

SUMMABILITY AND FRACTIONAL LINEAR PARTIAL DIFFERENTIAL EQUATIONS

SLAWOMIR MICHALIK

ABSTRACT. We consider the Cauchy problem for Kowalevskaya type fractional linear partial differential equations with constant coefficients in two complex variables. We show that the solutions can be analytically continued into certain sectors, and have at most exponential growth there, if and only if the Cauchy data have a corresponding property. Applying this result to the study of formal power series solutions of non-Kowalevskian linear partial differential equations, we obtain a characterisation of Borel summable solutions in terms of analytic continuation property and growth estimations of the Cauchy data. We also obtain a similar result in the case of non-Kowalevskian fractional equations.

1. INTRODUCTION

We study the initial value problem for Kowalevskaya type fractional linear partial differential equations with constant coefficients

$$(1) \quad \begin{cases} (\partial_t^{1/p})^m v(t, z) - \sum_{j=1}^m (\partial_t^{1/p})^{m-j} P_j(\partial_z^{1/p}) v(t, z) = 0 \\ (\partial_t^{1/p})^n v(0, z) = \varphi_n(z) \text{ for } n = 0, \dots, m-1 \end{cases},$$

where $m, p \in \mathbb{N}$, $P_j(\xi)$ are polynomials with complex coefficients of degree $\deg P_j(\xi) \leq j$, $\deg P_m(\xi) = m$ and the initial data $\varphi_n(z)$ are $1/p$ -analytic in some complex neighbourhood of the origin (i.e. the functions $z \mapsto \varphi_n(z^p)$ are analytic near the origin). By the fractional $1/p$ -derivative we mean the linear operator on the space of $z^{1/p}$ -power series defined by

$$\partial_z^{1/p} \left(\sum_{n=0}^{\infty} \frac{u_n z^{n/p}}{\Gamma(1+n/p)} \right) := \sum_{n=0}^{\infty} \frac{u_{n+1} z^{n/p}}{\Gamma(1+n/p)}.$$

Using the ideas of W. Balsler and M. Miyake [3] we find an integral representation of the solution of (1). Next we obtain a sufficient condition for analytic continuation with a growth estimate of the solution $v(t, z)$ of (1) in terms of the Cauchy data $\varphi_n(z)$. Moreover, by a symmetry of (1) with respect to t and z , we also prove the necessity of the condition. Roughly speaking, we have (see Theorem 1 in Section 6 for a precise formulation)

Theorem. *Let $\{\lambda_1, \dots, \lambda_l\}$ be the set of the characteristic roots of*

$$\lambda^m - \sum_{j=1}^m \lambda^{m-j} p_j = 0 \quad \text{with} \quad p_j := \lim_{\xi \rightarrow \infty} P_j(\xi) / \xi^j.$$

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Then the solution $v(t, z)$ of (1) can be analytically continued into a sector in a direction d and has the exponential growth of order at most $s > 1$ as $t \rightarrow \infty$ if and only if the Cauchy data $\varphi_n(z)$ are analytically continued into some sectors in directions $d + \text{parg } \lambda_j$ and have the same exponential growth at infinity.

Next we apply the above theorem to the study of summability properties of the formal solutions to non-Kowalevskian linear PDEs with constant coefficients. Namely, we consider the Cauchy problem

$$(2) \quad \begin{cases} \partial_t^p u(t, z) = \sum_{j=1}^p \partial_t^{p-j} P_{jq/p}(\partial_z) u(t, z) \\ \partial_t^n u(0, z) = \varphi_n(z) \text{ for } n = 0, \dots, p-1 \end{cases},$$

where $p, q \in \mathbb{N}$, $p < q$, $P_{jq/p}(\xi)$ are polynomials of degree not greater than jq/p ($j = 1, \dots, p$), $P_q(\xi)$ is a polynomial of degree q and $\varphi_n(z)$ ($n = 0, \dots, p-1$) are analytic in some complex neighbourhood of the origin.

In a similar way to [8], we use the operator $B^{1,1+1/k}$ with $k = p/(q-p)$, which after appropriate change of variables is equal to the modified k -Borel transform \mathcal{B}^k . Applying $B^{1,1+1/k}$ to the formal solution $\hat{u}(t, z)$ of (2), we conclude that the associated function $v(t, z) := B^{1,1+1/k} \hat{u}(t, z)$ satisfies (1) for certain Kowalevskaya type fractional equation. So, the problem of summability of solution to (2) is reduced to the analytic continuation property of solution of (1). This idea is a generalisation of the one given in [7], where the question about the summability of formal solutions to the heat equation is reduced to the investigation of the wave equation.

So, roughly speaking, we obtain (see Theorem 2 in Section 7 for a precise formulation)

Theorem. *The formal power series solution $\hat{u}(t, z)$ of the initial value problem (2) is $p/(q-p)$ -summable in a direction d if and only if the Cauchy data $\varphi_0(z), \dots, \varphi_{p-1}(z)$ can be analytically continued into some sectors in directions $(d + \arg \alpha_j + 2k\pi)p/q$ ($j = 1, \dots, l$, $k = 0, \dots, q-1$) and have the exponential growth of order at most $q/(q-p)$ there, where $\alpha_1, \dots, \alpha_l$ are the roots of the characteristic equation*

$$\alpha^p - \sum_{j=1}^p \alpha^{p-j} \tilde{p}_j = 0 \quad \text{with} \quad \tilde{p}_j := \lim_{\xi \rightarrow \infty} P_{jq/p}(\xi) / \xi^{jq/p}.$$

The above theorem extends the earlier results on the characterisation of Borel summable formal solutions of some special non-Kowalevskian linear PDEs obtained by: D.A. Lutz, M. Miyake and R. Schäfke [6] in the case of the heat equation, M. Miyake [9] in the case of the equation $\partial_t^p u - \partial_z^q u = 0$ with $p < q$, K. Ichinobe [4] in the case of the quasi-homogeneous equations and by S. Michalik [8] in the case of certain linear partial differential equations.

The sufficient condition for the Borel summability of formal solutions was also found by W. Balsler and M. Miyake [3] (for certain linear PDEs with constant coefficients) and by W. Balsler [2] (for general linear PDEs with constant coefficients). In the last paper W. Balsler conjectured that this sufficient condition is also necessary. Theorem 2 gives the positive answer to the conjecture in the case of equations in a normal form.

Finally, using Theorem 1 we also show a similar characterisation of Borel summability of formal power series solutions to the Cauchy problem for the non-Kowalevskian fractional linear partial differential equations (see Propositions 8–10 in Section 8).

2. NOTATION

The complex (resp. real) disc in \mathbb{C}^n (resp. \mathbb{R}^n) with a centre at the origin and a radius $r > 0$ is denoted by $D^n(r) := \{z \in \mathbb{C}^n : |z| < r\}$ (resp. $B^n(r) := \{x \in \mathbb{R}^n : |x| < r\}$). We write $D(r)$ for $n = 1$.

A sector $S(d, \varepsilon)$ of opening $\varepsilon > 0$ and in a direction $d \in \mathbb{R}$ in the universal covering space $\tilde{\mathbb{C}}$ of $\mathbb{C} \setminus \{0\}$ is defined by

$$S(d, \varepsilon) := \{z \in \tilde{\mathbb{C}} : z = re^{i\theta}, d - \varepsilon/2 < \theta < d + \varepsilon/2, r > 0\}.$$

By $\mathcal{O}(D)$ we denote the space of analytic functions on a domain $D \subseteq \mathbb{C}^n$. The Banach space of analytic functions on $D(r)$, continuous on its closure and equipped with the norm $\|\varphi\|_r := \max_{|z| \leq r} |\varphi(z)|$ is denoted by $\mathbb{E}(r)$.

