} s_{\lambda/\nu}(A_\bullet, 0)\, s_{\nu/\mu}(0, B_\bullet)$$

(The LLPT Notes p. 124)

Consider the flag $E_{\nu_\bullet}$. Let $E/E^{\leftarrow}_{\nu_\bullet}$ be the dual flag the jth member of which is $E/E_{\nu_{d+1-j}}$.

Thm 3 (Darondeau - P) For $\lambda \subseteq \square$, we have

$$[\Omega_\lambda(E_\bullet)] = \sum_{\beta \le \lambda} s^*_{\lambda/\beta}(E/E^{\leftarrow}_{\nu_\bullet})\, s_\beta(U).$$

Pf $[\Omega_\lambda(E_\bullet)] = \sum_{\beta \subseteq \square} m_\beta\, s_\beta(U)$.

$\iota : \Omega_\lambda(E_\bullet) \hookrightarrow G_d(E)$

$\iota_* s_\alpha(U) = [\Omega_\lambda(E_\bullet)]\, s_\alpha(U) = \sum_{\beta \subseteq \square} m_\beta\, s_\alpha(U)\, s_\beta(U)$

$$\begin{aligned}\omega_\lambda * s_\alpha(U) = (\pi \circ \iota)_* s_\alpha(U) &= \sum_{\beta \le \square} m_\beta\, \pi_*\, s_\alpha(U)\, s_\beta(U) \\ &= \sum_{\beta \le \square} m_\beta\, s_{\alpha/\beta^c}(E)\end{aligned}$$

By Prop. 3 $\omega_\lambda * s_\alpha(U) = s_{\alpha/\lambda^c}(0, E_{\nu_\bullet})$

So, for each partition α, we get an equation in the (8)
unknows m_β

$$(\alpha) \quad \sum_{\beta \subseteq \square} m_\beta \, s_{\alpha/\beta^c}(E) = s_{\alpha/\lambda^c}(E, E_{\nu_\bullet} - E)$$

System of equations (α) for $\alpha \subseteq \square$ ordered lexicographically is lower triangular with 1's on the diagonal. It is thus invertible and has a unique solution.

$$\sum_{\beta \subseteq \square} m_\beta \, s_{\alpha/\beta^c}(E) \overset{\text{Lemma 4}}{=} \sum_{\lambda^c \subseteq \beta^c \subseteq \alpha} s_{\alpha/\beta^c}(E, 0)\, s_{\beta^c/\lambda^c}(0, E_{\nu_\bullet} - E)$$

$$= \sum_{\lambda^c \subseteq \beta^c \subseteq \alpha} s_{\alpha/\beta^c}(E)\, s^*_{\lambda/\beta}(E - E^{\leftarrow}_{\nu_\bullet})$$

$$s^*_{\lambda/\beta}(E - E^{\leftarrow}_{\nu_\bullet}) = s^*_{\lambda/\beta}(E/E^{\leftarrow}_{\nu_\bullet})$$

Thus we obtain a solution (unique by above observations):

$$\forall \beta \subseteq \square, \quad m_\beta = s^*_{\alpha/\beta}(E/E^{\leftarrow}_{\nu_\bullet}) \qquad \text{Q.E.D.}$$

<u>Rk</u> (Classical Giambelli formula)

$$[\Omega_\lambda(E_\bullet)] = \det\left(c_{\lambda_i - i + j}(E - E_{\nu_{d+1-i}} - \mathcal{U})\right)_{1 \le i,j \le d}$$

can be obtained using these techniques.