PROJECTIONS OF FRACTAL PERCOLATIONS

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Abstract. In this paper we study the radial projection and the orthogonal projection of the random Cantor sets $E \subset \mathbb{R}^2$ which are called Mandelbrot percolation or percolation fractals. We prove that the following assertion holds almost surely: If the Hausdorff dimension of E is greater than 1 then both the orthogonal projection to **every** line and the radial projection with **every** center (see Figure 1 for the definitions) contain some intervals.

1. Introduction

In order to construct a model for turbulence Mandelbrot introduced [11] a random set which is now called Mandelbrot percolation or fractal percolations or canonical curdling. In the simplest case (we consider a more general case in this paper), we are given a natural number $M \geq 2$ and a probability $p \in (0, 1)$. First we partition the unit square $[0, 1]^2$ into M^2 congruent squares and then we retain any of them with probability p and discard them with probability $1-p$ independently. In the squares which were retained we repeat this process independently ad infinitum. The random set $E \subset [0, 1]^2$ resulted is the fractal percolation or canonical curdling. In fact in this paper sometimes we consider the more general setup where the M^2 congruent squares, mentioned above, are chosen with not necessarily the same probabilities.

These random Cantor sets have attracted considerable attention. In 1986 Falconer [4] and Mauldin, Williams [13] computed the almost sure Hausdorff dimension (conditioned on non-extinction). In 1988 Chayes, Chayes, Durett [2] proved that there is a critical probability p_c such for every $0 < p < p_c$ the random Cantors set E is totally disconnected, but for every $p > p_c$ with positive probability E percolates. This means

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that there is a connected component in E which connects the left hand side wall to the right hand side wall of the unit square $[0,1]^2$. Dekking and Meester [3] gave a simplified prove for the previously mentioned result and defined several phases such that as we increase p the process passes through all of these phases.

The most important conclusion of our result is that whenever the probability $p > 1/M$ then although the set E may be totally disconnected, almost surely conditioned on non-extinction, its all kind of projections (all orthogonal and all radial projections) contain some intervals. On the other hand, if $p \leq 1/M$ this cannot happen. We make all the calculations for the more general case when we choose the squares with non necessarily the same probabilities. In this more general case however we need to impose an additional assumption which always hold when all the probabilities are equal.

The paper is organized as follows: The new results of the paper are given in Section 2 accompanied with the most important notation which are necessary to state these results. The rest of the notation comes in Section 3. In Section 4 we derive Theorem 2 from Theorem 3. In Section 5 we derive Theorem 1 from Theorem 2. The most important arguments are left for Section 6 where we prove our Theorem 3. Finally, in Section 7 we consider the case $p \leq 1/M$, proving Theorem 4.

2. The results

First we provide a more intuitive but slightly heuristic definition of random Cantor set E which is the object of interest of this paper (see Section 3 for the precise definition). Given

$$M \geq 2 \text{ and } p_{i,j} \in (0,1) \text{ for every } i,j \in \{0,\dots,M-1\}.$$

We partition the unit square $K=[0,1]^2$ into M^2 congruent squares of side length $1/M$.

$$K = \bigcup_{i,j=0}^{M-1} K_{i,j} \text{ where } K_{i,j} := \left[\frac{i}{M},\frac{i+1}{M}\right] \times \left[\frac{j}{M},\frac{j+1}{M}\right]. \tag{2.1}$$

With probability $p_{i,j}$ we retain the square $K_{i,j}$ and with probability $1-p_{i,j}$ we discard $K_{i,j}$ for every $i,j \in \{0,\dots,M-1\}$ independently. The union of squares retained is called E_1. Within each squares $K_{i,j} \subset E_1$ we repeat the process described above independently. The squares of side length $1/M^2$ retained are called level two squares and their union is called E_2. Similarly, for every n we construct the n-th approximation of E called E_n. Clearly, $E_n \subset E_{n-1}$ for every $n \geq 2$. The object of interest in this paper is the random set $E := \bigcap_{n=1}^{\infty} E_n$. It follows easily from the general theory of branching processes that

$$\text{with positive probability } E \neq \emptyset \text{ if and only if } \sum_{0 \le i,j \le M-1} p_{i,j} > 1. \tag{2.2}$$

It was proved by Falconer and Mauldin, Williams that

$$\text{If } E \neq \emptyset \text{ then } \dim_{\mathrm{H}}(E) = \dim_{\mathrm{B}}(E) = \frac{\log\left(\sum_{i,j=0}^{M-1} p_{i,j}\right)}{\log M} \text{ a.s.} \tag{2.3}$$

The object of our study is the existence of intervals in different kind of projections of E. Using (2.3), except for the last section, we will assume that

$$\sum_{i,j=0}^{M-1} p_{i,j} > M \tag{2.4}$$

holds.
The main results of the paper are the following three theorems.

Theorem 1. *We assume that*

$$\forall 0 \le i,j \le M-1,\ p_{i,j} \equiv p > \frac{1}{M}. \tag{2.5}$$

Then the following assertions hold almost surely, conditioned on non-extinction:

- **(a):** *The orthogonal projections of E to **all** lines contains some intervals.*
- **(b):** *For **all** $t \in \mathbb{R}^2$ the radial projection with center t contains some intervals.*

To state a more general result for the case of different probabilities we need some notation. The nature of projections of angles 0 or $\pi/2$ is conspicuously different and these cases were treated by Falconer and Grimett. So, mostly we restrict our attention the the domain of angles

$$\mathfrak{D} := (0, \pi/2) \cup (\pi/2, \pi).$$

If $\alpha \in (0, \pi/2)$ then Δ^α denotes the decreasing diagonal of K (one connecting points $(0,1)$ and $(1,0)$). If $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ then Δ^α is the increasing diagonal of K. For an $\alpha \in (0,\pi) \setminus \left\{\frac{\pi}{2}\right\}$ we write $\Pi_\alpha : K \to \Delta^\alpha$ for the angle α projection to the diagonal Δ^α in K. That is

$$\text{for } (x,y) \in K, \quad \Pi_\alpha(x,y) = \begin{cases} \left(\frac{1+x\tan\alpha - y}{1+\tan\alpha}, \frac{(1-x)\tan\alpha + y}{1+\tan\alpha}\right), & \text{if } \alpha \in \left(0, \frac{\pi}{2}\right); \\ \left(\frac{y - x\tan\alpha}{1-\tan\alpha}, \frac{y - x\tan\alpha}{1-\tan\alpha}\right), & \text{if } \alpha \in \left(\frac{\pi}{2}, \pi\right). \end{cases}$$

Throughout the paper we frequently write $\underline{i}_n, \underline{j}_n$ for some elements of the set $\{0,\dots,M-1\}^n$, where $n \geq 1$ arbitrary. That is

$$\underline{i}_n := (i_1,\dots,i_n),\ \underline{j}_n := (j_1,\dots,j_n) \in \{1,\dots,M-1\}^n .$$

It is natural to associate such an $\underline{i}_n, \underline{j}_n$ with

$$t_{\underline{i}_n,\underline{j}_n} := \left(\sum_{\ell=1}^{n} i_\ell \cdot M^{-\ell}, \sum_{\ell=1}^{n} j_\ell \cdot M^{-\ell}\right) \in K. \tag{2.6}$$

The square of side length $1/M^n$ with left bottom corner $t_{\underline{i}_n,\underline{j}_n}$ is denoted by $K_{\underline{i}_n,\underline{j}_n}$ and we call it a level n square. Further, let $\varphi_{\underline{i}_n,\underline{j}_n}$ be the most natural contraction sending K onto $K_{\underline{i}_n,\underline{j}_n}$. That is

$$\varphi_{\underline{i}_n,\underline{j}_n}(x,y) = \frac{1}{M^n} \cdot (x,y) + t_{\underline{i}_n,\underline{j}_n}. \tag{2.7}$$

We also denote

$$p_{\underline{i}_n,\underline{j}_n} = \prod_{k=1}^{n} p_{i_k,j_k}.$$

We prove in Lemma 10 that if (2.5) holds then all $\alpha \in \mathfrak{D}$ satisfies the following condition:

Condition (A) There exist closed subintervals $\widetilde{\Delta}_1^\alpha, \widetilde{\Delta}_2^\alpha \subset \Delta^\alpha$ and an integer r_α such that

- **(a'):** $\widetilde{\Delta}_1^\alpha \subset \mathrm{int}\widetilde{\Delta}_2^\alpha$, $\widetilde{\Delta}_2^\alpha \subset \mathrm{int}\Delta^\alpha$ and
- **(b'):** $\displaystyle\sum_{x \in \Pi_\alpha(\varphi_{\underline{i}_{r_\alpha},\underline{j}_{r_\alpha}}(\widetilde{\Delta}_1^\alpha))} p_{\underline{i}_{r_\alpha},\underline{j}_{r_\alpha}} > 2 \cdot \mathbb{1}_{\widetilde{\Delta}_2^\alpha}(x)$ for all $x \in \widetilde{\Delta}_2^\alpha$.

Now we can state a generalization of Theorem 1:

Theorem 2. *Assume that all $\alpha \in \mathfrak{D}$ satisfy Condition (A). To handle the orthogonal projection to lines of angle 0 and $\frac{\pi}{2}$ we also require that*

$$\forall i,j \in \{0,\dots,M-1\}\,,\ \sum_{k=0}^{M-1} p_{i,k} > 1 \ \textit{and} \ \sum_{\ell=0}^{M-1} p_{\ell,j} > 1. \tag{2.8}$$

Then the following assertions hold almost surely, conditioned on non-extinction:

- **(a):** *The orthogonal projection of E to* ***all*** *lines contains some intervals.*
- **(b):** *For* ***all*** *$t \in \mathbb{R}^2$ the radial projection with center t contains same intervals.*

We remark that Lemma 9 provides a way to check if Condition (A) holds. It was proved by Falconer and Grimett [6] (see also [7]) that (2.8) implies that almost surely, conditioned on non-extinction, there is an interval in the orthogonal projections of E to lines of angle 0 and $\pi/2$. So, by assuming (2.8), we can restrict our selves to angles $\alpha \in \mathfrak{D}$ when we handle orthogonal projections. To state the most abstract version of our result we need to refer to Definition 11 where the notion of *regular family of almost linear projections* is introduced.

Theorem 3. *Let $\mathcal{S} = \{S_t\}_{t\in T}$ be a regular family of almost linear projections. Then the following assertion holds almost surely :*

$$\text{If } E \neq \emptyset \text{ then } \forall t \in T,\ S_t(E) \text{ contains an interval.} \tag{2.9}$$

For completeness, in the last section we will also consider the case when (2.4) does not hold.

Theorem 4. *Assume $\sum\limits_{i,j=0}^{M-1} p_{i,j} \leq M$. Then the orthogonal projection of E to **all** lines and the radial projection with center t for **all** $t \in \mathbb{R}^2$ has Lebesgue measure zero.*

3. Notation

Now we give the precise definition of the random set E which is called Fractal percolation, Mandelbrot percolation or canonical curdling. Let $\mathcal{T}$ be the M^2-array tree which is the set of all finite words over the alphabet $A = \{(0,0), \dots, (M-1, M-1)\}$. The root of $\mathcal{T}$ is the empty string $\emptyset$. The children of the node $(\underline{i}_n, \underline{j}_n) = ((i_1, j_1), \dots, (i_n, j_n))$ are $((i_1, j_1), \dots, (i_n, j_n), (i_{n+1}, j_{n+1}))$ for all $(i_{n+1}, j_{n+1}) \in A$. We write $\mathcal{T}_n$ for the collection of level n nodes, that is $\mathcal{T}_n = A^n$. It will be often useful to present a sequence from $\mathcal{T}_n$ as a pair of sequences $(\underline{i}_n, \underline{j}_n)$ of length n over an alphabet $B = \{0, \dots, M-1\}$.

For every node $(\underline{i}_n, \underline{j}_n)$ we are given a label $X_{\underline{i}_n, \underline{j}_n} \in \{0, 1\}$. We write Ω for the set of labeled trees

$$\Omega := \{0,1\}^{\mathcal{T}}.$$

For an $\omega \in \Omega$ and $n \in \mathbb{N}$ let

$$\mathcal{E}_n(\omega) := \left\{ (\underline{i}_n, \underline{j}_n) \in \mathcal{T}_n : X_{i_1, j_1} = \cdots = X_{i_1 \dots i_n, j_1 \dots j_n} = 1 \right\}$$

and

$$\mathcal{E}(\omega) := \left\{ \mathbf{i} = (i_1, i_2, \dots ; j_1, j_2, \dots) \in B^{\mathbb{N}} \times B^{\mathbb{N}} : \forall n,\ X_{i_1 \dots i_n, j_1 \dots j_n} = 1. \right\}$$

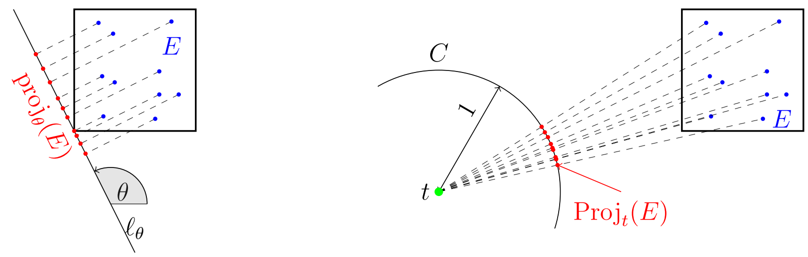


FIGURE 1. The radial projection to the line of angle θ and the radial projection with center t of the set E are $\text{proj}_\theta(E)$ and $\text{Proj}_t(E)$ respectively.