The space of formal power series

$$\hat{u}(t, z) = \sum_{j=0}^{\infty} u_j(z) t^j \quad \text{with } u_j(z) \in \mathbb{E}(r)$$

is denoted by $\mathbb{E}(r)[[t]]$. Moreover, we set $\mathbb{E}[[t]] := \bigcup_{r>0} \mathbb{E}(r)[[t]]$.

$P_m(\partial_t, \partial_z)$ is the principal part of the differential operator $P(\partial_t, \partial_z)$ of order m . In other words, if $P(\partial_t, \partial_z) = \sum_{j+k \leq m} a_{jk} \partial_t^j \partial_z^k$ then $P_m(\partial_t, \partial_z) = \sum_{j+k=m} a_{jk} \partial_t^j \partial_z^k$.

3. GEVREY FORMAL POWER SERIES AND BOREL SUMMABILITY

In this section we recall some fundamental facts about the Gevrey formal power series and the Borel summability. For more details we refer the reader to [1].

Definition 1. A function $u(t, z) \in \mathcal{O}(S(d, \varepsilon) \times D(r))$ is of *exponential growth of order at most $s > 0$ as $t \rightarrow \infty$ in $S(d, \varepsilon)$* if and only if for any $r_1 \in (0, r)$ and any $\varepsilon_1 \in (0, \varepsilon)$ there exist $A, B < \infty$ such that

$$\max_{|z| \leq r_1} |u(t, z)| < A e^{B|t|^s} \quad \text{for every } t \in S(d, \varepsilon_1).$$

Analogously, a function $\varphi(z) \in \mathcal{O}(S(d, \varepsilon))$ is of *exponential growth of order at most $s > 0$ as $z \rightarrow \infty$ in $S(d, \varepsilon)$* if and only if for any $\varepsilon_1 \in (0, \varepsilon)$ there exist $A, B < \infty$ such that

$$|\varphi(z)| < A e^{B|z|^s} \quad \text{for every } z \in S(d, \varepsilon_1).$$

Definition 2. Let $k > 0$. A formal power series

$$(3) \quad \hat{u}(t, z) := \sum_{j=0}^{\infty} u_j(z) t^j \quad \text{with } u_j(z) \in \mathbb{E}(r)$$

is *1/k-Gevrey formal power series in t* if its coefficients satisfy

$$\max_{|z| \leq r} |u_j(z)| \leq AB^j \Gamma(1 + j/k) \quad \text{for } j = 0, 1, \dots$$

with some positive constants A and B .

The set of 1/k-Gevrey formal power series in t over $\mathbb{E}(r)$ is denoted by $\mathbb{E}(r)[[t]]_{1/k}$. We also set $\mathbb{E}[[t]]_{1/k} := \bigcup_{r>0} \mathbb{E}(r)[[t]]_{1/k}$.

Definition 3. Let $k > 0$ and $d \in \mathbb{R}$. A formal series $\hat{u}(t, z) \in \mathbb{E}[[t]]_{1/k}$ defined by (3) is called *k-summable in a direction d* if and only if its *k-Borel transform*

$$\tilde{v}(t, z) := \sum_{j=0}^{\infty} u_j(z) \frac{t^j}{\Gamma(1 + j/k)}$$

is analytic in $S(d, \varepsilon) \times D(r)$ (for some $\varepsilon > 0$ and $r > 0$) and is of exponential growth of order at most k as $t \rightarrow \infty$ in $S(d, \varepsilon)$. The *k-sum of $\hat{u}(t, z)$ in the direction d* is represented by the Laplace transform of $\tilde{v}(t, z)$

$$u^\theta(t, z) := \frac{1}{t^k} \int_0^{\infty(\theta)} e^{-(s/t)^k} \tilde{v}(s, z) ds^k,$$

where the integration is taken over any ray $e^{i\theta}\mathbb{R}_+ := \{re^{i\theta} : r \geq 0\}$ with $\theta \in (d - \varepsilon/2, d + \varepsilon/2)$.

For every $k > 0$ and $d \in \mathbb{R}$, according to the general theory of moment summability (see Section 6.5 in [1]), a formal series (3) is *k-summable in a direction d* if and only if the same holds for the series

$$\sum_{j=0}^{\infty} u_j(z) \frac{j! \Gamma(1 + j/k)}{\Gamma(1 + j(1 + 1/k))} t^j.$$

Consequently, we obtain a characterisation of *k-summability* (analogous to Definition 3), if we replace the *k-Borel transform* by the *modified k-Borel transform*

$$v(t, z) := \mathcal{B}^k \hat{u}(t, z) := \sum_{j=0}^{\infty} u_j(z) \frac{j! t^j}{\Gamma(1 + j(1 + 1/k))}$$

and the Laplace transform by the *Ecalte acceleration operator*

$$u^\theta(t, z) = t^{-k/(1+k)} \int_0^{\infty(\theta)} v(s, z) C_{1+1/k}((s/t)^{k/(1+k)}) ds^{k/(1+k)}$$

with $\theta \in (d - \varepsilon, d + \varepsilon)$. Here integration is taken over the ray $e^{i\theta}\mathbb{R}_+$ and $C_{1+1/k}(\zeta)$ is defined by

$$C_{1+1/k}(\zeta) := \frac{1}{2\pi i} \int_\gamma u^{-1/(k+1)} e^{u - \zeta u^{k/(k+1)}} du$$

with a path of integration γ as in the Hankel integral for the inverse Gamma function (from ∞ along $\arg u = -\pi$ to some $u_0 < 0$, then on the circle $|u| = |u_0|$ to $\arg u = \pi$, and back to ∞ along this ray).

Hence the *k-summability* can be characterised as follows

Proposition 1. Let $k > 0$ and $d \in \mathbb{R}$. A formal series $\hat{u}(t, z)$ given by (3) is *k-summable in a direction d* if and only if its *modified k-Borel transform*

$$\mathcal{B}^k \hat{u}(t, z) = \sum_{j=0}^{\infty} u_j(z) \frac{j! t^j}{\Gamma(1 + j(1 + 1/k))}$$

satisfies the following conditions:

- $\mathcal{B}^k \hat{u}(t, z) \in \mathcal{O}(D(r_1) \times D(r_2))$ (for some $r_1 > 0$ and $r_2 > 0$), i.e. $\hat{u}(t, z) \in \mathbb{E}(r_2)[[t]]_{1/k}$.
- $\mathcal{B}^k \hat{u}(t, z)$ can be analytically continued to $S(d, \varepsilon) \times D(r)$ (for some $\varepsilon > 0$ and $r > 0$).
- $\mathcal{B}^k \hat{u}(t, z)$ is of exponential growth of order at most k as $t \rightarrow \infty$ in $S(d, \varepsilon)$.

