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ESTYMACJA WSPÓŁCZYNNIKA AUTOKORELACJI

W SZEREGU CZASOWYM AR(1)

XXXIX Ogólnopolska Konferencja Zastosowań Matematyki
Zakopane-Kościelisko 7 - 14 września 2010 r.

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Obserwacja

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Obserwacja

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Maximum Likelihood Estimator:

$$\hat{\rho}_{MLE} = arg \max_{\rho} \mathcal{L}(\rho; X_1, X_2, \ldots, X_n)$$

where

$$\mathcal{L}(\rho; x_1, x_2, \ldots, x_n) = \log(1-\rho^2) - x_1^2(1-\rho^2) - \sum_{i=2}^{n}(x_i - \rho x_{i-1})^2.$$

Observe that for every $x_1, x_2, \ldots, x_n$, function $\mathcal{L}(\rho; x_1, x_2, \ldots, x_n), \rho \in (-1, 1)$ is concave (second derivative is negative), $\mathcal{L}(\rho; x_1, x_2, \ldots, x_n) \to -\infty$ as $\rho \to \pm 1$ so that $\hat{\rho}_{MLE} \in (-1, 1)$ is uniquely defined. A disadvantage of the estimator is that no simple explicit formula for computing $\hat{\rho}_{MLE}$ is known.

Least Square Estimator:

$$\hat{\rho}_{LSE} = arg\min_{\rho} \sum_{i=2}^{n} (X_i - \rho X_{i-1})^2 = \frac{\sum_{i=2}^{n} X_i X_{i-1}}{\sum_{i=2}^{n} X_{i-1}^2}$$

Hurwicz estimator

$$\rho_{HUR} = Med\left(\frac{X_2}{X_1}, \frac{X_3}{X_2}, \ldots, \frac{X_n}{X_{n-1}}\right)$$

where $Med(\xi_1, \xi_2, \ldots, \xi_m)$ denotes a median of $\xi_1, \xi_2, \ldots, \xi_m$.
A nice property of the estimator is that it is median-unbiased which means that

$$P_\rho\{\hat{\rho}_{HUR} \leqslant \rho\} = P_\rho\{\hat{\rho}_{HUR} \geqslant \rho\} = \frac{1}{2}, \quad \text{for all} \quad \rho \in (-1, 1).$$

Haddad estimator:

$$\rho_{HAD} = \frac{Med(X_{t-1}X_t)}{Med(X_t^2)}$$

The estimator has been constructed as a robust counterpart of the least square estimator.

M-estimator with Huber loss function: 1̌3

$$\rho_{MHU} = arg \min_{\rho} \sum_{i=1}^{n-1} L(X_{i+1} - \rho X_i)$$

with

$$L(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } |x| \leqslant k, \\ k|x| - \frac{1}{2}k^2 & \text{if } |x| > k \end{cases}$$

Following Lehmann we assume $k = 3/2$

No simple explicit formula for computing $\hat{\rho}_{MHU}$ is known

Burg's estimator:

The estimator has been constructed as that minimizing the forward and backward prediction errors:

$$\rho_{BUR} = arg \min_{\rho} \sum_{i=2}^{n} ((X_i - \rho X_{i-1})^2 + (X_{i-1} - \rho X_i)^2)$$

Then

$$\rho_{BUR} = \frac{2\sum_{i=2}^{n} X_i X_{i-1}}{\sum_{i=2}^{n}(X_i^2 + X_{i-1}^2)}$$

It should be noted that the support of the estimator ρ_{BUR} is in the interval $(-1, 1)$

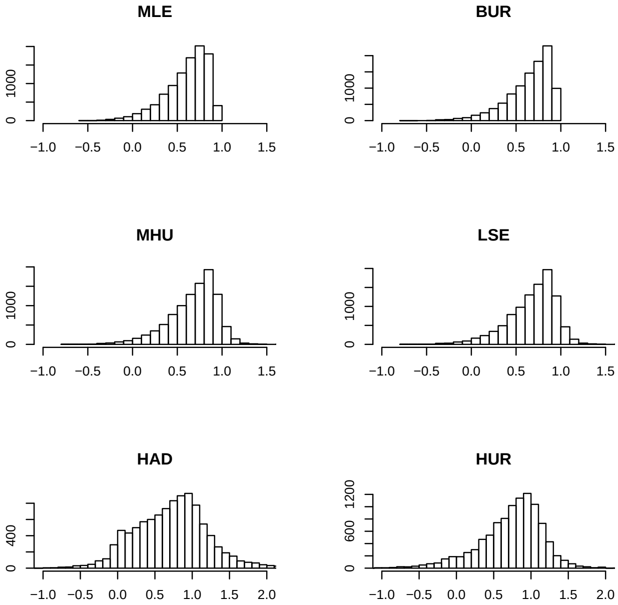


Fig. 2. Histograms

ENTROPY LOSS FUNCTION

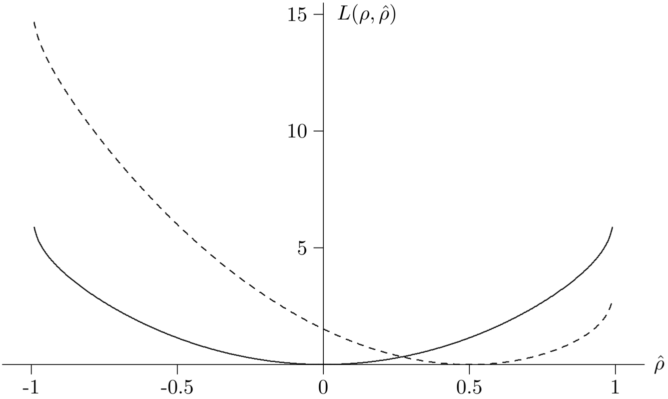
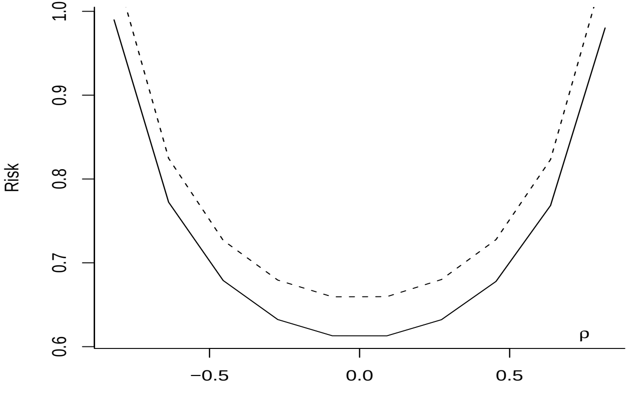


Fig.1. Entropy Loss Functions for $\rho = 0$ (solid) and $\rho = 0.5$ (dashed)

$$L(\theta, \hat{\theta}) = E_\theta \, log\left(\frac{f_\theta(X)}{f_{\hat{\theta}}(X)}\right)$$

$$L(\rho, \hat{\rho}) = \begin{cases} \frac{1}{2} log \frac{1-\rho^2}{1-\hat{\rho}^2} + \frac{\rho-\hat{\rho}}{2(1-\rho^2)}\left(n\rho - (n-2)\hat{\rho}\right), & \text{if } |\hat{\rho}| < 1, \\ +\infty, & \text{if } |\hat{\rho}| \geqslant 1. \end{cases}$$



Fig, 3. Risk functions of estimators LSE (solid) and BUR (dashes)