The probability measure $\mathbb{P}$ on Ω is defined such that for all $(\underline{i}_n, \underline{j}_n) \in \mathcal{T}$, $X_{\underline{i}_n, \underline{j}_n}$ are independent Bernoulli random variables with

$$\mathbb{P}(X_\emptyset) = 1 \text{ and } \mathbb{P}\left(X_{i_1 \dots i_n}\right) = p_{i_n}. \tag{3.1}$$

Using this we define the random set E as follows: For $(\underline{i}_n, \underline{j}_n) \in \mathcal{T}_n$ put

$$K_{\underline{i}_n, \underline{j}_n} := \left(\sum_{\ell=1}^{n} i_\ell \cdot M^{-\ell}, \sum_{\ell=1}^{n} j_\ell \cdot M^{-\ell}\right) + M^{-n} \cdot K,$$

and

$$p_{\underline{i}_n, \underline{j}_n} := \prod_{\ell=1}^{n} p_{i_\ell, j_\ell}.$$

For an $\omega \in \Omega$ let

$$E_n = E_n(\omega) := \bigcup_{(\underline{i}_n, \underline{j}_n) \in \mathcal{E}_n(\omega)} K_{\underline{i}_n, \underline{j}_n} \text{ and } E = E(\omega) := \bigcap_{n=1}^{\infty} E_n(\omega).$$

The orthogonal projection to the line of angle θ and the radial projection with center t of the set E are denoted by $\text{proj}_\theta(E)$ and $\text{Proj}_t(E)$ respectively. These definitions are presented on Figure 1. The following fact is immediate from (2.3) and from the definition of the box dimension:

Fact 5. *The following assertion holds almost surely:*
If $E(\omega) \neq \emptyset$ then

$$\lim_{n \to \infty} \frac{1}{n} \log \#\mathcal{E}_n(\omega) \to \log \sum_{i,j=0}^{M-1} p_{ij}. \tag{3.2}$$

We will frequently use the probability space $(\widehat{\Omega}, \widehat{\mathbb{P}})$, where

$$\widehat{\Omega} := \{\omega : \ (3.2) \text{ holds } \}, \qquad \widehat{\mathbb{P}}(\cdot) := \mathbb{P}(\cdot | \widehat{\Omega}). \tag{3.3}$$

This definition always makes sense because (2.2) and (2.3) imply that

$$\mathbb{P}(\widehat{\Omega}) > 0. \tag{3.4}$$

3.1. Π_α**-the angle** α **projection.** As we already mentioned above $\Pi_\alpha : K \to \Delta^\alpha$ is the angle α projection to Δ^α. Then for every α_1, α_2 and for every $(x,y) \in K$ we have

$$|\Pi_{\alpha_1}(x,y) - \Pi_{\alpha_2}(x,y)| \leq \sqrt{2}\, |\alpha_2 - \alpha_1| \,. \tag{3.5}$$

3.2. **The level** n **descendants.** We recall that $\varphi_{\underline{i}_n,\underline{j}_n}$ was defined in (2.7). The angle α projection of the diagonal $\Delta^\alpha_{\underline{i}_n,\underline{j}_n} := \varphi_{\underline{i}_n,\underline{j}_n}(\Delta^\alpha)$ of the level n square $K_{\underline{i}_n,\underline{j}_n}$ to the diagonal Δ^α is denoted by $\Delta_{\alpha,\underline{i}_n,\underline{j}_n}$.

$$\Delta_{\alpha,\underline{i}_n,\underline{j}_n} := \Pi_\alpha(\Delta^\alpha_{\underline{i}_n,\underline{j}_n}) = \Pi_\alpha \circ \varphi_{\underline{i}_n,\underline{j}_n}(\Delta^\alpha).$$

In what follows we will define a family of subintervals $\Delta^\alpha_i \subset \Delta^\alpha$, we will use notation $\Delta_{\underline{i}_n,\underline{j}_n;i} := \varphi_{\underline{i}_n,\underline{j}_n}(\Delta^\alpha_i)$.
The map $\Pi_\alpha \circ \varphi_{\underline{i}_n,\underline{j}_n} : \Delta \to \Delta_{\alpha,\underline{i}_n,\underline{j}_n}$ is a linear contraction of ratio M^{-n}. The inverse of this map is denoted by $\psi_{\alpha,\underline{i}_n,\underline{j}_n}$. That is

$$\psi_{\alpha,\underline{i}_n,\underline{j}_n} : \Delta_{\alpha,\underline{i}_n,\underline{j}_n} \to \Delta^\alpha, \qquad \psi_{\alpha,\underline{i}_n,\underline{j}_n} := \left(\Pi_\alpha \circ \varphi_{\underline{i}_n,\underline{j}_n}\right)^{-1}$$

is onto, linear and $\left|\psi_{\alpha,\underline{i}_n,\underline{j}_n}(x) - \psi_{\alpha,\underline{i}_n,\underline{j}_n}(y)\right| = M^n \cdot |x-y|$ for all $x, y \in \Delta_{\alpha,\underline{i}_n,\underline{j}_n}$. For an $\alpha \in \mathfrak{D}$ and $x \in \Delta^\alpha$ the collection of the labels of those level n squares $K_{\underline{i}_n,\underline{j}_n}$ whose Π_α projection to Δ^α contains x is called $D_n(x,\alpha)$.

$$D_n(x,\alpha) := \left\{ (\underline{i}_n, \underline{j}_n) : x \in \Pi_\alpha(K_{\underline{i}_n,\underline{j}_n}) \right\}.$$

If we restrict ourselves to those level n squares that appear in E_n we obtain

$$\mathcal{E}_n[x,\alpha](\omega) := \left\{ (\underline{i}_n, \underline{j}_n) \in \mathcal{E}_n(\omega) : x \in \Pi_\alpha(\Delta^\alpha_{\underline{i}_n,\underline{j}_n}) \right\}. \tag{3.6}$$

The family of all of the level n descendants of a point $x \in \Delta^\alpha$ is

$$\Psi^n_\alpha(x) := \left\{ (\psi_{\alpha,\underline{i}_n,\underline{j}_n}(x), \underline{i}_n, \underline{j}_n) \right\}_{(\underline{i}_n,\underline{j}_n) \in \mathcal{E}_n(\omega,x,\alpha)}$$

3.3. **A random inverse Markov operator and its expected value.** For an $\alpha \in \mathfrak{D}$ and every $\omega \in \Omega$ we introduce the random operator $G_{\alpha,\omega}$ defined on the set of real valued functions on Δ^α.

$$G_{\alpha,\omega} f(x) := \sum_{(i,j) \in \mathcal{E}_1[x,\alpha](\omega)} f \circ \psi_{\alpha,i,j}(x).$$

The n-th iterate of $G_{\alpha,\omega}$ is denoted by $G^n_{\alpha,\omega}$. Clearly,

$$G^n_{\alpha,\omega} f(x) = \sum_{(\underline{i}_n,\underline{j}_n)\in\mathcal{E}_n[x,\alpha](\omega)} f \circ \psi_{\alpha,\underline{i}_n,\underline{j}_n}(x).$$

In particular for any $H \subset \Delta^\alpha$ we have

$$G^n_{\alpha,\omega} \mathbb{1}_H(x) = \#\left\{(\underline{i}_n,\underline{j}_n) \in \mathcal{E}_n(\omega) : x \in \Pi_\alpha\left(\varphi_{\underline{i}_n,\underline{j}_n}(H)\right)\right\}. \tag{3.7}$$

The expected value of the random operator G_α is called F_α. That is

$$F_\alpha := \mathbb{E}\left[G_{\alpha,\omega}\right]. \tag{3.8}$$

We denote the n-th iterate of F_α by F^n_α. Then for every real valued function f we have

$$F^n_\alpha f(x) = \sum_{(\underline{i}_n,\underline{j}_n)\in D_n(x,\alpha)} p_{\underline{i}_n,\underline{j}_n} \cdot f \circ \psi_{\alpha,\underline{i}_n,\underline{j}_n}(x). \tag{3.9}$$

In what follows we need to apply F_α also for an arbitrary function h whose domain is a proper subset $H \subset \Delta^\alpha$. To define $F_\alpha h$ in this case first we extend the definition of h from H to Δ^α by

$$\widetilde{h}(x) := \begin{cases} h(x), & \text{if } x \in H; \\ 0, & \text{otherwise.} \end{cases}$$

We define $F_\alpha h(x) := F_\alpha \widetilde{h}(x)$. Then F_α is a positive linear operator which preserves $\mathcal{C}_0(\Delta^\alpha)$ the set of continuous real functions on Δ vanishing at the end points. Clearly, for an arbitrary $H \subset \Delta$, $n \geq 1$ and an $\beta \in \left(0, \frac{\pi}{2}\right)$ we have

$$F^n_\beta\left(\mathbb{1}_H(x)\right) = \sum_{x\in\Pi_\beta\left(\varphi_{\underline{i}_n,\underline{j}_n}(H)\right)} p_{\underline{i}_n,\underline{j}_n}. \tag{3.10}$$

Using this we obtain

Fact 6. *Part (b') of Condition (A) is equivalent to*

$$F^{r_\alpha}_\alpha\left(\mathbb{1}_{\widetilde{\Delta}^\alpha_1}(x)\right) > 2 \cdot \mathbb{1}_{\widetilde{\Delta}^\alpha_2}(x) \textit{ for all } x \in \widetilde{\Delta}^\alpha_2. \tag{3.11}$$

3.4. **A family of almost linear projections.** Here we consider projections which include the angle α projections for every $\alpha \in \left(0, \frac{\pi}{2}\right)$ and the radial projections with center away from K.
First we need to introduce two proper sub intervals $\Delta^\alpha_1, \Delta^\alpha_2$ of Δ^α which are going to be used very frequently throughout this paper:

Definition 7.

(1) *We define the set* $\mathfrak{A} \subset \mathfrak{D}$ *as follows: An angle* $\alpha \in \mathfrak{D}$ *belongs to the set* $\mathfrak{A}$ *iff there exist an open neighborhood*

$$V_\alpha = (\alpha - \xi_\alpha, \alpha + \xi_\alpha) \subset \mathfrak{D} \tag{3.12}$$

of α *with radius* $\xi_\alpha > 0$ *and closed subintervals* $\Delta_1^\alpha, \Delta_2^\alpha \subset \Delta^\alpha$ *and an integer* r_α *such that*

(a): $\Delta_1^\alpha \subset \mathrm{int}\Delta_2^\alpha,\ \Delta_2^\alpha \subset \mathrm{int}\Delta^\alpha$,

(b): $F_\beta^{r_\alpha} \mathbb{1}_{\Delta_1^\alpha}(x) > 2 \cdot \mathbb{1}_{\Delta_2^\alpha}(x)$ *for all* $\beta \in V_\alpha,\ x \in \Delta_2^\alpha$.

(2) *Assume that* $\alpha \in \mathfrak{A}$. *Then we define*

$$\delta_\alpha := \min\left\{\mathrm{dist}(\Delta_1^\alpha, \mathbb{R} \setminus \Delta_2^\alpha), \mathrm{dist}(\Delta_2^\alpha, \mathbb{R} \setminus \Delta^\alpha)\right\}. \tag{3.13}$$

Actually it is enough to require that condition (b) in Definition 7 holds for a single α then it also holds for a sufficiently small neighborhood of α.

Fact 8. *An angle* $\alpha \in \mathfrak{D}$ *belongs to the set* $\mathfrak{A}$ *if and only if condition (A) holds for* α.

Proof. By Fact 6 it is clear that for every $\alpha \in \mathfrak{A}$ satisfies Condition (A). To verify the opposite inclusion we fix such an α which satisfies condition (A). For notational simplicity we write $n := r_\alpha$. We can actually choose $\Delta_2^\alpha := \widetilde{\Delta}_2^\alpha$ and we define Δ_1^α as an arbitrary closed interval satisfying

$$\mathrm{int}\left(\widetilde{\Delta}_1^\alpha\right) \subset \mathrm{int}(\Delta_1^\alpha) \subset \Delta_1^\alpha \subset \mathrm{int}(\Delta_2^\alpha). \tag{3.14}$$

Clearly, we can a find small open neighborhood $V_\alpha \subset \mathfrak{D}$ of α such that

$$\forall \beta \in V_\alpha,\ \forall \underline{i}_n, \underline{j}_n,\ \Pi_\beta(\varphi_{\underline{i}_n,\underline{j}_n}(\Delta_1^\alpha)) \supset \Pi_\alpha(\varphi_{\underline{i}_n,\underline{j}_n}(\widetilde{\Delta}_1^\alpha)) \tag{3.15}$$

holds. Hence for an $x \in \Delta_2^\alpha$ we have

$$\begin{aligned} F_\beta^n\left(\mathbb{1}_{\Delta_1^\alpha}(x)\right) &= \sum_{x \in \Pi_\beta(\varphi_{\underline{i}_n,\underline{j}_n}(\Delta_1^\alpha))} p_{\underline{i}_n,\underline{j}_n} \\ &\geq \sum_{x \in \Pi_\alpha(\varphi_{\underline{i}_n,\underline{j}_n}(\widetilde{\Delta}_1^\alpha))} p_{\underline{i}_n,\underline{j}_n} = F_\alpha^n\left(\mathbb{1}_{\widetilde{\Delta}_1^\alpha}(x)\right) \\ &> 2 \cdot \mathbb{1}_{\widetilde{\Delta}_2^\alpha}(x) = 2 \cdot \mathbb{1}_{\Delta_2^\alpha}(x), \end{aligned}$$

where in the first and in the last step we used (3.10). □

A sufficient condition for $\alpha \in \mathfrak{A}$ is as follows:

Lemma 9. *Assume that for an* $\alpha \in \mathfrak{D}$ *the following holds:*

$$\exists f_\alpha \in \mathcal{C}_0(\Delta^\alpha),\ \exists \varepsilon > 0 \text{ such that } F_\alpha f_\alpha \geq (1+\varepsilon) f_\alpha \text{ and } f_\alpha|_{\mathrm{int}\Delta^\alpha} > 0. \tag{3.16}$$

Then $\alpha \in \mathfrak{A}$.

The proofs of Lemmas 9 and 10 below are given in Section (5.1). We will verify in the proof of Lemma 10 that if all probabilities are the same and greater than $1/M$ then (3.16) always holds. In this way we obtain that

Lemma 10. *Assume that there exists a* $p \in (0,1)$ *such that*

$$\forall 0 \le i,j \le M-1 \text{ we have } p = p_{ij} \text{ and } M \cdot p > 1. \tag{3.17}$$

Then for all $\alpha \in \mathfrak{D}$ *we have* $\alpha \in \mathfrak{A}$.

Now we define the notion of regular family of almost linear projections. This will have crucial role in the rest of the paper because we prove all of our results in such a way that both of the family of angle α projection for all α and the family of radial projections with all possible centers can be covered by countably many regular family of almost linear projections.

Definition 11.