4. α -ANALYTIC FUNCTIONS AND α -DERIVATIVES

In this section we introduce some kind of fractional derivatives ∂_z^α of the formal power series in $\mathbb{C}[[z^\alpha]]$. These operators are a natural generalisation of the derivative ∂_z defined into the space $\mathbb{C}[[z]]$. Namely, we have

Definition 4. Let $\alpha > 0$. The linear operator on the space of formal power series

$$\partial_z^\alpha : \mathbb{C}[[z^\alpha]] \rightarrow \mathbb{C}[[z^\alpha]]$$

given by

$$(4) \quad \partial_z^\alpha \left(\sum_{n=0}^{\infty} \frac{u_n}{\Gamma(1 + \alpha n)} z^{\alpha n} \right) := \sum_{n=0}^{\infty} \frac{u_{n+1}}{\Gamma(1 + \alpha n)} z^{\alpha n}$$

is called an α -derivative.

Definition 5. We say that a function $u(z)$ is α -analytic on $D \subseteq \mathbb{C}$ (or, generally, on $D \subseteq \mathbb{C}^n$) if and only if the function $z \mapsto u(z^{1/\alpha})$ is analytic for every $z^{1/\alpha} \in D$. The space of α -analytic functions will be denoted by $\mathcal{O}_\alpha(D)$.

If the formal power series $\hat{u}(z) \in \mathbb{C}[[z^\alpha]]$ is convergent in some complex neighbourhood of the origin, then its sum $u(z)$ is α -analytic near the origin. For such functions we have well defined α -derivative given by (4), which coincides with the Caputo fractional derivative (see [5]).

We also define the α -Taylor series of $u(z) \in \mathcal{O}_\alpha(D)$ by

$$(5) \quad u(z) = \sum_{n=0}^{\infty} \frac{(\partial_z^\alpha)^n u(0)}{\Gamma(1 + \alpha n)} z^{\alpha n}.$$

In the case of α -analytic functions, the role of the exponential function e^z is played by

$$e_\alpha(z) := E_\alpha(z^\alpha) = \sum_{n=0}^{\infty} \frac{z^{\alpha n}}{\Gamma(1 + \alpha n)},$$

where $E_\alpha(z)$ denotes the Mittag-Leffler function. By the definition of $e_\alpha(z)$ and by the results on the Mittag-Leffler function (see [10]), we have

Proposition 2. *The function $e_\alpha(z)$ satisfies the following properties:*

- $e_\alpha(z) \in \mathcal{O}_\alpha(\mathbb{C})$ and there exists $C < \infty$ such that $|e_\alpha(z)| \leq Ce^{|z|}$ for every $z \in \mathbb{C}$,
- for every $a \in \mathbb{C}$ we have $\partial_z^\alpha e_\alpha(az) = a^\alpha e_\alpha(az)$ (in particular $\partial_z^\alpha e_\alpha(z) = e_\alpha(z)$),
- if $\alpha < 2$ and $\arg z \in (\pi/2, 2\pi/\alpha - \pi/2)$ then $e_\alpha(z) \rightarrow 0$ as $z \rightarrow \infty$.

Let us assume that $\alpha \in \mathbb{Q}_+$. Since every q/p -analytic function is also $1/p$ -analytic, without loss of generality we may take $\alpha = 1/p$, where $p \in \mathbb{N}$. Observe that $1/p$ -analytic function is in fact an analytic function defined on the Riemann surface of $\sqrt[p]{z}$. Hence we find the integral representation

Lemma 1. *Let $\varphi(z) \in \mathcal{O}_{1/p}(D(r))$. Then for every $|z| < \varepsilon < r$ and $k \in \mathbb{N}$ we have*

$$(\partial_z^{1/p})^k \varphi(z) = \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) \int_0^{\infty(\theta)} \zeta^{k/p} e_{1/p}(z\zeta) e^{-w\zeta} d\zeta dw$$

for $\theta \in (-\arg w - \pi/2, -\arg w + \pi/2)$, where $\oint_{|w|=\varepsilon}^p$ denotes the integration p times around the positively oriented circle of radius ε .

Moreover, there exist $\varrho > 0$ and $A, B < \infty$ such that

$$\sup_{|z| < \varrho} |(\partial_z^{1/p})^k \varphi(z)| \leq AB^{k/p} \Gamma(1 + k/p) \quad \text{for } k = 0, 1, \dots$$

Proof. By the Cauchy integral formula on the Riemann surface of $\sqrt[p]{z}$ we have

$$\varphi(0) = \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \frac{\varphi(w)}{w} dw$$

and more generally for every $n \in \mathbb{N}$

$$\begin{aligned} (\partial_z^{1/p})^n \varphi(0) &= \frac{\Gamma(1 + n/p)}{2p\pi i} \oint_{|w|=\varepsilon}^p \frac{\varphi(w)}{w^{n/p+1}} dw \\ &= \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) \int_0^{\infty(\theta)} \zeta^{n/p} e^{-w\zeta} d\zeta dw \end{aligned}$$

with $\theta \in (-\arg w - \pi/2, -\arg w + \pi/2)$.

Hence, by (5)

$$\begin{aligned} \varphi(z) &= \sum_{n=0}^{\infty} \frac{(\partial_z^{1/p})^n \varphi(0)}{\Gamma(1 + n/p)} z^{n/p} = \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) \int_0^{\infty(\theta)} \sum_{n=0}^{\infty} \frac{(z\zeta)^{n/p}}{\Gamma(1 + n/p)} e^{-w\zeta} d\zeta \\ &= \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) \int_0^{\infty(\theta)} e_{1/p}(z\zeta) e^{-w\zeta} d\zeta. \end{aligned}$$

The first part of the proof is finished by the observation that $(\partial_z^{1/p})^k e_{1/p}(z\zeta) = \zeta^{k/p} e_{1/p}(z\zeta)$.

To show the second part, note that by Proposition 2

$$\begin{aligned} \left| \int_0^{\infty(\theta)} \zeta^{k/p} e_{1/p}(z\zeta) e^{-w\zeta} d\zeta \right| &\leq C \int_0^{\infty} \zeta^{k/p} e^{(|z|-|w|)\zeta} d\zeta \leq C \int_0^{\infty} \zeta^{k/p} e^{-\varepsilon\zeta/2} d\zeta \\ &= C \frac{\Gamma(1 + k/p)}{(\varepsilon/2)^{k/p+1}}. \end{aligned}$$

for $|z| < \varepsilon/2$, $|w| = \varepsilon$ and $\theta = -\arg w$. It means that for $\varrho := \varepsilon/2$ we have

$$\sup_{|z| < \varrho} |(\partial_z^{1/p})^k \varphi(z)| \leq \frac{C}{2p\pi} \oint_{|w|=\varepsilon}^p |\varphi(w)| \frac{\Gamma(1 + k/p)}{(\varepsilon/2)^{k/p+1}} dw \leq AB^{k/p} \Gamma(1 + k/p)$$

with some positive constants $A, B < \infty$. \square

5. OPERATORS $B^{\alpha, \beta}$

In this section we introduce the operators $B^{\alpha, \beta}$ with $\alpha, \beta > 0$, which are related to the modified k -Borel operators \mathcal{B}^k . Using the operators $B^{\alpha, \beta}$ we reduce the question about summability to the study of the solution of the appropriate Kowalevskaya type equation.

Definition 6. Let $\alpha, \beta > 0$. We define a linear operator on the space of formal power series

$$B^{\alpha, \beta} : \mathbb{E}[[t^\alpha]] \rightarrow \mathbb{E}[[t^\beta]]$$

by the formula

$$(6) \quad B^{\alpha, \beta} \hat{u}(t, z) = B^{\alpha, \beta} \left(\sum_{n=0}^{\infty} \frac{u_n(z)}{\Gamma(1 + \alpha n)} t^{\alpha n} \right) := \sum_{n=0}^{\infty} \frac{u_n(z)}{\Gamma(1 + \beta n)} t^{\beta n}.$$

Observe that for any formal series $\hat{u}(t, z) \in \mathbb{E}[[t]]$ and $k > 0$ holds

$$\mathcal{B}^k \hat{u}(t, z) = B^{1,s} \hat{u}(t^{1/s}, z) \quad \text{with } s = 1 + 1/k.$$

In particular, taking $s = q/p$ and using the operator $B^{1,q/p}: \mathbb{E}[[t]] \rightarrow \mathbb{E}[[t^{q/p}]]$ given by

$$B^{1,q/p} \left(\sum_{j=0}^{\infty} \frac{u_n(z)}{n!} t^n \right) = \sum_{n=0}^{\infty} \frac{u_n(z)}{\Gamma(1 + nq/p)} t^{nq/p},$$

one can reformulate Proposition 1 as follows

Proposition 3. *Let $p, q \in \mathbb{N}$, $p < q$ and $d \in \mathbb{R}$. Then the formal series $\hat{u}(t, z) \in \mathbb{E}[[t]]$ is $p/(q-p)$ -summable in a direction d if and only if the function $v(t, z) := B^{1,q/p} \hat{u}(t, z)$ satisfies the following conditions*

- a) $z \mapsto v(t, z)$ is analytic in some complex neighbourhood of the origin,
- b) $t \mapsto v(t, z)$ is q/p -analytic in some complex neighbourhood of the origin,
- c) $t \mapsto v(t, z)$ can be q/p -analytically continued to some sectors in directions $(d + 2k\pi)p/q$ ($k = 0, \dots, q-1$) and has at most the exponential growth of order $q/(q-p)$ there.

The following properties of the operators $B^{\alpha,\beta}$ play crucial role in our study of summability.

Proposition 4. *Let $\alpha, \beta > 0$ and $\hat{u}(t, z) \in \mathbb{E}[[t^\alpha]]$. Then operators $B^{\alpha,\beta}$ and derivatives satisfy the commutation formulas:*

- a) $B^{\alpha,\beta} \partial_t^\alpha \hat{u}(t, z) = \partial_t^\beta B^{\alpha,\beta} \hat{u}(t, z);$
- b) $B^{\alpha,\beta} \partial_z \hat{u}(t, z) = \partial_z B^{\alpha,\beta} \hat{u}(t, z);$
- c) $B^{\alpha,\beta} P(\partial_t^\alpha, \partial_z) \hat{u}(t, z) = P(\partial_t^\beta, \partial_z) B^{\alpha,\beta} \hat{u}(t, z)$ for any polynomial $P(\tau, \zeta) := \sum_{j=1}^m \sum_{l=1}^n a_{jl} \tau^j \zeta^l$ with constant coefficients $a_{jl} \in \mathbb{C}$.

Proof. From (6) we see that

$$B^{\alpha,\beta} \partial_t^\alpha \hat{u}(t, z) = B^{\alpha,\beta} \left(\sum_{n=0}^{\infty} \frac{u_{n+1}(z)}{\Gamma(1 + \alpha n)} t^{\alpha n} \right) = \sum_{n=0}^{\infty} \frac{u_{n+1}(z)}{\Gamma(1 + \beta n)} t^{\beta n} = \partial_t^\beta B^{\alpha,\beta} \hat{u}(t, z)$$

and

$$B^{\alpha,\beta} \partial_z \hat{u}(t, z) = \sum_{n=0}^{\infty} \frac{\partial_z u_n(z)}{\Gamma(1 + \beta n)} t^{\beta n} = \partial_z B^{\alpha,\beta} \hat{u}(t, z).$$

Consequently

$$\begin{aligned} B^{\alpha,\beta} P(\partial_t^\alpha, \partial_z) \hat{u}(t, z) &= B^{\alpha,\beta} \left(\sum_{j=1}^m \sum_{l=1}^n a_{jl} (\partial_t^\alpha)^j \partial_z^l \hat{u}(t, z) \right) \\ &= \sum_{j=1}^m \sum_{l=1}^n a_{jl} B^{\alpha,\beta} \left((\partial_t^\alpha)^j \partial_z^l \hat{u}(t, z) \right) = \sum_{j=1}^m \sum_{l=1}^n a_{jl} (\partial_t^\beta)^j \partial_z^l B^{\alpha,\beta} \hat{u}(t, z) \\ &= P(\partial_t^\beta, \partial_z) B^{\alpha,\beta} \hat{u}(t, z). \end{aligned}$$

□

By Proposition 4 we have

Proposition 5. *A formal series $\hat{u}(t, z)$ is a solution of the Cauchy problem (2) for the non-Kowalevskian linear partial differential equation with constant coefficients if and only if the formal series $\hat{v}(t, z) := B^{1-q/p}\hat{u}(t, z)$ satisfies the following fractional equation of Kowalevskaya type*

$$(7) \quad \begin{aligned} (\partial_t^{1/p})^{qp}v(t, z) &= \sum_{j=1}^p (\partial_t^{1/p})^{q(p-j)} P_{jq/p}((\partial_z^{1/p})^p)v(t, z), \\ (\partial_t^{1/p})^j v(0, z) &= \varphi_n(z) \text{ for } j = nq, \\ (\partial_t^{1/p})^j v(0, z) &= 0 \text{ for } j \neq nq, j < qp, n = 0, \dots, p-1. \end{aligned}$$

6. THE SOLUTION OF FRACTIONAL EQUATION

In this section we consider the initial value problem for fractional equation

$$(8) \quad P(\partial_t^{1/p}, \partial_z^{1/p})v(t, z) = 0,$$

$$(9) \quad (\partial_t^{1/p})^n v(0, z) = \varphi_n(z) \in \mathcal{O}_{1/p}(D(r)) \quad \text{for some } r > 0 \quad (n = 0, \dots, m-1),$$

where

$$P(\partial_t^{1/p}, \partial_z^{1/p}) := (\partial_t^{1/p})^m - \sum_{j=1}^m (\partial_t^{1/p})^{m-j} P_j(\partial_z^{1/p})$$

with $\deg P_j(\xi) \leq j$ and $\deg P_m(\xi) = m$. We would like to find the relation between the Cauchy data and solution. For this purpose we will use an integral representation of the solution of (8) with the initial data $\varphi_n(z)$ given by the recurrence relations

$$(10) \quad \varphi_0(z) := \varphi(z), \quad \varphi_n(z) := \sum_{j=1}^n P_j(\partial_z^{1/p})\varphi_{n-j}(z) \quad \text{for } n = 1, \dots, m-1,$$

where $\varphi(z) \in \mathcal{O}_{1/p}(D(r))$. The construction of this integral representation is based upon the results of W. Balsler and M. Miyake [3]. This method is also similar to that used in our previous paper [8], where the case of differential equations (i.e. for $p = 1$) was considered.

We start with the version of the Cauchy-Kowalevskaya theorem

Lemma 2. *The formal power series solution*

$$(11) \quad \hat{v}(t, z) = \sum_{n=0}^{\infty} v_n(z) \frac{t^{n/p}}{\Gamma(1 + n/p)}$$

of (8) with the initial data (9) is convergent in some neighbourhood of the origin.