(a): *We say that a function* $S : K \to \Delta$ *is an* **almost linear projection with direction function** $\alpha : K \to \mathfrak{D}$ *if*

$$S(x,y) = \Pi_{\alpha(x,y)}(x,y) \qquad (x,y) \in K, \tag{3.18}$$

where the direction function α *assigns an angle* $\alpha(x,y)$ *to the point* $(x,y) \in K$ *such that the following four conditions hold:*

- **(a0):** *The set* $\{\alpha(x,y) : (x,y) \in K\}$ *is contained in either of the two connected components of* $\mathfrak{D}$.
- **(a1):**

$$\alpha(x,y) \in \mathfrak{A}, \quad \forall (x,y) \in K. \tag{3.19}$$

In this way $\delta_{\alpha(x,y)}$ *is defined for all* $(x,y) \in K$ *by (3.13).*

- **(a2):** *Further, we require that*

$$\delta_\alpha := \inf\left\{\delta_{\alpha(x,y)} : (x,y) \in K\right\} > 0. \tag{3.20}$$

- **(a3):** *We assume that* $\exists \gamma > 0$ *such that*

$$\forall \underline{i}_n, \underline{j}_n \text{ if } (x,y),(x',y') \in K_{\underline{i}_n,\underline{j}_n}, \text{ then } |\alpha(x,y) - \alpha(x',y')| \le \gamma M^{-n}. \tag{3.21}$$

The smallest such γ *is denoted by* γ_α *and called the* ***distortion of direction function*** α. *That is*

$$\gamma_\alpha := \min\{\gamma : \gamma \text{ satisfies (3.21)}\}. \tag{3.22}$$

(b): *Given a family* $\mathcal{S} = \{S_t\}_{t \in T}$ *of almost linear projections with corresponding direction functions* $\{\alpha_t(x,y)\}_{t \in T}$. *We say that* $\mathcal{S}$ ***is a regular family of almost linear projections*** S_t *if the following conditions hold:*

(b1): *There exists* $t_{\mathcal{S}} \in T$, $(x_{\mathcal{S}}, y_{\mathcal{S}}) \in K$ *such that for the angle*

$$\alpha_{\mathcal{S}} := \alpha_{t_{\mathcal{S}}}(x_{\mathcal{S}}, y_{\mathcal{S}}) \in \mathfrak{D} \tag{3.23}$$

we have

$$\forall t \in T,\ \forall (x,y) \in K,\ \alpha_t(x,y) \in V_{\alpha_{\mathcal{S}}}, \tag{3.24}$$

where V_α *was defined in Definition 7.*

(b2): *Put* $\delta_{\mathcal{S}} := \delta_{\alpha_{\mathcal{S}}}$ *and* $r_{\mathcal{S}} := r_{\alpha_{\mathcal{S}}}$. *We require that*

$$\gamma_{\mathcal{S}} := \max\{\gamma_{\alpha_t} : t \in T\} < M^{-r_{\mathcal{S}}} \cdot \delta_{\mathcal{S}}/64. \tag{3.25}$$

(b3): *The parameter domain* T *has finite box dimension in the sup metric:*

$$\rho(t_1, t_2) := \sup_{(x,y)\in K} |\alpha_{t_1}(x,y) - \alpha_{t_2}(x,y)|. \tag{3.26}$$

In particular, what we use about T *it is that there exists a* $\vartheta > 64/\delta_{\mathcal{S}}$ *such that for every* n *big enough we can find a* $T_n \subset T$ *satisfying*

$$T_n \text{ is } \delta_{\mathcal{S}} M^{-n}/64 \text{ dense in } T, \text{ and } \#T_{n+r_{\mathcal{S}}} \cdot \frac{64 M^{n+r_{\mathcal{S}}}}{\delta} \leq \vartheta^n. \tag{3.27}$$

(b4): *We assume that the diameter of* T *(in the metric* ρ*) is smaller than* $M^{-r_{\mathcal{S}}} \cdot \delta_{\mathcal{S}}/64$.

Remark 12. *We remark that putting together parts (a3), (b2) and (b4) of Definition 11 yields:*

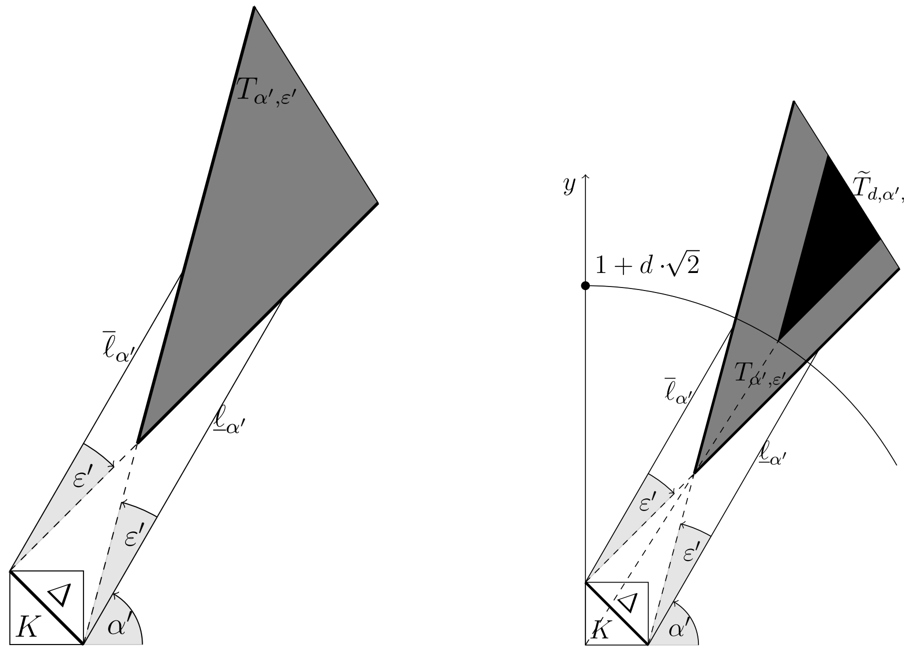
$$\forall t \in T, \forall (x,y) \in K,\ |\alpha_t(x,y) - \alpha_{t_{\mathcal{S}}}(x_{\mathcal{S}}, y_{\mathcal{S}})| < \delta_{\mathcal{S}} M^{-r_{\mathcal{S}}}/32. \tag{3.28}$$

3.5. **Linear projections.** For every $\tau \in \mathfrak{A}$ the family of angle t projections $\{\Pi_t\}_{t\in(\tau-\varepsilon,\tau+\varepsilon)}$ is a regular family of almost linear (actually linear) projections whenever ε is so small that, using the notation of Definition 7, both $(\tau-\varepsilon, \tau+\varepsilon) \subset V_\tau$ and $\varepsilon < M^{-r_\tau}\delta_\tau/(64\sqrt{2})$.
In this case

$$\alpha_t(x,y) \equiv t,\ \gamma = 0 \text{ and } \rho(t_1,t_2) \leq \sqrt{2} \cdot |t_2 - t_1|. \tag{3.29}$$

3.6. **Radial projections.** Here we assume that the assumptions of Theorem 2 hold.
For a $t \in \mathbb{R}^2 \setminus K$ and $(x,y) \in K$, $(x,y) \neq t$ we define $\alpha_t(x,y)$ as the angle between the positive half of the x-axis and the line which connects the point (x,y) to t. Put

$$R_t(x,y) := \Pi_{\alpha_t(x,y)}(x,y). \tag{3.30}$$



(a) The diameter (in metric ρ) of the cone $T_{\alpha',\varepsilon'}$ is smaller than $2\varepsilon'$.

(b) For every $t \in \widetilde{T}_{d,\alpha',\varepsilon'}$ (the black cone) the distortion of the direction function $\alpha_t(x,y)$, defined in Section 3.6, is smaller than $1/d$.

FIGURE 2. R_t is the radial projection for $t \in \mathbb{R}^2$

By elementary geometry for an arbitrary $t \in \mathbb{R}^2 \setminus K$:
(3.31)

$$\mathrm{Proj}_t(E) \text{ contains interval if and only if } R_t(E) \text{ contains inerval},$$

where we remind the reader that $\mathrm{Proj}_t(x,y)$ was defined as the radial projection with center t of the point (x,y).
In the rest of this Section we find some necessary conditions on the parameter domain $T \subset \mathbb{R}^2 \setminus K$ so that

$$\mathcal{R} = \{R_t\}_{t\in T}$$

is a regular family of almost linear projections. For an angle $\alpha' \in \mathfrak{D}$ and small $0 < \varepsilon' < \alpha'$, the infinite cone on Figure 2 (a) is denoted by $T_{\alpha',\varepsilon'}$. Note that

$$\text{if } t \in T_{\alpha',\varepsilon'} \text{ then } \forall (x,y) \in K,\ |\alpha_t(x,y) - \alpha'| < \varepsilon' \tag{3.32}$$

Further $\forall d>0,\ (x,y),(x',y') \in K_{\underline{i}_n,\underline{j}_n}$ for some $\underline{i}_n, \underline{j}_n$ then

$$\text{if } |t| > 1+\sqrt{2}\cdot d \text{ then } |\alpha_t(x,y)-\alpha_t(x',y')| < \frac{1}{d}M^{-n}. \tag{3.33}$$

That is

$$\forall d>0, \quad \text{if } |t| > 1+\sqrt{2}\cdot d \text{ then } \gamma_{\alpha_t} \le \frac{1}{d}. \tag{3.34}$$

Hence we obtain that

Fact 13. *Assume that the assumptions of Theorem 2 hold. Using the notation of definition 7 if*

$$\varepsilon' < \xi_{\alpha'}/4 \text{ and } d > \frac{64M^{r_{\alpha'}}}{\delta_{\alpha'}} \tag{3.35}$$

then $\{R_t\}_{t\in T}$ is a regular family of almost linear projections whenever $T \subset T_{\alpha',\varepsilon'} \setminus B(0, 1+d\cdot\sqrt{2})$. We write $\widetilde{T}_{d,\alpha',\varepsilon'}$ for the unbounded black cone region on Figure 2 (b). If d,α',ε' satisfy (3.35) then $\{R_t\}_{t\in\widetilde{T}_{d,\alpha',\varepsilon'}}$ is a regular family of almost linear projections.

We introduce the set $\widetilde{T} \subset \mathbb{R}^2$ such that family of radial projection with center $t \in \widetilde{T}$ is a countable union of regular families of almost linear projections. Furthermore, for each $\alpha \in (0,2\pi)\setminus\{0,\pi/2,\pi,3\pi/2\}$ the half line through the origin with angle α is contained $\widetilde{T}$ apart from its initial segment.

Definition 14 (The definition of $\widetilde{T}$)**.** *Let $\{\alpha'_\ell,\varepsilon'_\ell\}$ be defined such that for all ℓ, $\alpha'_\ell \in (0,\pi/2)$, $0<\varepsilon'_\ell<\xi_{\alpha'_\ell}/4$ and $(0,\pi/2)\subset \cup_{\ell=1}^{\infty}(\alpha'_\ell-\varepsilon'_\ell, \alpha'_\ell+\varepsilon'_\ell)$ in such a way that every $\tau\in(0,\pi/2)$ is covered only L times, where L is a constant which can be derived from the Besicovich Covering Theorem* [12, Theorem 2.7]. *For all ℓ we fix a d_ℓ such that (3.35) holds. We define*

$$\widetilde{T}^1 := \bigcup_{\ell=1}^{\infty} \widetilde{T}_{d_\ell,\alpha'_\ell,\varepsilon'_\ell}. \tag{3.36}$$

Substituting the interval $(0,\pi/2)$ with $(\pi/2,\pi)$, $(\pi,3\pi/2)$ and $(3\pi/2,2\pi)$ and substituting the origin $\mathbf{c}_1 := (0,0)$ with the appropriate corner of K: $\mathbf{c}_2 := (1,0)$, $\mathbf{c}_3 := (1,1)$ and $\mathbf{c}_4 := (0,1)$ we can define analogously the sets $\widetilde{T}^2$, $\widetilde{T}^3$ and $\widetilde{T}^4$ respectively. That is, we get $\widetilde{T}_{d_\ell,\alpha'_\ell,\varepsilon'_\ell}$ for $\alpha'\in(\pi/2,\pi)$, $\alpha'\in(\pi,3\pi/2)$ and $\alpha'\in(3\pi/2,2\pi)$ by reflecting the black cone on part (b) of Figure 2 to the line $x=1/2$, to the center of K and to the line $y=1/2$ respectively.