Proof. By the principle of superposition of solutions of linear equations, we may assume that the initial data satisfy (10). Note that coefficients $v_n(z)$ satisfy the recurrence relation

$$v_n(z) = \sum_{j=1}^m P_j(\partial_z^{1/p})v_{n-j}(z) \quad \text{for } n = 1, 2, \dots$$

with $v_{-m+1}(z) = \dots = v_{-1}(z) = 0$ and $v_0(z) = \varphi(z)$. It means that

$$(12) \quad v_n(z) = q_n(\partial_z^{1/p})\varphi(z) \quad \text{for } n = 1, 2, \dots,$$

where $q_n(\xi)$ is a solution of the difference equation

$$(13) \quad q_n(\xi) = \sum_{j=1}^m P_j(\xi) q_{n-j}(\xi) \quad \text{for } n = 1, 2, \dots$$

with the initial conditions

$$(14) \quad q_0(\xi) = 1 \quad \text{and} \quad q_{-1}(\xi) = \dots = q_{-m+1}(\xi) = 0.$$

Observe that $q_n(\xi)$ is a polynomial of degree less than or equal to n , so $q_n(\xi) = \sum_{k=0}^n q_{nk} \xi^k$ with some constant coefficients q_{nk} .

Put $Q_n(\xi) := \sum_{k=0}^n |q_{nk}| \xi^k$. Since $Q_n(\xi)$ is a polynomial of degree n , there exists $K > 0$ such that

$$(15) \quad |Q_n(\xi)| \leq (K^{1/p} \xi)^n \quad \text{for every } n \in \mathbb{N} \text{ and } \xi > 1.$$

By Lemma 1, there exist $\varrho > 0$ and $A, B < \infty$ such that

$$\begin{aligned} \sup_{|z| < \varrho} |\hat{v}_n(z)| &\leq \sum_{k=0}^n |q_{nk}| \sup_{|z| < \varrho} |(\partial_z^{1/p})^k \varphi(z)| \leq \sum_{k=0}^n |q_{nk}| AB^{k/p} \Gamma(1 + k/p) \\ &\leq A \Gamma(1 + n/p) Q_n(B^{1/p}) \leq A (KB)^{n/p} \Gamma(1 + n/p). \end{aligned}$$

Therefore, the formal series (11) is convergent for $|t| < (KB)^{-1}$ and $|z| < \varrho$. \square

Lemma 3. *Let $s > 1$, $d \in \mathbb{R}$ and $(t, z) \in D^2(r)$ (for some $r > 0$). Suppose that $v(t, z)$ is a solution of (8) with the initial data (10) and that $\{\lambda_1, \dots, \lambda_l\}$ is the set of the roots of the characteristic equation*

$$P_m(\lambda, 1) = \lambda^m - \sum_{j=1}^m \lambda^{m-j} p_j = 0 \quad \text{with} \quad p_j := \lim_{\xi \rightarrow \infty} P_j(\xi) / \xi^j.$$

Then $v(t, z) = \sum_{j=1}^l v_j(t, z)$, where $v_j(t, z) \in \mathcal{O}_{1/p}(D^2(r))$ satisfies the following property: if $\varphi(z) \in \mathcal{O}_{1/p}(S(d + \arg \lambda_j, \tilde{\delta}))$ (for some $\tilde{\delta} > 0$) is of exponential growth of order at most s at infinity then $v_j(t, z) \in \mathcal{O}_{1/p}(S(d, \delta) \times D(r))$ (for some $\delta > 0$ and $r > 0$) has the same exponential growth as $t \rightarrow \infty$.

Proof. By Lemma 2, for sufficiently small t and z , the power series (11) is convergent. Hence, using (12) and applying Lemma 1, one can choose $\varepsilon > |z|$ such that

$$(16) \quad v(t, z) = \sum_{n=0}^{\infty} \frac{t^{n/p}}{\Gamma(1 + n/p)} \sum_{k=0}^n q_{nk} (\partial_z^{1/p})^k \varphi(z) = \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) k(t, w, z) dw,$$

where the kernel function is defined by

$$\begin{aligned} k(t, w, z) &:= \sum_{n=0}^{\infty} \frac{t^{n/p}}{\Gamma(1 + n/p)} \sum_{k=0}^n q_{nk} \int_0^{\infty(\theta)} \zeta^{k/p} e_{1/p}(z\zeta) e^{-w\zeta} d\zeta \\ &= \int_0^{\infty(\theta)} q(t, \zeta) e_{1/p}(z\zeta) e^{-w\zeta} d\zeta \end{aligned}$$

with $\theta = -\arg w$ and

$$q(t, \zeta) := \sum_{n=0}^{\infty} \frac{t^{n/p}}{\Gamma(1 + n/p)} \sum_{k=0}^n q_{nk} \zeta^{k/p} = \sum_{n=0}^{\infty} \frac{t^{n/p}}{\Gamma(1 + n/p)} q_n(\zeta^{1/p}).$$

By (15) we estimate

$$|q(t, \zeta)| \leq \sum_{n=0}^{\infty} \frac{|t|^{n/p} (K|\zeta|)^{n/p}}{\Gamma(1 + n/p)} = e_{1/p}(K|t||\zeta|).$$

Hence, by Proposition 2, $q(t, \zeta) \in \mathcal{O}_{1/p}(\mathbb{C}^2)$ is of exponential growth of order at most 1. Moreover, for $|z| < |w|$ and $\theta = -\arg w$ we have

$$\left| \int_0^{\infty(\theta)} q(t, \zeta) e_{1/p}(z\zeta) e^{-w\zeta} d\zeta \right| < \infty \quad \text{for } K|t| + |z| < |w|.$$

It means that under above conditions on w and z , the function $t \mapsto k(t, w, z)$ is $1/p$ -analytic on the set $\{t \in \mathbb{C} : K|t| + |z| < |w|\}$.

On the other hand, using the characteristic equation

$$(17) \quad P(\lambda, \xi) = \lambda^m - \sum_{j=1}^m P_j(\xi) \lambda^{m-j} = 0$$

one can find the solution of (13). We may assume that for sufficiently large $|\xi|$, say $|\xi| > |\zeta_0^{1/p}|$, the characteristic equation (17) has exactly l distinct holomorphic solutions $\lambda_1(\xi), \dots, \lambda_l(\xi)$ of multiplicity m_1, \dots, m_l ($\sum_{j=1}^l m_j = m$). Since $P_m(\xi) \neq 0$, we conclude that $\lambda_j(\xi) \neq 0$. Moreover, $\deg P_j(\xi) \leq j$ and $\deg P_m(\xi) = m$, which gives

$$(18) \quad \lim_{\xi \rightarrow \infty} \lambda_j(\xi)/\xi = \lambda_j \in \mathbb{C} \setminus \{0\},$$

where λ_j are the roots of the characteristic equation

$$(19) \quad P_m(\lambda, 1) = \lambda^m - \sum_{j=1}^m p_j \lambda^{m-j} = 0 \quad \text{with } p_j := \lim_{\xi \rightarrow \infty} P_j(\xi)/\xi^j.$$

Note that λ_j are not necessarily the distinct roots of (19). From (18) we can also assume that for $|\xi| > |\zeta_0^{1/p}|$ the functions $\lambda_j(\xi)$ are invertible, where the inverse functions $\lambda_j^{-1}(\tau)$ are the roots of the characteristic equation $P(\tau, \lambda^{-1}) = 0$.