The following properties of $\widetilde{T}^k$, $k = 1, \ldots, 4$ are straightforward from the definition:

Fact 15. *Note that the boundary of* $\widetilde{T}^k$ *is a continuous curve that consists of countably many straight lines. Further, let* $P_1, \ldots, P_4$ *be the OPEN plane quarter defined as the set of points on the plane that are NE, NW, SW, SE direction from the corners* $\mathbf{c}_1, \ldots, \mathbf{c}_4$ *respectively. For any* $k = 1, \ldots, 4$ *let* h_θ *be the half line through* $\mathbf{c}_k$ *with angle* θ *chosen in such a way that* $h_\theta \subset P_k$*. We write* M_θ *for the point where* h_θ *enters into* $\widetilde{T}^k$*. Then the length of the segment* $[\mathbf{c}_k, M_\theta]$ *connecting the corner* $\mathbf{c}_k$ *with* M_θ *is a continuous function of* θ *and* $h_\theta \setminus [O, M_\theta) \subset \widetilde{T}^k$*.*

Because of the self-similar nature of the construction it will be important to consider sets which are constructed like $\widetilde{T}^i$ but the construction is carried out for a scaled version of K. To introduce these sets we need some definitions. For a $w \in \mathbb{R}^2$ and $0 < \lambda$ let

$$\Upsilon_{w,\lambda}(x) := \lambda(x - w)$$

and we write

$$\widetilde{T}_{\underline{i}_n, \underline{j}_n} := \bigcup_{q=1}^{4} \Upsilon_{\mathbf{c}_q, M^{-n}} \left(\widetilde{T}^q\right) + \varphi_{\underline{i}_n, \underline{j}_n}(\mathbf{c}_q). \tag{3.37}$$

By the Fact 15 it is immediate that:

Fact 16. *Let* $u \in P_i$ *be arbitrary points. Then there exists a* k_0 *such that for all* $k \geq k_0$ *we have*

$$u \in \operatorname{int}\left(\Upsilon_{\mathbf{c}_i, M^{-k}}(\widetilde{T}^i)\right), \ i = 1, \ldots, 4. \tag{3.38}$$

Actually in Section 4 we will need a stronger result:

Lemma 17. *Given a point* $z \in K$ *such that* z *does not lie on the boundary of any cylinder squares. We write* $\mathbf{i}, \mathbf{j} \in \{0, \ldots, M-1\}^{\mathbb{N}}$ *such that* $z = \left(\sum_{k=1}^{\infty} i_k M^{-k}, \sum_{k=1}^{\infty} j_k M^{-k}\right)$ *and as usual we denote* $\underline{i}_n = (i_1, \ldots, i_n)$*,* $\underline{j}_n = (j_1, \ldots, j_n)$*. Then*

$$\bigcup_{n=1}^{\infty} \widetilde{T}_{\underline{i}_n, \underline{j}_n} = \mathbb{R}^2 \setminus (\text{ horizontal line through } z \cup \text{ vertical line through } z). \tag{3.39}$$

Proof. Fix an arbitrary $u \in \mathbb{R}^2$ which lies neither on the same horizontal nor on the same vertical line as z. Without loss of generality we may assume that

$$u \text{ lies SE direction from } z. \tag{3.40}$$

It is enough to prove that for every n big enough we have

$$u \in \Upsilon_{\mathbf{c}_2, M^{-n}}(\widetilde{T}^2) + \varphi_{\underline{i}_n, \underline{j}_n}(\mathbf{c}_2). \tag{3.41}$$

This is equivalent to

$$M^n\left(u - \varphi_{\underline{i}_n, \underline{j}_n}(\mathbf{c}_2)\right) + \mathbf{c}_2 \in \widetilde{T}^2. \tag{3.42}$$

Clearly,

$$M^n\left(u - \varphi_{\underline{i}_n, \underline{j}_n}(\mathbf{c}_2)\right) + \mathbf{c}_2 \in \Upsilon^{-1}_{\mathbf{c}_2, M^{-n}}(u - z) + K \tag{3.43}$$

The assertion of the Lemma follows from Fact 16.

□

4. We derive Theorem 2 from Theorem 3

Throughout this section we assume that the assertion of Theorem 3 holds.

Proof of part (a) of Theorem 2. As we have already mentioned the orthogonal projection to lines of angle 0 and $\pi/2$ was settled by Falconer and Grimett [6] and [7]. We consider Π_α instead of proj_α since $\text{proj}_\alpha(E)$ contains interval if and only if $\Pi_\alpha(E)$ contains an interval for every angle α. So, for symmetry it is enough to verify that almost surely, conditioned on non-extinction:

$$\forall \alpha \in (0, \pi/2)\,,\ \Pi_\alpha(E) \text{ contains some intervals.} \tag{4.1}$$

This is so because using the notation of Section 3.5 we can cover $(0, \pi/2)$ by countably many intervals $(\tau_n - \varepsilon_n, \tau_n + \varepsilon_n) \subset (0, \pi/2)$ such that for each n, $\{\Pi_t\}_{t \in (\tau_n - \varepsilon_n, \tau_n + \varepsilon_n)}$ is a regular family of almost linear projections. It follows from Theorem 3 that almost surely conditioned on non-extinction for all n and for all $t \in (\tau_n - \varepsilon_n, \tau_n + \varepsilon_n)$ $\Pi_t(E)$ contains some intervals. Since $(0, \pi/2) \subset \cup_{n=1}^{\infty}(\tau_n - \varepsilon_n, \tau_n + \varepsilon_n)$ this completes the proof of (4.1). □

Proof of part (b) of Theorem 2. We use the self-similar nature of the process. Namely, for

$$E_{\underline{i}_n, \underline{j}_n} := E \cap K_{\underline{i}_n, \underline{j}_n}$$

we have

$$\text{if } (\underline{i}_n, \underline{j}_n) \in \mathcal{E}_n \text{ then } \varphi^{-1}_{\underline{i}_n, \underline{j}_n}(E_{\underline{i}_n, \underline{j}_n}) \stackrel{d}{=} E. \tag{4.2}$$

Consider the family $R_t(x, y)$ defined in Section 3.6. Using Fact 13 we obtain that for $k = 1, \ldots, 4$, $\{R_t(x, y)\}_{t \in \widetilde{T}^k}$ is a countable union of regular families of almost linear projections. Hence it follows from

Theorem 3 that for any $k \in \{1, \ldots, 4\}$ almost surely the following holds:

(4.3) $$\text{If } E \neq \emptyset \text{ then } \forall t \in \widetilde{T}^k,\ R_t(E) \text{ contains an interval.}$$

Using the self-similar nature of the process mentioned above we obtain that for all $\underline{i}_n, \underline{j}_n$ the following assertion holds almost surely:

(4.4)
$$\text{If } E_{\underline{i}_n,\underline{j}_n}(\omega) \neq \emptyset \text{ then } \forall t \in \widetilde{T}_{\underline{i}_n,\underline{j}_n},\ R_t(E_{\underline{i}_n,\underline{j}_n}) \text{ contains an interval.}$$

This implies that for

$$\widetilde{\Omega} := \left\{ \omega \in \Omega : (4.4) \text{ holds for all } (\underline{i}_n, \underline{j}_n) \right\}$$

we have

(4.5) $$\mathbb{P}\left(\widetilde{\Omega}\right) = 1.$$

Let

$$\widetilde{\Omega}' := \left\{ \omega \in \widetilde{\Omega} :\ \text{either } E(\omega) = \emptyset \text{ or } \dim_{\mathrm{H}}(E(\omega)) > 1. \right\}$$

It follows from (2.8) that

$$\mathbb{P}(\widetilde{\Omega}') = 1$$

Whenever $\omega \in \widetilde{\Omega}'$ and $E(\omega)$ is not empty then we can find $z_1, z_2, z_3 \in E(\omega)$ which do not lie on the same rectangle with sides parallel to the coordinate axis, and z_1, z_2 and z_3 satisfy the assumption of Lemma 17. Applying the Lemma 17 for z_k the set on the left hand side of (3.39) is called $\widetilde{T}^k_{\underline{i}_n,\underline{j}_n}$. Clearly $\mathbb{R}^2 = \cup_{k=1}^3 \widetilde{T}^k_{\underline{i}_n,\underline{j}_n}$. This completes the proof of Part (b) of Theorem 2.

□

5. We derive Theorem 1 from 2

The fact that Theorem 1 follows from Theorem 2 is an easy consequence of Lemmas 9 and 10.

5.1. **The proof of Lemmas 9 and 10.** First we prove

Lemma 18. *Assume that α is chosen such that (3.16) holds. We fix an arbitrary f_α and $\varepsilon > 0$ which satisfy (3.16). Then we can choose closed intervals*

(5.1) $$\Delta_1^\alpha \subset \mathrm{int}\Delta_2^\alpha \text{ and } \Delta_2^\alpha \subset \mathrm{int}\Delta,$$

such that for

(5.2) $$g_1^\alpha := f_\alpha|_{\Delta_1^\alpha},\ g_2^\alpha := f_\alpha|_{\Delta_2^\alpha}$$

we have

$$F_\alpha \widetilde{g}_1^\alpha(x) > \left(1+\frac{\varepsilon}{2}\right)\cdot \widetilde{g}_2^\alpha(x) \text{ for } x \in \Delta_2^\alpha. \tag{5.3}$$

It follows from (3.16) and (5.1) that

$$0 < g_1^\alpha(x) \le g_2^\alpha(x) \text{ holds for all } x \in \Delta_1^\alpha. \tag{5.4}$$

Proof. We fix an α and f_α and ε satisfying (3.16) and suppress index α in this proof. For a set $H \subset \Delta$, put $B_r(H)$ for the radius r open neighborhood of H in Δ.

$$B_r(H) := \{y \in \Delta : \exists h \in H,\ |h-y| < \delta\}.$$

We call $W \subset \Delta$ the Π-projection of the mesh $1/M$ grid points in K.

$$W := \left\{x \in \Delta : \exists 0 \le i,j \le M,\ x = \Pi\left(\frac{i}{M}, \frac{j}{M}\right)\right\}.$$

We partition W into the set of the endpoints of Δ called $W_0 := \{(0,1),(1,0)\}$ and $W_1 := W \setminus W_0$. Fix $\eta > 0$ which satisfies:

$$\frac{\varepsilon}{2}\cdot \min_{x\in B_{\eta/M}(W_1)} f(x) > (M+1)^2 \sup_x \left\{f(x) : x \in B_{\eta/M}(W_0)\right\}. \tag{5.5}$$

and we define the two subintervals of Δ

$$\Delta_1 := \Delta \setminus B_\eta(W_0) \text{ and } \Delta_2 := \Delta \setminus B_{\eta/M}(W_0). \tag{5.6}$$

Let

$$B := B_{\eta/M}(W) \text{ and } B_i := B_{\eta/M}(W_i), i = 0,1. \tag{5.7}$$

Note that

$$x \in (\Delta \setminus B), (i,j) \in D_1(x,\alpha) \Longrightarrow \psi_{\alpha,i,j}(x) \in \Delta_1. \tag{5.8}$$

Fix an arbitrary $x \in \Delta_2 = \Delta \setminus B_0$. We divide the proof (5.3) into two cases among which the first is obvious:

$\mathbf{x} \in \mathbf{\Delta} \setminus \mathbf{B}$: Using the definition of F and then (3.16) we obtain

$$Fg_1(x) = Ff(x) \ge (1+\varepsilon)f(x) \ge \left(1+\frac{\varepsilon}{2}\right)\cdot g_2(x).$$

$\mathbf{x} \in \mathbf{B_1}$: By the definition of F:

$$Fg_1(x) \ge Ff - (M+1)^2\|f - \widetilde{g}_1\|_\infty,\ \forall x \in \Delta. \tag{5.9}$$

From this and from the fact that $Ff(x) \ge (1+\varepsilon)f(x)$, we obtain

$$Fg_1(x) \ge \left(1+\frac{\varepsilon}{2}\right) f(x) + \left(\frac{\varepsilon}{2} f(x) - (M+1)^2\|f - \widetilde{g}_1\|_\infty\right)$$

The definition of η yields that the expression in the second bracket is positive. This implies that

$$Fg_1(x) > \left(1 + \frac{\varepsilon}{2}\right) g_2(x) \text{ for } x \in \Delta_2.$$

□

Lemma 9 easily follows from Lemma 18.

Proof of Lemma 9. Using the notation of Lemma 18 we define r_α as the smallest integer satisfying

$$L := \left(1 + \frac{\varepsilon}{2}\right)^{r_\alpha} > 2 \cdot \frac{\max\limits_{x \in \Delta_1^\alpha} g_1^\alpha(x)}{\min\limits_{x \in \Delta_2^\alpha} g_2^\alpha(x)} \tag{5.10}$$

Then clearly,

$$F_\alpha^{r_\alpha} \mathbb{1}_{\Delta_1^\alpha}(x) > 2 \cdot \mathbb{1}_{\Delta_2^\alpha}(x) \text{ for all } x \in \Delta_2^\alpha.$$

□

5.2. **The case of equal probabilities.**

Proof of Lemma 10. We assume that

$$\forall i, j \qquad p_{ij} = p > \frac{1}{M}. \tag{5.11}$$

Fix an arbitrary $\alpha \in \mathcal{D}$. For an arbitrary $x \in \Delta^\alpha$ we define $f_\alpha(x)$ as the length of the segment that is cut out from the square K by the line through x of slope $\tan \alpha$. See Figure (3) for the graph of f_α. It is straightforward that f_α satisfies (3.16) with $\varepsilon := M \cdot p - 1 > 0$. Hence by Lemma 9 we obtain that $\alpha \in \mathfrak{A}$. □

The proof of Theorem 1 assuming Theorem 2. It follows from Lemma 10 that (2.5) implies that the conditions of Theorem 2 hold. Since the conclusions of these theorems are the same this completes the proof.

□

6. The proof of Theorem 3

In this section we fix a family of almost linear projections $\{S_t\}_{t \in T}$ with direction function $\alpha_t(x, y)$. For symmetry from now on without loss of generality we may assume that

$$\forall t, \forall (x, y) \in K \quad \alpha_t(x, y) \in (0, \pi/2) \tag{6.1}$$

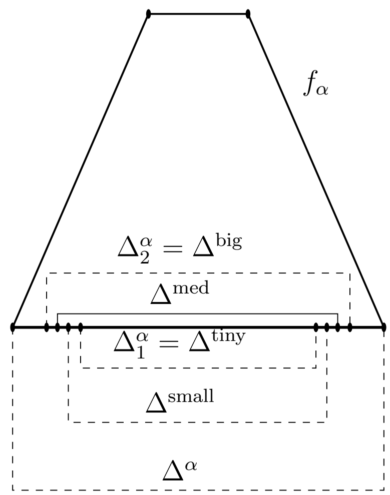


FIGURE 3. The function f_a and the intervals Δ_i^α.

and we write simply Δ for the diagonal of K that connects the corners $(0,1)$ and $(1,0)$. Using the notation from part **(b)** of Definition 11, for notational simplicity we write

$$\delta := \delta_{\mathcal{S}},\ \gamma := \gamma_{\mathcal{S}}, r := r_{\alpha_{\mathcal{S}}} \text{ and } \Delta^{\text{tiny}} := \Delta_1^{\alpha_{\mathcal{S}}}, \Delta^{\text{big}} := \Delta_2^{\alpha_{\mathcal{S}}}.$$

6.1. **The four subintervals of Δ.** Here we define two more subintervals of Δ and we study the relations between their image by $\psi_{\beta,\underline{i}_n,\underline{j}_n}$ for some $\beta \in V_{\alpha_{\mathcal{S}}}$, $\underline{i}_n, \underline{j}_n$. Let

$$\Delta^{\text{small}} := \text{closure}(B_{\delta/3}(\Delta^{\text{tiny}})),\ \Delta^{\text{med}} := \text{closure}(B_{2\delta/3}(\Delta^{\text{tiny}})).$$

By (3.13) we have $B_\delta\left(\Delta^{\text{big}}\right) \subset \Delta$ and

(6.2)
$$B_{\delta/3}\left(\Delta^{\text{tiny}}\right) \subset \Delta^{\text{small}}, B_{\delta/3}\left(\Delta^{\text{small}}\right) \subset \Delta^{\text{med}}, B_{\delta/3}\left(\Delta^{\text{med}}\right) \subset \Delta^{\text{big}}.$$

Consequently, we write $\Delta_{\underline{i}_n,\underline{j}_n}^{\text{tiny}} := \varphi_{\underline{i}_n,\underline{j}_n}(\Delta^{\text{tiny}})$ and

$$\Delta_{\underline{i}_n,\underline{j}_n}^{\text{small}} := \varphi_{\underline{i}_n,\underline{j}_n}(\Delta^{\text{small}}), \Delta_{\underline{i}_n,\underline{j}_n}^{\text{med}} := \varphi_{\underline{i}_n,\underline{j}_n}(\Delta^{\text{med}}), \Delta_{\underline{i}_n,\underline{j}_n}^{\text{big}} := \varphi_{\underline{i}_n,\underline{j}_n}(\Delta^{\text{big}})$$

Lemma 19. *Let $I_1 \subsetneq I_2$ be any two distinct intervals from*

$$\left\{\Delta_{\underline{i}_r,\underline{j}_r}^{\text{tiny}}, \Delta_{\underline{i}_r,\underline{j}_r}^{\text{small}}, \Delta_{\underline{i}_r,\underline{j}_r}^{\text{med}}, \Delta_{\underline{i}_r,\underline{j}_r}^{\text{big}}, \Delta_{\underline{i}_r,\underline{j}_r}\right\}.$$

Then for every $t_1, t_2 \in T$, $(x_1,y_1),(x_2,y_2) \in K$ we have

$$B_{\delta M^{-r}/8}\left(\Pi_{\alpha_{t_1}(x_1,y_1)} I_1\right) \subset \Pi_{\alpha_{t_2}(x_2,y_2)} I_2. \tag{6.3}$$

Proof. By (3.28) we know that $|\alpha_{t_1}(x_1,y_1) - \alpha_{t_2}(x_2,y_2)| < \delta M^{-r}/16$. Using (3.5) we obtain that the distance between the image of the appropriate end points of I_1 and I_2 by the maps $\Pi_{\alpha_{t_1}(x_1,y_1)}$ and $\Pi_{\alpha_{t_2}(x_2,y_2)}$

is bigger than $\delta M^{-r}\left(\frac{1}{3}-\frac{2\sqrt{2}}{16}\right) > \delta M^{-r}\frac{1}{8}$. This immediately implies that the assertion of the Lemma holds. □

For any n and $t \in T$ we approximate $S_t(x,y) = \Pi_{\alpha_t(x,y)}(x,y)$ on $K_{\underline{i}_n,\underline{j}_n}$ with the linear projection with angle corresponding to the left upper corner of $K_{\underline{i}_n,\underline{j}_n}$. That is

$$\text{for } (x,y) \in K_{\underline{i}_n,\underline{j}_n} \text{ let } S_t^{(n)}(x,y) := \Pi_{\alpha_{\underline{i}_n,\underline{j}_n}(t)}(x,y), \tag{6.4}$$

where

$$\alpha_{\underline{i}_n,\underline{j}_n}(t) := \alpha_t(\varphi_{\underline{i}_n,\underline{j}_n}((0,1))). \tag{6.5}$$

To avoid ambiguity, if (x,y) belongs to more than one level n squares then we count the one which has lexicographically the highest order.

Lemma 20. *Given $n > r$. Assume that $\rho(t_1,t_2) < \delta M^{-n}/32$. For an arbitrary $\underline{i}_n, \underline{j}_n$ let $I_1 \subsetneq I_2$ be any two distinct intervals from*

$$\left\{\Delta^{\text{tiny}}_{\underline{i}_n,\underline{j}_n}, \Delta^{\text{small}}_{\underline{i}_n,\underline{j}_n}, \Delta^{\text{med}}_{\underline{i}_n,\underline{j}_n}, \Delta^{\text{big}}_{\underline{i}_n,\underline{j}_n}, \Delta_{\underline{i}_n,\underline{j}_n}\right\}.$$

Further, and let k_1, k_2 (not necessarily distinct) elements of $\{n-r, n\}$. Then we have

$$B_{\delta M^{-n}/4}\left(S_{t_1}^{(k_1)}(I_1)\right) \subset S_{t_2}^{(k_2)}(I_2). \tag{6.6}$$

The proof is an immediate consequence of the following assertion.

Fact 21. *For every $0 \le k$, $0 < m$, $t_1, t_2 \in T$ and $(x,y) \in K$ we have*

$$\left|S_{t_1}^{(m+k)}(x,y) - S_{t_2}^{(m)}(x,y)\right| \le \sqrt{2}\rho(t_1,t_2) + \sqrt{2}\gamma M^{-m}. \tag{6.7}$$

Proof. It is enough to prove that

$$\forall t_1, t_2 \in T \text{ and } (x,y) \in K, \ \left|S_{t_1}^{(n)}(x,y) - S_{t_2}^{(n)}(x,y)\right| \le \sqrt{2}\rho(t_1,t_2). \tag{6.8}$$

and

$$\forall m, \forall k \ge 0, (x,y) \in K, \ \left|S_t^{(m+k)}(x,y) - S_t^{(m)}(x,y)\right| \le \sqrt{2}\gamma M^{-m}. \tag{6.9}$$

hold. By definition, for any $n, t_1, t_2 \in T, \underline{i}_n, \underline{j}_n$ we have

$$|\alpha_{\underline{i}_n,\underline{j}_n}(t_1) - \alpha_{\underline{i}_n,\underline{j}_n}(t_2)| \le \rho(t_1,t_2). \tag{6.10}$$

Using first (3.5) then this we obtain that $t_1, t_2 \in T$, $(x,y), (x',y') \in K$,

$$\left|\Pi_{\alpha_{t_1}(x',y')}(x,y) - \Pi_{\alpha_{t_2}(x',y')}(x,y)\right| \le \sqrt{2}\rho(t_1,t_2). \tag{6.11}$$

Applying this in the case when (x',y') is equal to the left upper corner of $K_{\underline{i}_n,\underline{j}_n} \ni (x,y)$ yields (6.8). The proof of (6.9) is similar. □

Proof of Lemma 20. We use Fact 21 with $m = n - r$ and $k = r$. It follows from (3.25) that $\forall(x,y),\ k_1,k_2 \in \{n, n-r\}$,

$$\rho(t_1,t_2) < \delta M^{-n}/32 \Rightarrow \left| S_{t_1}^{(k_1)}(x,y) - S_{t_2}^{(k_2)}(x,y) \right| < \delta M^{-n}/8. \tag{6.12}$$

We know by (6.2) that $B_{\delta M^{-n}/3}(I_1) \subset I_2$. By definition, for all $k \le n$ the linear projection $S_t^{(k)}|_{\Delta_{\underline{i}_n,\underline{j}_n}}$ preserves distance. Hence,

$$B_{\delta M^{-n}/3}\left(S_{t_1}^{(k_1)}(I_1)\right) \subset S_{t_1}^{(k_1)}(I_2). \tag{6.13}$$

Now we apply (6.12) for the endpoints of the interval I_2. This immediately yields that (6.6) holds. □

In the rest of this argument the following sets will play a central role:

Definition 22. *For an $n > r$, $x \in \Delta$, $t \in T$ and $\mathfrak{s} \in \{\text{tiny,small, med, big}\}$ we define the random sets*

$$K_{\mathfrak{s}}^n[x,t](\omega) := \left\{ (\underline{i}_n, \underline{j}_n) \in \mathcal{E}_n(\omega) : x \in S_t^{(n)}\left(\Delta_{\underline{i}_n,\underline{j}_n}^{\mathfrak{s}}\right)\right\}$$

and

$$\widetilde{K}_{\mathfrak{s}}^n[x,t](\omega) := \left\{ (\underline{i}_n, \underline{j}_n) \in \mathcal{E}_n(\omega) : x \in S_t^{(n-r)}\left(\Delta_{\underline{i}_n,\underline{j}_n}^{\mathfrak{s}}\right)\right\}.$$

The cardinality of these random sets are the random variables:

$$H_{\mathfrak{s}}^n[x,t](\omega) := \# K_{\mathfrak{s}}^n[x,t](\omega) \text{ and } \widetilde{H}_{\mathfrak{s}}^n[x,t](\omega) := \# \widetilde{K}_{\mathfrak{s}}^n[x,t](\omega).$$

Finally, for an $(\underline{i}_n, \underline{j}_n) \in K_{\mathfrak{s}}^n[x,t](\omega)$ we define $z = z_{\mathfrak{s}}^n[x,t,\underline{i}_n,\underline{j}_n](\omega)$ such that $z \in \Delta_{\underline{i}_n,\underline{j}_n}^{\mathfrak{s}}$ and $\Pi_{\alpha_{\underline{i}_n,\underline{j}_n}(t)}(z) = x$.

Fact 23. *For every $\beta \in V_{\alpha_{\mathcal{S}}}$ we have*

$$\forall x \in \Delta^{\text{big}}, \quad \mathbb{E}\left[\# \left\{ (\underline{i}_r, \underline{j}_r) \in \mathcal{E}_r : x \in \Pi_\beta(\Delta_{\underline{i}_r,\underline{j}_r}^{\text{tiny}})\right\}\right] > 2. \tag{6.14}$$

Proof. It follows from (3.7) and (3.8) (see also (24)) that the left hand side of the inequality in (6.14) is equal to $F_\beta^r \mathbb{1}_{\Delta^{\text{tiny}}}(x)$. Then part 1 (b) of Definition 7 yields the assertion of the Fact. □

Definition 24. *Our aim here is to define the analogue of the set $\mathcal{E}_r[x,\alpha](\omega)$ (see (3.6)) within $K_{\underline{i}_n,\underline{j}_n}$ for an arbitrary $\underline{i}_n, \underline{j}_n$.*

- **(a):** *Let $\Pi_\beta^{\underline{i}_n,\underline{j}_n}$ be the angle β projection in $K_{\underline{i}_n,\underline{j}_n}$ to the diagonal $\Delta_{\underline{i}_n,\underline{j}_n}$. That is $\Pi_\beta^{\underline{i}_n,\underline{j}_n} : K_{\underline{i}_n,\underline{j}_n} \to \Delta_{\underline{i}_n,\underline{j}_n}$ is defined by*

$$\Pi_\beta^{\underline{i}_n,\underline{j}_n} := \varphi_{\underline{i}_n,\underline{j}_n} \circ \Pi_\beta \circ \varphi_{\underline{i}_n,\underline{j}_n}^{-1}.$$

- **(b):** *We write $\underline{i}_{n+k}|_n = \underline{i}_n$ if the restriction of $\underline{i}_{n+k}$ to its first n elements is equal to $\underline{i}_n$.*

(c): $\mathcal{E}_r^{\underline{i}_n,\underline{j}_n} := \left\{ (\widetilde{\underline{i}}_{n+r}, \widetilde{\underline{j}}_{n+r}) \in \mathcal{E}_{n+r}(\omega) : \widetilde{\underline{i}}_{n+k}|_n = \underline{i}_n, \widetilde{\underline{j}}_{n+k}|_n = \underline{j}_n \right\}.$

(d): *For an arbitrary* $(\underline{i}_n, \underline{j}_n)$ *and* $x \in \Delta_{\underline{i}_n,\underline{j}_n}$, *angle* β *and* $\mathfrak{s} \in \{\text{tiny}, \text{small}, \text{med}, \text{big}, \{\ \}\}$ *we define*

(6.15)
$$\mathcal{E}_{r,\mathfrak{s}}^{\underline{i}_n,\underline{j}_n}[x,\beta](\omega) := \left\{ (\widetilde{\underline{i}}_{n+r}, \widetilde{\underline{j}}_{n+r}) \in \mathcal{E}_r^{\underline{i}_n,\underline{j}_n}(\omega) : x \in \Pi_\beta^{\underline{i}_n,\underline{j}_n}(\Delta^{\mathfrak{s}}_{\underline{i}_n,\underline{j}_n}) \right\},$$

where in case of $\mathfrak{s} = \{\ \}$ *we mean* $\Delta^{\mathfrak{s}}_{\underline{i}_n,\underline{j}_n} = \Delta_{\underline{i}_n,\underline{j}_n}$. *When* $n = 0$ *we write* $\Delta^{\mathfrak{s}}_{\underline{i}_{n+r},\underline{j}_{n+r}} = \Delta^{\mathfrak{s}}$ *and*

$$\mathcal{E}_{r,\mathfrak{s}}[x,\beta](\omega) = \left\{ (\underline{i}_r, \underline{j}_r) \in \mathcal{E}_r(\omega) : x \in \Pi_\beta(\Delta^{\mathfrak{s}}) \right\}.$$

Fact 25. *Let* $\mathfrak{s}_1, \mathfrak{s}_2 \in \{\text{tiny}, \text{small}, \text{med}, \text{big}, \{\ \}\}$ *and* $\mathfrak{s}_1 < \mathfrak{s}_2$. *Then for any* $\underline{i}_n, \underline{j}_n$, $x \in \Delta_{\underline{i}_n,\underline{j}_n}$ *we have*

$$\mathcal{E}_{r,\mathfrak{s}_1}^{\underline{i}_n,\underline{j}_n}[x,\beta_1](\omega) \subset \mathcal{E}_{r,\mathfrak{s}_2}^{\underline{i}_n,\underline{j}_n}[x,\beta_2](\omega), \tag{6.16}$$

where $\beta_k := \alpha_{t_k}(x_k, y_k)$ *for arbitrary* $t_1, t_2 \in T$ *and* $(x_1,y_1),(x_2,y_2) \in K$. *In particular,*

$$\mathcal{E}_{r,\text{tiny}}^{\underline{i}_n,\underline{j}_n}[x,\alpha_{\mathcal{S}}](\omega) \subset \mathcal{E}_{r,\text{small}}^{\underline{i}_n,\underline{j}_n}[x,\alpha_{\underline{i}_n,\underline{j}_n}(t)](\omega). \tag{6.17}$$

Proof. The assertion of the Fact immediately follows from Lemma 19. □

Further, note that

Fact 26. *It follows from the definition and (6.17) that for* $(\underline{i}_n, \underline{j}_n) \in K^n_{\mathfrak{big}}[x,t](\omega)$

(6.18)
$$\#\mathcal{E}_{r,\text{small}}^{\underline{i}_n,\underline{j}_n}[z^n_{\mathfrak{big}}[x,t,\underline{i}_n,\underline{j}_n](\omega), \alpha_{\underline{i}_n,\underline{j}_n}(t)](\omega) \geq \#\mathcal{E}_{r,\text{tiny}}^{\underline{i}_n,\underline{j}_n}[z^n_{\mathfrak{big}}[x,t,\underline{i}_n,\underline{j}_n](\omega), \alpha_{\mathcal{S}}](\omega)$$

Hence,

$$\begin{aligned} \widetilde{H}^{n+r}_{\text{small}}[x,t](\omega) &\geq \sum_{(\underline{i}_n,\underline{j}_n) \in K^n_{\mathfrak{big}}[x,t](\omega)} \#\mathcal{E}_{r,\text{small}}^{\underline{i}_n,\underline{j}_n}[z^n_{\mathfrak{big}}[x,t,\underline{i}_n,\underline{j}_n](\omega), \alpha_{\underline{i}_n,\underline{j}_n}(t)](\omega) \\ &\geq \sum_{(\underline{i}_n,\underline{j}_n) \in K^n_{\mathfrak{big}}[x,t](\omega)} \#\mathcal{E}_{r,\text{tiny}}^{\underline{i}_n,\underline{j}_n}[z^n_{\mathfrak{big}}[x,t,\underline{i}_n,\underline{j}_n](\omega), \alpha_{\mathcal{S}}](\omega) \end{aligned}$$

See Figure 4.