Using the roots of the characteristic equation (17) one can find m linear independent solutions of the difference equation (13)

$$\lambda_j^n(\xi), n\lambda_j^n(\xi), \dots, n^{m_j-1}\lambda_j^n(\xi) \quad \text{for } j = 1, \dots, l.$$

Hence for $|\zeta| > |\zeta_0|$ and $\zeta^{1/p} = \xi$, the solution of (13) is given by

$$(20) \quad q_n(\zeta^{1/p}) = \sum_{j=1}^l \sum_{k=0}^{m_j-1} c_{jk}(\zeta^{1/p}) n^k \lambda_j^n(\zeta^{1/p}).$$

We can calculate the coefficients $c_{jk}(\zeta^{1/p})$ using the initial conditions (14) and solving the system of linear equations. Observe that for sufficiently large $|\zeta|$, say $|\zeta| > |\zeta_0|$, the coefficients $c_{jk}(\zeta^{1/p})$ are holomorphic and have the polynomial growth as $|\zeta| \rightarrow \infty$.

Moreover, since

$$nt^{n/p} = ((\partial_t^{1/p})^p t - 1)pt^{n/p} =: r(t, \partial_t^{1/p})t^{n/p},$$

we have

$$\sum_{n=0}^{\infty} \frac{n^k t^{n/p} \lambda_j^n (\zeta^{1/p})}{\Gamma(1+n/p)} = \sum_{n=0}^{\infty} \frac{r^k(t, \partial_t^{1/p}) t^{n/p} \lambda_j^n (\zeta^{1/p})}{\Gamma(1+n/p)} = r^k(t, \partial_t^{1/p}) e_{1/p}(t \lambda_j^p (\zeta^{1/p}))$$

and consequently

$$q(t, \zeta) = \sum_{j=1}^l \sum_{k=0}^{m_j-1} c_{jk}(\zeta^{1/p}) r^k(t, \partial_t^{1/p}) e_{1/p}(t \lambda_j^p (\zeta^{1/p})) \quad \text{for } |\zeta| > |\zeta_0|.$$

Hence

$$\begin{aligned} k(t, w, z) &= \int_0^{\zeta_0} q(t, \zeta) e_{1/p}(z\zeta) e^{-w\zeta} d\zeta \\ &+ \sum_{j=1}^l \sum_{k=0}^{m_j-1} \int_{\zeta_0}^{\infty(\theta)} c_{jk}(\zeta^{1/p}) r^k(t, \partial_t^{1/p}) e_{1/p}(t \lambda_j^p (\zeta^{1/p})) e_{1/p}(z\zeta) e^{-w\zeta} d\zeta. \end{aligned}$$

Fix $w, z \in \mathbb{C}$ such that $|z|$ is small relative to $|w| = \varepsilon$. It means that $|w| - |z| \approx |w|$ and $\arg(w - z) \approx \arg w$. To show analytic continuation of $k(t, w, z)$ with respect to t , we consider the functions

$$(21) \quad a_j(t, z, w) := \int_{\zeta_0}^{\infty(\theta)} e_{1/p}(t \lambda_j^p (\zeta^{1/p})) e_{1/p}(z\zeta) e^{-w\zeta} d\zeta \quad \text{for } j = 1, \dots, l.$$

To estimate $a_j(t, w, z)$ for $j = 1, \dots, l$, observe that by (15), (18) and (20) we have $|\lambda_j| < K^{1/p}$ and consequently the function $t \mapsto a_j(t, w, z)$ is analytic on $\{t \in \mathbb{C} : |t| < |w|/K\}$ for $j = 1, \dots, l$. Moreover, by (18), for sufficiently large ζ we have $\arg \lambda_j^p (\zeta^{1/p}) \approx \arg \zeta + p \arg \lambda_j$. Now we want to replace a direction θ in (21) by θ_j satisfying:

- $\arg t + p \arg \lambda_j + \theta_j \in (\frac{\pi}{2}, 2p\pi - \frac{\pi}{2})$ modulo $2p\pi$ (in this case by Proposition 2 we have $e_{1/p}(t \lambda_j^p (\zeta^{1/p})) \rightarrow 0$ as $\zeta \rightarrow \infty$, $\arg \zeta = \theta_j$),
- $\arg w + \theta_j \in (-\frac{\pi}{2}, \frac{\pi}{2})$ modulo 2π (in this case there exists $\epsilon > 0$ such that $|e_{1/p}(z\zeta) e^{-w\zeta}| \leq e^{-\epsilon|\zeta|}$ as $\zeta \rightarrow \infty$, $\arg \zeta = \theta_j$).

Observe that these requirements may be together satisfied under the condition that $\arg t + p \arg \lambda_j \neq \arg z$ modulo $2p\pi$. Therefore the function $t \mapsto a_j(t, w, z)$ can be $1/p$ -analytically continued to $\{t \in \tilde{\mathbb{C}} : \arg t \neq \arg w - p \arg \lambda_j \text{ modulo } 2p\pi\}$ and has the exponential growth of order at most 1 there. It means that also the function

$$t \mapsto k_j(t, w, z) := \sum_{k=0}^{m_j-1} \int_{\zeta_0}^{\infty(\theta_j)} c_{jk}(\zeta^{1/p}) r^k(t, \partial_t^{1/p}) e_{1/p}(t \lambda_j^p (\zeta^{1/p})) e_{1/p}(z\zeta) e^{-w\zeta} d\zeta$$

can be $1/p$ -analytically continued to the same sector and has the exponential growth of order at most 1 there.

By the Cauchy integral theorem

$$\frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) \int_0^{\zeta_0} q(t, \zeta) e_{1/p}(z\zeta) e^{-w\zeta} d\zeta dw = 0.$$

Hence, by (16), for t and z close to the origin, $|z| < \varepsilon$ and for sufficiently small $\varepsilon > 0$, the solution of (8) with the initial conditions (10) satisfies

$$\begin{aligned} v(t, z) &= \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) k(t, w, z) dw = \sum_{j=1}^l \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) k_j(t, w, z) dw \\ &= \sum_{j=1}^l v_j(t, z), \end{aligned}$$

where

$$(22) \quad \begin{aligned} v_j(t, z) &:= \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) \times \\ &\times \sum_{k=0}^{m_j-1} \int_{\zeta_0}^{\infty(\theta_j)} c_{jk}(\zeta^{1/p}) r^k(t, \partial_t^{1/p}) e_{1/p}(t\lambda_j^p(\zeta^{1/p})) e_{1/p}(z\zeta) e^{-w\zeta} d\zeta dw. \end{aligned}$$

To estimate $v_j(t, z)$, fix z such close to the origin, that $\arg(w - z) \approx \arg w$ along a circle $|w| = \varepsilon$. Repeating the proof of Theorem 3.1 in [3], we split this circle into 2 arcs γ and $\tilde{\gamma}$, where γ extends between points of argument $d + p \arg \lambda_j - \tilde{\delta}/3$ and $d + p \arg \lambda_j + \tilde{\delta}/3$. Finally, since $\varphi(z) \in \mathcal{O}_{1/p}(S(d + p \arg \lambda_j, \tilde{\delta}))$, we may deform γ into a path γ_R along the ray $\arg w = d + \arg \lambda_j - \tilde{\delta}/3$ to a point with modulus R (which can be chosen arbitrarily large), then along the circle $|w| = R$ to the ray $\arg w = d + \arg \lambda_j + \tilde{\delta}/3$ and back along this ray to the original circle. So, we have

$$\begin{aligned} v_j(t, z) &= \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) k_j(t, w, z) dw \\ &= \frac{1}{2p\pi i} \oint_{\gamma_R} \varphi(w) k_j(t, w, z) dw + \frac{1}{2p\pi i} \oint_{\tilde{\gamma}} \varphi(w) k_j(t, w, z) dw. \end{aligned}$$

Note that R may be chosen arbitrarily large and the function $t \mapsto k_j(t, w, z)$ is analytic on $|t| < |w|/K$, where $|z|$ is small relative to $|w|$. Hence, one can find $\delta > 0$ such that the first integral on the right-hand side is $1/p$ -analytically continued to $S(d, \delta) \times D(r)$. Estimating this integral we see that it is of exponential growth of order at most s as $t \rightarrow \infty$.

Moreover, since the function $t \mapsto k_j(t, w, z)$ is analytically continued into the region $\{t \in \tilde{\mathbb{C}} : \arg t \neq \arg w - p \arg \lambda_j \text{ modulo } 2p\pi\}$, we see that the second integral on the right-hand side is also $1/p$ -analytically continued to $S(d, \delta) \times D(r)$ and satisfies the appropriate growth estimate there as $t \rightarrow \infty$. It means that $v_j(t, z)$ is $1/p$ -analytically continued to $S(d, \delta) \times D(r)$ and is of exponential growth of order at most s as $t \rightarrow \infty$. \square

Now, in a similar way to [8], we introduce some kind of pseudodifferential operators connected with $1/p$ -derivative. For this purpose, let $\lambda(\zeta^{1/p})$ be a non-vanishing $1/p$ -analytic function for $|\zeta| \geq |\zeta_0|$ with the polynomial growth at infinity. First, we define an operator $\lambda(\partial_z^{1/p})$ acting on $e_{1/p}(z\zeta)$ as follows

$$\lambda(\partial_z^{1/p}) e_{1/p}(z\zeta) := \lambda(\zeta^{1/p}) e_{1/p}(z\zeta) \quad \text{for } |\zeta| \geq |\zeta_0|.$$

Moreover, observe that for any $\varphi(z) \in \mathcal{O}_{1/p}(D(\tilde{r}))$ (with some $\tilde{r} > 0$) we have

$$\lambda(\partial_z^{1/p})\varphi(z) = \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) \int_{\zeta_0}^{\infty(\theta)} \lambda(\zeta^{1/p}) e_{1/p}(z\zeta) e^{-w\zeta} d\zeta dw.$$

Analogously, if $f(\zeta, t, w) \in \mathcal{O}_{1/p}(\mathbb{C} \times D(r) \times D(\tilde{r}))$ (with some $r > 0$ and $\tilde{r} > 0$) and

$$v(t, z) = \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) \int_{\zeta_0}^{\infty(\theta)} f(\zeta, t, w) e_{1/p}(z\zeta) d\zeta dw$$

is well-defined and $1/p$ -analytic in some complex neighbourhood of the origin, then we have

$$(23) \quad \lambda(\partial_z^{1/p})v(t, z) = \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) \int_{\zeta_0}^{\infty(\theta)} \lambda(\zeta^{1/p}) f(\zeta, t, w) e_{1/p}(z\zeta) d\zeta dw.$$

Remark 1. Let $s > 1$. Observe that $v(t, z)$ is $1/p$ -analytic on the the set $D(r) \times S(d, \delta)$ and is of the exponential growth of order at most s as $z \rightarrow \infty$ if and only if $\lambda(\partial_z^{1/p})v(t, z)$ has the same properties.

Remark 2. Using the pseudodifferential operators we have

$$P(\partial_t^{1/p}, \partial_z^{1/p})v(t, z) = (\partial_t^{1/p} - \lambda_1(\partial_z^{1/p}))^{m_1} \dots (\partial_t^{1/p} - \lambda_l(\partial_z^{1/p}))^{m_l} v(t, z) = 0.$$

We show

Lemma 4. *The functions $v_j(t, z)$ given by (22) are the solutions of pseudodifferential equations*

$$(\partial_t^{1/p} - \lambda_j(\partial_z^{1/p}))^{m_j} v_j(t, z) = 0 \quad \text{for } j = 1, \dots, l.$$

Proof. By (22) and (23) we have

$$\begin{aligned} & (\partial_t^{1/p} - \lambda_j(\partial_z^{1/p}))^{m_j} v_j(t, z) \\ &= \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) \int_{\zeta_0}^{\infty(\theta_j)} \sum_{k=0}^{m_j-1} c_{jk}(\zeta^{1/p}) b_{jk}(t, \zeta) e_{1/p}(z\zeta) e^{-w\zeta} d\zeta dw, \end{aligned}$$

where

$$b_{jk}(t, \zeta) = (\partial_t^{1/p} - \lambda_j(\zeta^{1/p}))^{m_j} \sum_{n=0}^{\infty} \frac{n^k \lambda_j^n(\zeta^{1/p}) t^{n/p}}{\Gamma(1 + n/p)}.$$

Next, for any fixed $k \in \mathbb{N}_0$ we define a family of polynomials

$$P_{k,0}(n) := n^k, \quad P_{k,j}(n) := P_{k,j-1}(n+1) - P_{k,j-1}(n) \quad \text{for } j \in \mathbb{N}.$$

Observe that $\deg P_{k,j}(\xi) = \max\{k - j, -1\}$, where $\deg P_{k,j}(\xi) = -1$ if and only if $P_{k,j}(\xi) \equiv 0$.

Using these polynomials we have

$$\begin{aligned} b_{jk}(t, \zeta) &= (\partial_t^{1/p} - \lambda_j(\zeta^{1/p}))^{m_j} \sum_{n=0}^{\infty} \frac{P_{k,0}(n) \lambda_j^n(\zeta^{1/p}) t^{n/p}}{\Gamma(1 + n/p)} = \dots \\ &= (\partial_t^{1/p} - \lambda_j(\zeta^{1/p}))^{m_j - k - 1} \sum_{n=0}^{\infty} \frac{P_{k,k+1}(n) \lambda_j^{n+k}(\zeta^{1/p}) t^{n/p}}{\Gamma(1 + n/p)} = 0. \end{aligned}$$

□

Lemma 5. *Let $v(t, z) \in \mathcal{O}_{1/p}(D^2(r))$ with some $r > 0$. Then $v(t, z)$ satisfies the pseudodifferential equation*

$$(24) \quad \left(\partial_t^{1/p} - \lambda_j(\partial_z^{1/p}) \right) v(t, z) = 0$$

if and only if $v(t, z)$ is a solution of

$$(25) \quad \left(\partial_z^{1/p} - \lambda_j^{-1}(\partial_t^{1/p}) \right) v(t, z) = 0.$$

Proof. Since the equations (24) and (25) are symmetric, it is sufficient to show one-way implication. So, let us assume that $v(t, z)$ is a solution of (24). Then

$$(26) \quad v(t, z) = \sum_{n=0}^{\infty} \frac{\lambda_j^n (\partial_z^{1/p})^n \varphi(z)}{\Gamma(1+n/p)} t^{n/p} \quad \text{with} \quad \varphi(z) := v(0, z) \in \mathcal{O}_{1/p}(D(r)).$$