For a fixed t, the angle $\alpha_{\mathcal{S}}$ projection of the level r squares partition Δ into finitely many intervals. Observe that the random variable $\#\mathcal{E}_{r,\text{tiny}}^{\underline{i}_n,\underline{j}_n}[z^n_{\mathfrak{big}}[x,t,\underline{i}_n,\underline{j}_n], \alpha_{\mathcal{S}}]$ is the same if we substituted x with an x' from the same interval of this partition. These random variables will

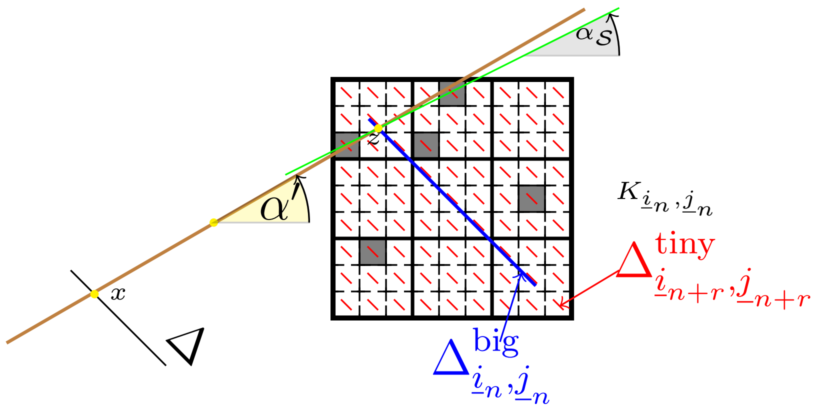


FIGURE 4. $\alpha' = \alpha_{\underline{i}_n,\underline{j}_n}(t)$, $z = z^n_{\mathtt{big}}[x,t,\underline{i}_n,\underline{j}_n]$. If $r=2$, and $\mathcal{E}_r^{\underline{i}_n,\underline{j}_n}(\omega)$ correspond the 5 gray squares of the figure then $\#\mathcal{E}^{\underline{i}_n,\underline{j}_n}_{r,\text{tiny}}[z^n_{\mathtt{big}}[x,t,\underline{i}_n,\underline{j}_n],\alpha_{\mathcal{S}}] = 2$.

be called as Z_i in the corollary of the Cramer theorem below. First we observe that for the self similar nature of the construction:

Fact 27. *Conditional on $(\underline{i}_n,\underline{j}_n) \in \mathcal{E}_n$, for every $z \in \Delta^{\text{big}}_{\underline{i}_n,\underline{j}_n}$ we have $\mathcal{E}^{\underline{i}_n,\underline{j}_n}_{r,\text{tiny}}[z,\alpha_{\mathcal{S}}] \stackrel{d}{=} \mathcal{E}_{r,\text{tiny}}[\varphi^{-1}_{\underline{i}_n,\underline{j}_n} z, \alpha_{\mathcal{S}}]$.*

Using this we can easily verify that

Lemma 28. *For every $z \in \Delta^{\text{big}}_{\underline{i}_n,\underline{j}_n}$ we have*

$$\mathbb{E}\left[\#\mathcal{E}^{\underline{i}_n,\underline{j}_n}_{r,\text{tiny}}[z,\alpha_{\mathcal{S}}] | (\underline{i}_n,\underline{j}_n) \in \mathcal{E}_n\right] > 2. \tag{6.19}$$

Proof. The inequality is immediate from Facts 27 and 23. □

6.2. **The corollary of Crammer's theorem and its applications.** The corollary of Crammer's Large Deviation Theorem stated bellow plays a key role in the proof Theorem 3.

Lemma 29. *We are given the real valued bounded random variables $Z_1,\dots,Z_R$ such that*

$$\forall j,\ \mathbb{E}[Z_j] \geq \gamma > 0. \tag{6.20}$$

We write $A := \max_j(\sup Z_j - \inf Z_j)$. Let us fix a number $0<\eta<\gamma$. Using Crammer theorem for every $1 \leq j \leq R$ we can find $0 < \tau_j < 1$ such that for every n and for every independent $Z_j^{(1)},\dots,Z_j^{(n)} \stackrel{d}{=} Z_j$ we have

$$\mathbb{P}\left(Z_j^{(1)} + \cdots + Z_j^{(n)} < n\eta\right) < \tau_j^n. \tag{6.21}$$

Let $\tau := \frac{1}{2}\left(1 + \max_j \tau_j^{\frac{\gamma-\eta}{RA}}\right)$. *We choose* n_1 *such that*

$$\forall k \geq n_1,\ \tau^k > R \cdot \left(\sup_{1\leq j\leq R} \tau_j^{\frac{\gamma-\eta}{RA}}\right)^k. \tag{6.22}$$

We are also given $C \geq n_1$ *independent random variables* $Y_1, \ldots, Y_C$ *such that for each* $i \in \{1, \ldots, C\}$ *there exists a* $j(i) \in \{1, \ldots, R\}$ *with*

$$Y_i \stackrel{d}{=} Z_{j(i)}.$$

Then

$$\mathbb{P}\left\{\sum_{i=1}^{C} Y_i < C \cdot \eta\right\} \leq \tau^C. \tag{6.23}$$

Observe that (6.23) hods independently of

$$n_j := \#\left\{i : Y_i \stackrel{d}{=} Z_j\right\}.$$

Proof. The event $\left\{\sum_{i=1}^{C} Y_i < C \cdot \eta\right\}$ implies that for some j_0 the event

$$\left\{\sum_{j(i)=j_0} Y_i < n_{j_0} \cdot \gamma - \frac{1}{R} \cdot C \cdot (\gamma - \eta)\right\}$$

holds. In return this implies that $\frac{1}{R} \cdot C \cdot (\gamma - \eta) \leq n_{j_0} \cdot A$. Hence

$$n_{j_0} \geq \frac{\gamma - \eta}{RA} C. \tag{6.24}$$

$$\begin{aligned}
&\mathbb{P}\left\{\sum_{i=1}^{C} Y_i < C \cdot \eta\right\} \qquad (6.25)\\
&\leq \sum_{j_0 : n_{j_0} \geq \frac{\gamma-\eta}{RA} C} \mathbb{P}\left\{\sum_{j(i)=j_0} Y_i < n_{j_0}\mathbb{E}\left[Z_{j_0}\right] - n_{j_0} \cdot \left(\mathbb{E}\left[Z_{j_0}\right] - \gamma + \frac{C}{n_{j_0}}\frac{\gamma-\eta}{R}\right)\right\}\\
&\leq \sum_{j_0 : n_{j_0} \geq \frac{\gamma-\eta}{RA} C} \mathbb{P}\left\{\sum_{j(i)=j_0} Y_i < n_{j_0}\left(\mathbb{E}\left[Z_{j_0}\right] - \frac{\gamma-\eta}{R}\right)\right\}\\
&\leq \sum_{j_0 : n_{j_0} \geq \frac{\gamma-\eta}{RA} C} \tau_{j_0}^{n_{j_0}} \leq R \cdot \left(\sup \tau_{j_0}^{\frac{\gamma-\eta}{RA}}\right)^C < \tau^C.
\end{aligned}$$

Where in the one but last line we used (6.20) and that $C/n_{j_0} \geq 1$. Further, in the last line we used Crammer's Large Deviation Theorem and (6.22). □

Corollary 30. *Using the notation of Lemma 29, if the number of random variables C is a random variable itself that is independent of $Y_1, Y_2, \dots$ and takes only values from the set of integers that are greater than n_1 then a standard argument yields that under the hypothesis of Lemma 29 we have*

$$\mathbb{P}\left(\sum_{i=1}^{C} Y_i < C \cdot \eta\right) \leq \mathbb{E}\left[\tau^C\right]. \tag{6.26}$$

We apply this lemma in the following way:

Definition 31. *The angle $\alpha_{\mathcal{S}}$ projections $\Pi_{\alpha_{\mathcal{S}}}$ of the level r squares $K_{\underline{i}_r,\underline{j}_r}$ determines a partition of Δ^{bigg} into the left closed right open intervals $I_1, \dots, I_R$. This defines R. Note that for every ω if $x_1, x_2 \in I_k$ for some $1 \leq k \leq R$ then $\mathcal{E}_{r,\text{tiny}}[x_1, \alpha_{\mathcal{S}}] = \mathcal{E}_{r,\text{tiny}}[x_2, \alpha_{\mathcal{S}}]$. Hence, we can define the random variable*

$$Z_i(\omega) := \#\mathcal{E}_{r,\text{tiny}}[x, \alpha_{\mathcal{S}}](\omega) \qquad \text{for some } x \in I_i. \tag{6.27}$$

By Fact 23 if we choose $\gamma = 2$ then (6.20) holds. Clearly $A < M^{2r}$. Let $\eta := 3/2$. Then we define $0 < \tau_j < 1$ for every $1 \leq j \leq R$ so that (6.21) holds. Similarly, we define n_1 which satisfies (6.22). We fix an $x \in \Delta^{\text{small}}, t \in T$ and n. Put

$$C = C[x,t](\omega) := H^n_{\text{big}}[x,t](\omega). \tag{6.28}$$

Now our aim is to define the random variables

$$Y_i(\omega) := Y_i^{(n)}[x,t](\omega), \qquad i = 1, \dots, C.$$

To do so, we arrange the elements of $K^n_{\text{big}}[x,t](\omega)$ into lexicographic order. Assume that $(\widetilde{\underline{i}}_n, \widetilde{\underline{j}}_n)$ is the k-th element. Then we write $y_k = y_k[x,t](\omega)$ for the intersection of interval $\Delta_{\widetilde{\underline{i}}_n,\widetilde{\underline{j}}_n}$ with the line through x of angle $\alpha_{\widetilde{\underline{i}}_n,\widetilde{\underline{j}}_n}(t)$. Then we define

$$Y_k(\omega) := \#\mathcal{E}^{\widetilde{\underline{i}}_n,\widetilde{\underline{j}}_n}_{r,\text{tiny}}[y_k, \alpha_{\mathcal{S}}](\omega). \tag{6.29}$$

That is Y_k is the random variable on the right hand side of the inequality in (6.18). Clearly

$$Y_k \stackrel{d}{=} Z_\ell \text{ if } \varphi^{-1}_{\underline{i}_n,\underline{j}_n}(y_k) \in I_\ell. \tag{6.30}$$

Note that the random variables $C, Y_1, \dots, Y_C$ are independent because the value of Y_k (using the notation of Definition 31) depends only the way the process runs within $\widetilde{\underline{i}}_n, \widetilde{\underline{j}}_n$.

Definition 32. *For the rest of the section we fix a $\xi > 0$ and a number*

$$1 < v < \frac{\sum_{i,j=1}^{M} p_{i,j}}{M}.$$

We recall that the threshold n_1 was defined in Definition 31.

(a): *It follows from the definition of $\widehat{\Omega}$ and from the Egorov Theorem that we can choose $n_2 \geq n_1$ such that for*

$$\widetilde{\Omega} := \left\{ \omega \in \widehat{\Omega} : \forall n \geq n_2,\ \#\mathcal{E}^n(\omega) > v^n \cdot M^n \right\},\ \widetilde{\mathbb{P}}(\cdot) := \mathbb{P}(\cdot | \widetilde{\Omega}) \tag{6.31}$$

we have

$$\widehat{\mathbb{P}} \left\{ \widehat{\Omega} \setminus \widetilde{\Omega} \right\} < \frac{\xi}{100}. \tag{6.32}$$

(b): *The following sequences have an important role in the rest of the paper (see the right hand side of (6.37):*

$$c_1 := \frac{|\Delta^{\text{tiny}}|}{2|\Delta|}, \quad N(n) := c_1 \cdot v^n, \quad a_k(n) := \vartheta^{n+kr} \tau^{(3/2)^{k-1} N(n)}. \tag{6.33}$$

where ϑ was defined in part (b3) of Definition 11.

(c): *The definition of $n_3 \geq n_2$ below is motivated by the goal to provide that*

$$\forall n \geq n_3, \qquad \vartheta^n \cdot \sum_{k=1}^{\infty} a_k(n) < \xi/100. \tag{6.34}$$

A straightforward but tedious calculation yields that (6.34) holds if we choose n_3 such that all $n \geq n_3$ satisfies:

$$v^n \geq \frac{1}{c_1} \left[\frac{\log \frac{300}{\xi}}{\log \frac{1}{\tau}} + (n + 4r) \frac{\log \vartheta}{\log \frac{1}{\tau}} \right],\ \vartheta^{2n+r} \left(4 + 2\vartheta^{3r} \right) \tau^{v^n \cdot c_1} < \frac{\xi}{100}. \tag{6.35}$$

6.3. **The proof of Theorem 3 assuming Proposition 33.** First we state a Proposition which contains the difficult part of the proof of Theorem 3. Then we prove Theorem 3 assuming this Proposition.

Proposition 33. *Let $\widetilde{t} \in T$ be arbitrary. Then with positive probability, the projection $S_t(E)$ contains an interval for all $t \in T$ satisfying*

$\rho(t,\widetilde{t}) < \delta M^{-n}/64$. More precisely, let $T' \subset T$

(6.36)
$$\mathfrak{I}(T') := \left\{\omega \in \widetilde{\Omega} : S_t(E(\omega)) \text{ contains some intervals } \forall t \in T'\right\}.$$

Then

$$\forall \widetilde{t} \in T_n,\ \forall n \geq n_3,\ \widetilde{\mathbb{P}}\left(\mathfrak{I}\left(B(\widetilde{t}, \delta M^{-n}/64)\right)\right) > 1 - 2\sum_{k=1}^{\infty} a_k(n), \tag{6.37}$$

where $B(\widetilde{t}, \delta M^{-n}/64)$ is the $\delta M^{-n}/64$ neighborhood of $\widetilde{t}$ in T with respect to the metric ρ. We remind the reader that the set T_n was defined in part (b3) of Definition 11.

Assuming Proposition 33 now we prove Theorem 3.

The proof of Theorem 3. First we verify that

$$\widetilde{\mathbb{P}}\left(\mathfrak{I}(T)\right) > 1 - \frac{\xi}{50}. \tag{6.38}$$

To see this we fix an $n \geq n_3$. By part (b3) of Definition 11 we can find a set $T_n \subset T$ which is $\delta M^{-n}/64$-dense in T and $\#T_n \leq \vartheta^n$.

$$\widetilde{\mathbb{P}}\left(\mathfrak{I}^c(T)\right) \leq \sum_{\widetilde{t} \in T_n} \mathbb{P}\left(\mathfrak{I}^c(B(\widetilde{t}, \delta M^{-n}/64))\right) < \vartheta^n \cdot 2 \cdot \sum_{k=1}^{\infty} a_k(n) < 2 \cdot \frac{\xi}{100},$$

where in the second inequality we used Proposition 33 and in the last step we used (6.34). So, (6.38) holds. It is easy to check that on any probability space for any two events E, F the following identity holds:

$$|\mathbb{P}(E) - \mathbb{P}(E|F)| \leq \mathbb{P}(F^c). \tag{6.39}$$

By (6.32) we have

$$\left|\widehat{\mathbb{P}}(\mathfrak{I}(T)) - \widetilde{\mathbb{P}}(\mathfrak{I}(T))\right| = \left|\widehat{\mathbb{P}}(\mathfrak{I}(T)) - \widehat{\mathbb{P}}\left(\mathfrak{I}(T)|\widetilde{\Omega}\right)\right| \leq \widehat{\mathbb{P}}(\widetilde{\Omega}^c) < \frac{\xi}{100}.$$

Hence by (6.38)

$$\widehat{\mathbb{P}}(\mathfrak{I}(T)) > 1 - \frac{3 \cdot \xi}{100}.$$

Since ξ was arbitrary, this completes the proof of Theorem 3 assuming that Proposition 33 holds. □

In the rest of this Section we prove that Proposition 33 holds.

6.4. **The proof of Proposition 33.** We start with a simple Lemma which shows the role of $N(n)$ (defined in (6.33)).

Lemma 34. *For all $\omega \in \widetilde{\Omega}$, for every $\widetilde{t} \in T$ there exists an interval $J = J(\widetilde{t}, \omega) \subset \Delta$ of length $|J| = M^{-n}|\Delta^{\text{tiny}}|$ such that the $\delta M^{-n}/64$ neighborhood of $\widetilde{t}$, (we already denoted it by $B(\widetilde{t}, \delta M^{-n}/64)$) satisfies:*

$$\forall t \in B(\widetilde{t}, \delta M^{-n}/64), \ \forall x \in J, \quad H^n_{\text{small}}[x,t](\omega) > N(n). \tag{6.40}$$

Proof. Fix an arbitrary $\omega \in \widetilde{\Omega}$. By (6.31) the number of elements of $\mathcal{E}_n(\omega)$ grows at least as fast as $(vM)^n$. By definition, $H^n_{\text{tiny}}[x,t](\omega)$ is the sum of $|\mathcal{E}_n(\omega)|$ of characteristic functions of intervals of length $\Delta^{\text{tiny}} \cdot M^{-n}$ each. Hence

$$\frac{1}{|\Delta|} \int\limits_{\Delta} H^n_{\text{tiny}}[x,t](\omega) dx = \frac{1}{|\Delta|} v^n M^n |\Delta^{\text{tiny}}| M^{-n} = 2N(n). \tag{6.41}$$

In this way there is an $x = x(t,\omega) \in \Delta$ such that

$$H^n_{\text{tiny}}[x,t](\omega) \geq 2N(n).$$

This implies that there exists an interval $J = J(\omega, t)$ of length $\Delta^{\text{tiny}} \cdot M^{-n}$ such that

$$\forall x \in J, \quad H^n_{\text{tiny}}[x,t](\omega) > N(n). \tag{6.42}$$

Let $(\underline{i}_n, \underline{j}_n)$ be an arbitrary element of $K^n_{\text{tiny}}[x,t](\omega)$. Then by Definition 22 and Lemma 20, for all $t' \in T$ satisfying $\rho(t,t') < \delta M^{-n}/32$ we have

$$x \in S_t^{(n)}(\Delta^{\text{tiny}}_{\underline{i}_n, \underline{j}_n}) \subset S_{t'}^{(n)}(\Delta^{\text{small}}_{\underline{i}_n, \underline{j}_n}).$$

Hence we obtain that $(\underline{i}_n, \underline{j}_n) \in K^n_{\text{small}}[x,t'](\omega)$ which together with (6.42) completes the proof of the Lemma. □

Let

$$L := \left\lceil \frac{3|\Delta|}{M^{-n}|\Delta^{\text{tiny}}|} \right\rceil = \left\lceil \frac{\Delta}{|J|/3} \right\rceil.$$

We partition Δ into $J_1, \dots, J_L$ with $|J_k| = |\Delta|/L$. Clearly $|J_k| \leq |J|/3$. For a $t \in T$ and $\omega \in \widetilde{\Omega}$ we define $k(t,\omega) \in \{1, \dots, L\}$ such that

$$J_{k(t,\omega)} \subset J(t,\omega), \qquad J_{k(t,\omega)-1} \not\subset J(t,\omega), \tag{6.43}$$

where $J(t,\omega)$ is the interval defined in Lemma 34. Put

$$\widetilde{\Omega}_{t,\ell} := \left\{ \omega \in \widetilde{\Omega} : \ell = k(t,\omega) \right\}. \tag{6.44}$$

Then by Lemma 34, for every $\widetilde{t} \in T_n$, $\widetilde{\Omega} = \bigcup_{k=1}^{L} \widetilde{\Omega}_{\widetilde{t},k}$. Let

$$\mathcal{I}_{\widetilde{t}} := \left\{ \ell : \mathbb{P}\left(\widetilde{\Omega}_{\widetilde{t},\ell}\right) > 0 \right\}. \tag{6.45}$$

Put

$$\widetilde{\mathbb{P}}_{\widetilde{t},\ell}(\cdot) := \widetilde{\mathbb{P}}(\cdot|\widetilde{\Omega}_{\widetilde{t},\ell}) \text{ for } \ell \in \mathcal{I}_{\widetilde{t}}. \tag{6.46}$$

The following lemma is the core of the proof of Proposition 33.

Lemma 35. *Fix an arbitrary $n \geq n_3$, $\widetilde{t} \in T_n$ and $\ell \in \mathcal{I}_{\widetilde{t}}$. For simplicity here we write*

$$B := B(\widetilde{t}, \delta M^{-n})/64, J = J_\ell \text{ and } \mathfrak{P} := \widetilde{\mathbb{P}}_{\widetilde{t},\ell}.$$

Then

$$\forall t \in B, \forall x \in J \text{ and } \forall \omega \in \Omega_{\widetilde{t},\ell},\ H^n_{\text{big}}[x,t](\omega) > N(n). \tag{6.47}$$

Further,

$$\mathfrak{P}\left(H^{n+kr}_{\text{big}}[x,t] > \left(\frac{3}{2}\right)^k \cdot N(n), \forall k,\ \forall (x,t) \in B \times J\right) > \prod_{k=0}^{\infty}(1-a_k(n)). \tag{6.48}$$

Proof. Clearly, (6.47) is merely a combination of a weakened version of Lemma 34 and the previous definitions. It follows from (3.27) that for every $k \geq 0$ we can find sets X_k, B_k satisfying

- **(a):** $X_k \subset J$ and X_k is a $\delta M^{-(n+(k+1)r)}/64$-dense set in J,
- **(b):** $B_k \subset B$ and B_k is a $\delta M^{-(n+(k+1)r)}/64$-dense set in B,
- **(c):** $\#X_k \cdot \#B_k \leq \vartheta^{n+kr}$.

The proof is carried out by induction for k. The two important ingredients of this proof are: Fact 23 and (6.47) (in the case of $k = 0$ and a corresponding statement for higher k). To settle the case when $k = 0$ first we verify that:

Fact 36. *Let $(x,t) \in X_k \times B_k$ arbitrary. Then*

$$\mathfrak{P}\left(\widetilde{H}^{n+r}_{\text{small}}[x,t] \leq \frac{3}{2}N(n)\right) < \tau^{N(n)}. \tag{6.49}$$

Proof. In the proof of the Fact we use the notation of Definition 31. Putting together (6.28), (6.47) and (6.26) (in this order) we obtain that

$$\begin{aligned}\mathfrak{P}\left(\sum_{i=1}^{C} Y_i \leq \frac{3}{2}N(n)\right) &\leq \mathfrak{P}\left(\sum_{i=1}^{C} Y_i \leq C \cdot \frac{3}{2}\right) \leq \mathfrak{E}\left(\tau^C\right) \\ &< \tau^{N(n)},\end{aligned} \tag{6.50}$$

where we wrote $\mathfrak{E}$ for the expectation corresponding to the probability $\mathfrak{P}$. On the other hand Fact 26 yields that

$$\sum_{k=1}^{C} Y_k \leq \sum_{k=1}^{C} \#\mathcal{E}_{r,\text{small}}^{\underline{i}_n,\underline{j}_n}\left[y_k, \alpha_{\underline{i}_n,\underline{j}_n}(t)\right](\omega) \leq \widetilde{H}_{\text{small}}^{n+k}[x,t](\omega). \tag{6.51}$$

Putting together the inequalities (6.50) and (6.51) completes the proof of the Fact. □

Applying Fact 36 for all elements of $X_0 \times B_0$ leads to

$$\mathfrak{P}\left(\widetilde{H}_{\text{small}}^{n+r}[x,t] > \frac{3}{2}N(n),\ \forall (x,t) \in X_0 \times B_0\right) \geq 1 - \vartheta^n \tau^{N(n)}. \tag{6.52}$$

Note that for every $t \in B$ there exists a $t' \in B_0$ with $\rho(t,t') < \delta M^{-(n+r)}/64$. Hence by Lemma 20 we obtain that for every $\underline{i}_{n+r}, \underline{j}_{n+r}$,

$$S_t^{(n+r)}\left(\Delta_{\underline{i}_{n+r},\underline{j}_{n+r}}^{\text{med}}\right) \supset S_{t'}^{(n)}\left(\Delta_{\underline{i}_{n+r},\underline{j}_{n+r}}^{\text{small}}\right)$$

This yields that

$$H_{\text{med}}^{n+r}[x,t] \geq \widetilde{H}_{\text{small}}^{n+r}[x,t']. \tag{6.53}$$

So, by (6.52) we have

$$\mathfrak{P}\left(H_{\text{med}}^{n+r}[x,t] > \frac{3}{2}N(n),\ \forall (x,t) \in X_0 \times B\right) \geq 1 - \vartheta^n \tau^{N(n)}. \tag{6.54}$$

Finally, let $x \in J$ arbitrary. Then we can find $x' \in X_0$ with $|x - x'| < \delta M^{-(n+r)}/64$. Using (6.2), for every $\underline{i}_n, \underline{j}_n$ we obtain that the $\delta M^{-(n+r)}/3$-neighborhood $\Delta_{\underline{i}_n,\underline{j}_n}^{\text{med}}$ is contained in $\Delta_{\underline{i}_n,\underline{j}_n}^{\text{big}}$. Thus for all ω:

$$K_{\text{med}}^{n+r}[x,t](\omega) \subset K_{\text{big}}^{n+r}[x',t](\omega).$$

That is $H_{\text{med}}^{n+r}[x,t](\omega) \leq H_{\text{big}}^{n+r}[x',t](\omega)$ This and (6.54) follows that

$$\mathfrak{P}\left(H_{\text{big}}^{n+r}[x,t] > \frac{3}{2}N(n), \forall (x,t) \in J \times B\right) \geq 1 - \vartheta^n \tau^{N(n)}. \tag{6.55}$$

Which completes the proof in the case when $k = 0$. For $k = 1$ we argue similarly. Instead of $\mathfrak{P}$ we use its conditional measure to the subset of $\widetilde{\Omega}_{\tilde{t},k}$ which appears in (6.55). That is we substitute $\mathfrak{P}$ with with the conditional probability of $\mathfrak{P}$ to $\left\{\omega \in \widetilde{\Omega}_{\tilde{t},k} : H_{\text{big}}^{n+r}[x,t] > \frac{3}{2}N(n), \forall (x,t) \in J \times B\right\}$.

Following the analogous steps as in case of $k=0$ we obtain that
(6.56)
$$\mathfrak{P}\left(H_{\text{big}}^{n+kr}[x,t] > \left(\frac{3}{2}\right)^k \cdot N(n), k=0,1,\ \forall (x,t)\in B\times J\right) > \prod_{k=0}^{1}(1-a_k(n)).$$
Continuing this for all $k\in\mathbb{N}$ completes the proof of the Lemma. □

Proof of Proposition 33. It follows from (6.48) that for all $n \geq n_3$ and $k \in \mathcal{I}_{\widetilde{t}}$ (defined in (6.45)) we have

$$\widetilde{\mathbb{P}}\left(\mathfrak{I}\left(B(\widetilde{t},\delta M^{-n}/64)\right)|\widetilde{\Omega}_{\widetilde{t},k}\right) > \prod_{k=0}^{\infty}(1-a_k(n)) > 1-2\sum_{k=0}^{\infty} a_k(n), \tag{6.57}$$

where the second inequality follows from the fact that by the first inequality of (6.35) we have $a_k(n) < \frac{1}{2}$ for all k. Hence

$$\begin{aligned}\widetilde{\mathbb{P}}\left(\mathfrak{I}\left(B(\widetilde{t},\delta M^{-n}/64)\right)\right) &= \sum_{k\in\mathcal{I}_{\mathfrak{e}}}\widetilde{\mathbb{P}}\left(\mathfrak{I}\left(B(\widetilde{t},\delta M^{-n}/64)\right)|\widetilde{\Omega}_{\widetilde{t},k}\right)\cdot\widetilde{\mathbb{P}}\left(\widetilde{\Omega}_{\widetilde{t},k}\right)\\ &\geq 1-2\sum_{k=0}^{\infty}a_k(n).\end{aligned}$$

□

7. The proof of Theorem 4

Let us note first that Condition (A) implies that the random operators $G_{\alpha,\omega}$ are statistically L^1-norm expanding, so (2.4) follows. In this Section we consider the case when (2.4) does not hold. Hence, we cannot use regular families of almost linear projections as defined in Definition 11.

Definition 37. *We say that a function $S: K\to\Delta$ is a* ***near linear projection with direction function*** $\alpha : K \to [0,\pi/2]$ *if there exists $\gamma>0$ such that*
(7.1)
$$\forall \underline{i}_n, \underline{j}_n \text{ if } (x,y),(x',y')\in K_{\underline{i}_n,\underline{j}_n}, \text{ then } |\alpha(x,y)-\alpha(x',y')| \leq \gamma M^{-n}.$$
The smallest such γ is called the ***distortion of direction function*** *α.*
A family $\mathcal{S}=\{S_t\}_{t\in T}$ of near linear projections with corresponding direction functions $\{\alpha_t(x,y)\}_{t\in T}$ is ***a regular family of near linear projections*** *S_t if the parameter domain T has finite box dimension in the sup metric:*

$$\rho(t_1,t_2) := \sup_{(x,y)\in K}|\alpha_{t_1}(x,y)-\alpha_{t_2}(x,y)|. \tag{7.2}$$

We are going to prove the following result:

Theorem 38. *Let $\mathcal{S} = \{S_t\}_{t\in T}$ be a regular family of near linear projections. Assume that $\sum_{i,j=0}^{M-1} p_{i,j} \leq M$. Then the following assertion holds almost surely :*

$$\forall t \in T,\ Leb(S_t(E)) = 0. \tag{7.3}$$

Theorem 4 will follow as in Section 4.
Let us begin from some basic properties. For a branching process with expected number of children of each parent equal to N, almost surely there exist a number K such that for all $n > 0$ number of children in generation n is not greater than KN^n. In our setting this implies a stronger version of Fact 5:

Fact 39. *Assuming that $\sum_{i,j=0}^{M-1} p_{i,j} \leq M$, the following assertion holds almost surely: there exists $K > 0$ such that for all $n > 0$*

$$\#\mathcal{E}_n(\omega) \leq KM^n.$$

Let $(\underline{i}_n, \underline{j}_n) \in \mathcal{E}_n$ and $t \in T$. We denote

$$V_n(\underline{i}_n, \underline{j}_n, t) = \bigcup_{\rho(t,\tilde{t})<M^{-n}} S_{\tilde{t}}(K_{\underline{i}_n,\underline{j}_n}).$$

Clearly, for $K_{\underline{i}_{n+1},\underline{j}_{n+1}} \subset K_{\underline{i}_n,\underline{j}_n}$ and $\rho(t,\tilde{t}) \leq (M-1)M^{-n-1}$ we have

$$V_{n+1}(\underline{i}_{n+1}, \underline{j}_{n+1}, \tilde{t}) \subset V_n(\underline{i}_n, \underline{j}_n, t). \tag{7.4}$$

By definition of a regular family of near linear projections, $V_n(\underline{i}_n, \underline{j}_n, t)$ is contained in some interval of length at most $(\sqrt{2}+1+\gamma)M^{-n}$, where γ is the supremum of distortions of linear projections $S_t; t \in T$. Let us choose for all $n > 0$ a finite family B_n M^{-n}-dense in T in such a way that every element of B_{n+1} is in distance at most $(M-1)M^{-n-1}$ from some element of B_n. It can be done in such a way that the number of elements of B_n increases only exponentially fast in n.
Let $a_0 = 1$ and

$$a_{k+1} = a_k - \frac{a_{k+1}}{10+2\gamma} p^{K(5+\gamma)^2/a_{k+1}}.$$

where

$$p = \prod_{i,j=0}^{M-1} (1 - p_{ij}) > 0.$$

Observe that p is the probability that for an arbitrary $(\underline{i}_n, \underline{j}_n) \in \mathcal{E}_n$ the square $K_{\underline{i}_n,\underline{j}_n}$ does not have any children on level $n+1$. It is easy to see that the sequence $\{a_k\}_{k=0}^{\infty}$ is well defined and converges to zero monotonically.

Let us divide Δ into M^n equal intervals $\Delta_{i,n}$. For a given $t \in B_n$ denote by $V_{i,n}(t)$ the number of sets $V_n(\underline{i}_n, \underline{j}_n, t)$ intersecting $\Delta_{i,n}$. Put

$$G_n(t) := \{\Delta_{i,n} : V_{i,n}(t) > 0\} \text{ and } Z_n(t) := \#G_n(t).$$

It follows from (7.4) that

(7.5)

$$\text{If } \rho(\widetilde{t}, t) < (M-1)M^{-(n+1)} \text{ and } \widetilde{\Delta} \in G_{n+1}(\widetilde{t}) \text{ then } \exists \Delta \in G_n(t),\ \widetilde{\Delta} \subset \Delta.$$

The assertion of Theorem 38 will follow if we prove that

$$\lim_{n\to\infty} \sup_{t\in B_n} M^{-n} Z_n(t) = 0.$$

To get contradiction we assume that this does not hold. By (7.4) the sequence $\left\{\sup_{t\in B_n} M^{-n} Z_n(t)\right\}_{n=1}^{\infty}$ is nonincreasing. Hence, there must exist some k such that

$$\exists n_0,\ \forall n > n_0,\ \forall t \in B_n, \quad M^{-n} Z_n(t) < a_k \tag{7.6}$$

but

$$\forall n,\ \exists t \in B_n, \quad M^{-n} Z_n(t) \geq a_{k+1}. \tag{7.7}$$

For an $n > n_0$ put

$$R_n := \{t \in B_n : Z_n(t) \geq a_{k+1} M^n\}.$$

It was our assumption above that

$$R_n \neq \emptyset \qquad \forall n \geq n_0. \tag{7.8}$$

Now we fix an $n > n_0$ and $t \in R_n$. Further, for every i satisfying $\Delta_{i,n} \in G_n(t)$ we introduce the event

$$A_{i,n}(t) := \left\{\Delta_{i,n} \cap S_{\tilde{t}}(E_{n+1}) = \emptyset \text{ whenever } \rho(\widetilde{t}, t) < M^{-n}\right\}.$$

The $Z_n(t)$ events $\{A_{i,n}(t)\}_{\Delta_{i,n}\in G_n(t)}$ are not independent. However, if we restrict ourselves to an arbitrary subfamily where any two indices i and i' differ at least by $5+\gamma$ then this subfamily of events will be independent. By the definition of p we have

$$\mathbb{P}\left(A_{i,n}(t)\right) \geq p^{V_{i,n}(t)}. \tag{7.9}$$

Let G be an arbitrary subset of $G_n(t)$ and we write Z for the cardinality of G. Then using that each set $V_n(\underline{i}_n, \underline{j}_n, t)$ intersects at most $5+\gamma$

intervals $\Delta_{i,n}$ by Fact 39 we obtain from the Arithmetic-Geometric Mean Inequality that

$$\begin{aligned}(7.10)\qquad \sum_{\Delta_{i,n}\in G}\mathbb{P}\left(A_{i,n}(t)\right) &\geq \sum_{\Delta_{i,n}\in G} p^{V_{i,n}(t)} \geq Z\cdot p^{\sum V_{i,n}(t)/Z}\\ &\geq Z\cdot p^{K(5+\gamma)M^n/Z}.\end{aligned}$$

It follows from our remark above concerning the independency of certain subsystems of $\{A_{i,n}(t)\}_{\Delta_{i,n}\in G_n(t)}$ that we can choose a $H_n(t)\subset G_n(t)$ with the following properties

(a): for $I_n(t) := \{i : \Delta_{i,n}(t)\in H_n(t)\}$, the events $\{A_{i,n}(t)\}_{i\in I_n(t)}$ are independent,

(b): $\#I_n(t) = \frac{1}{5+\gamma}\cdot a_{k+1}M^n$

(c): $\sum_{i\in I_n(t)}\mathbb{P}\left(A_{i,n}(t)\right) \geq \frac{1}{5+\gamma}a_{k+1}M^n p^{K(5+\gamma)^2/a_{k+1}}$.

We write

$$P := \mathbb{P}\left(\sum_{\Delta_{i,n}\in G_n(t)} \mathbb{1}_{A_{i,n}(t)} < \frac{1}{10+2\gamma}a_{k+1}M^n p^{K(5+\gamma)^2/a_{k+1}}\right).$$

Then

$$P \leq \mathbb{P}\left(\sum_{i\in I_n(t)} \mathbb{1}_{A_{i,n}(t)} < \sum_{i\in I_n(t)}\mathbb{P}\left(A_{i,n}(t)\right) - \#I_n(t)T\right),$$

where $T = \frac{1}{2}p^{K(5+\gamma)^2/a_{k+1}}$. Using this and the Azuma-Hoeffding inequality [1], Thm 7.2.1 we obtain that

$$P \leq \exp\left[-2T^2\#I_n(t)\right].$$

That is for $\tau = \exp\left[-2T^2 a_{k+1}/(5+\gamma)\right] < 1$ we obtain that

$$(7.11)\qquad \mathbb{P}\left(\sum_{\Delta_{i,n}\in G_n(t)} \mathbb{1}_{A_{i,n}(t)} \leq a_{k+1}M^n\frac{T}{5+\gamma}\right) \leq \tau^{M^n}.$$

Hence

(7.12)

$$\mathbb{P}\left(\forall t\in R_n \text{ we have } \sum_{\Delta_{i,n}\in G_n(t)} \mathbb{1}_{A_{i,n}(t)} > a_{k+1}M^n\frac{T}{5+\gamma}\right) > 1-\#B_n\cdot\tau^{M^n}.$$

By the definition of B_n this converges to 1. So, almost surely we can find an $n > n_0$ such that (7.12) holds. Then for every $t \in R_n$ we have (7.13)

$$\#\left\{i : V_{i,n}(t) > 0,\ \Delta_{i,n}(t) \cap S_{\widetilde{t}}(E_{n+1}) = \emptyset,\ \forall \rho(\widetilde{t}, t) < M^{-n}\right\} \geq \frac{a_{k+1}M^n T}{5+\gamma}.$$

Using this, (7.6) and (7.4) we obtain that for every $\widetilde{t} \in B_{n+1}$ satisfying $\rho(\widetilde{t}, t) < (M-1)M^{-(n+1)}$ for some $t \in R_n$ we have

$$Z_{n+1}(\widetilde{t}) < a_k M^{n+1} - a_{k+1}M^{n+1}\frac{T}{5+\gamma} = M^{n+1}a_{k+1}. \tag{7.14}$$

That is $\widetilde{t} \in B_{n+1} \setminus R_{n+1}$. On the other hand for those $\widetilde{t} \in B_{n+1}$ for which there exist $t \in B_n$ such that $\rho(\widetilde{t}, t) < (M-1)M^{-(n+1)}$ and $Z_n(t) < a_{k+1}M^n$ we obtain by (7.4) that $Z_{n+1}(\widetilde{t}) < a_{k+1}M^{n+1}$. This implies that $R_{n+1} = \emptyset$ which contradicts to our assumption (7.8). This completes the proof of the Theorem.

References

[1] N. Alon, J. H. Spencer, and P. Erdos, The Probabilistic Method, John Wiley And Sons, Inc, New York, 1992.

[2] J. T. Chayes, L. Chayes, and R. Durrett, Connectivity properties of Mandelbrot's percolation process, *Prob. Theory Rel. Fields* **77**,307-324 (1988).

[3] F.M. Dekking, R.W.J. Meester, On the structure of Mandelbrot's Percolation Process and Other Random Cantor Sets *Journal of Statistical Physics* **58**, Nos. 5/6, 1990.

[4] Falconer, K.J., *Random fractals*, Math. Proc. Cambridge Philos. Soc., Vol. 100 (1986), No.3, 559-582.

[5] Falconer, K.J., *Projections of random Cantor sets*, J. Theoret. Probab., no. 2, (1989), no. 1, 65–70.

[6] K.J. Falconer, and G.R. Grimmett, On the geometry of random Cantor sets and fractal percolation. *J. Theoret. Probab.* Vol. 5. (1992), No.3, 465-485.

[7] K.J. Falconer, and G.R. Grimmett, On the geometry of random Cantor sets and fractal percolation Correction. *J. Theoret. Probab.* Vol. 7. (1994), No.1, 209-210.

[8] Carlos Gustavo T. de A. Moreira and Jean-Christophe Yoccoz. Stable intersections of regular Cantor sets with large Hausdorff dimensions. *Ann. of Math. (2)*, 154(1):45–96, 2001.

[9] F.M. Dekking, and G. R. Grimmett. Superbranching processes and projections of random Cantor sets. Probab. Theory Related Fields, 78, (1988), 3, 335–355.

[10] F.M. Dekking and K. Simon. On the size of the algebraic diffrence of two random Cantor sets. *Random Structures and Algorithms*, **32**, (2008) 205-222.

[11] B. B. Mandelbrot, The Fractal Geometry of Nature. *Freeman, San Francisco, 1983.*

[12] P. Mattila Geometry of Sets and Measures in Euclidean Scapces. *Cambridge 1995.*

[13] Mauldin, R. Daniel,Williams, S. C., *Random recursive constructions: asymptotic geometric and topological properties*, Trans. Amer. Math. Soc., Vol. 295. (1986), no. 1, 325-346.

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