Since

$$\lambda_j(\partial_z^{1/p})\varphi(z) = \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) \int_{\zeta_0}^{\infty(\theta)} \lambda_j(\zeta^{1/p}) e_{1/p}(z\zeta) e^{-w\zeta} d\zeta dw,$$

by (26) we have

$$\begin{aligned} v(t, z) &= \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) \int_{\zeta_0}^{\infty(\theta)} \sum_{n=0}^{\infty} \frac{\lambda_j^n (\zeta^{1/p})^n t^{n/p}}{\Gamma(1+n/p)} e_{1/p}(z\zeta) e^{-w\zeta} d\zeta dw \\ &= \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) \int_{\zeta_0}^{\infty(\theta)} e_{1/p}(\lambda_j^p(\zeta^{1/p})t) e_{1/p}(z\zeta) e^{-w\zeta} d\zeta dw. \end{aligned}$$

Using the substitution $\tau := \lambda_j^p(\zeta^{1/p})$ we obtain

$$\begin{aligned} v(t, z) &= \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) \times \\ &\quad \times \int_{\tau_0}^{\infty(\tilde{\theta})} e_{1/p}(\tau t) e_{1/p}(z\lambda_j^{-p}(\tau^{1/p})) e^{-w\lambda_j^{-p}(\tau^{1/p})} d\lambda_j^{-p}(\tau^{1/p}) dw \end{aligned}$$

with $\tau_0 := \lambda_j^p(\zeta_0^{1/p})$ and $\tilde{\theta} := \theta + p \arg \lambda_j$. Hence we have

$$\begin{aligned} \partial_z^{1/p} v(t, z) &= \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^p \varphi(w) \times \\ &\quad \times \int_{\tau_0}^{\infty(\tilde{\theta})} \lambda_j^{-1}(\tau^{1/p}) e_{1/p}(\tau t) e_{1/p}(z\lambda_j^{-p}(\tau^{1/p})) e^{-w\lambda_j^{-p}(\tau^{1/p})} d\lambda_j^{-p}(\tau^{1/p}) dw \\ &= \lambda_j^{-1}(\partial_t^{1/p}) v(t, z). \end{aligned}$$

□

Lemma 6. *Let $v(t, z) \in \mathcal{O}_{1/p}(D^2(r))$ with some $r > 0$. Then for every $n \in \mathbb{N}$, $v(t, z)$ satisfies the pseudodifferential equation*

$$\left(\partial_t^{1/p} - \lambda_j(\partial_z^{1/p}) \right)^n v(t, z) = 0$$

if and only if $v(t, z)$ is a solution of

$$\left(\partial_z^{1/p} - \lambda_j^{-1}(\partial_t^{1/p}) \right)^n v(t, z) = 0.$$

Proof. The proof is by induction on n . By Lemma 5, the statement holds for $n = 1$. So, we may assume that assertion holds for $n = k$. We have

$$\begin{aligned}
 & (\partial_t^{1/p} - \lambda_j(\partial_z^{1/p}))^{k+1}v(t, z) = 0 \iff \\
 & (\partial_t^{1/p} - \lambda_j(\partial_z^{1/p}))\tilde{v}(t, z) = 0 \text{ with } \tilde{v}(t, z) := (\partial_t^{1/p} - \lambda_j(\partial_z^{1/p}))^k v(t, z) \\
 & \text{by Lemma 5 } \iff (\partial_z^{1/p} - \lambda_j^{-1}(\partial_t^{1/p}))\tilde{v}(t, z) = 0 \iff \\
 & (\partial_t^{1/p} - \lambda_j(\partial_z^{1/p}))^k \bar{v}(t, z) = 0 \text{ with } \bar{v}(t, z) := (\partial_z^{1/p} - \lambda_j^{-1}(\partial_t^{1/p}))v(t, z) \\
 & \text{by the inductive assumption } \iff (\partial_z^{1/p} - \lambda_j^{-1}(\partial_t^{1/p}))^k \bar{v}(t, z) = 0 \iff \\
 & (\partial_z^{1/p} - \lambda_j^{-1}(\partial_t^{1/p}))^{k+1}v(t, z) = 0.
 \end{aligned}$$

□

Now, we are ready to prove the main result

Theorem 1. *Let $s > 1$ and $d \in \mathbb{R}$. Suppose that $v(t, z)$ is a solution of (8) with the initial data (9). Moreover, suppose that $\{\lambda_1, \dots, \lambda_l\}$ is the set of the characteristic roots of*

$$P_m(\lambda, 1) = \lambda^m - \sum_{j=1}^m \lambda^{m-j} p_j = 0 \quad \text{with } p_j := \lim_{\xi \rightarrow \infty} P_j(\xi)/\xi^j.$$

Then $v(t, z) \in \mathcal{O}_{1/p}(S(d, \delta) \times D(r))$ (for some $\delta > 0$ and $r > 0$) is of exponential growth of order at most s as $t \rightarrow \infty$ if and only if for every $j = 1, \dots, l$ and $n = 0, \dots, m-1$, the Cauchy data $\varphi_n(z) \in \mathcal{O}_{1/p}(S(d + p \arg \lambda_j, \tilde{\delta}))$ (with some $\tilde{\delta} > 0$) has the same exponential growth of order at most s at infinity.

Proof. (\Leftarrow) Without loss of generality we may assume that the initial data satisfy (10) for $\varphi(z) \in \mathcal{O}_{1/p}(S(d + p \arg \lambda_j, \tilde{\delta}))$, which has the exponential growth of order at most s as $z \rightarrow \infty$. By Lemma 3, $v(t, z) = \sum_{j=1}^l v_j(t, z)$, where $v_j(t, z) \in \mathcal{O}_{1/p}(S(d, \delta) \times D(r))$ have the exponential growth of order at most s as $t \rightarrow \infty$. It means that also $v(t, z) \in \mathcal{O}_{1/p}(S(d, \delta) \times D(r))$ satisfies the same condition of the growth.

(\Rightarrow) If $v(t, z) \in \mathcal{O}_{1/p}(S(d, \delta) \times D(r))$ is a solution of (8) and is of exponential growth of order s , then $v(t, z)$ also satisfies the following Cauchy problem in z -direction

$$P(\partial_t^{1/p}, \partial_z^{1/p})v(t, z) = 0, \quad (\partial_t^{1/p})^n v(t, 0) = \psi_n(t) \quad \text{for } n = 0, \dots, m-1,$$

where $\psi_n(t) \in \mathcal{O}_{1/p}(S(d, \delta))$ are of exponential growth of order at most s and

$$\begin{aligned}
 P(\partial_t^{1/p}, \partial_z^{1/p}) &= (\partial_t^{1/p})^m - \sum_{j=1}^m (\partial_t^{1/p})^{m-j} P_j(\partial_z^{1/p}) \\
 &= c((\partial_z^{1/p})^m - \sum_{j=1}^m (\partial_z^{1/p})^{m-j} \tilde{P}_j(\partial_t^{1/p}))
 \end{aligned}$$

with some polynomials $\tilde{P}_j(\xi)$ satisfying $\deg \tilde{P}_j(\xi) \leq j$ and $\deg \tilde{P}_m(\xi) = m$.

As in a previous case, we may put

$$\psi_0(t) := \psi(t), \quad \psi_n(t) := \sum_{j=1}^n \tilde{P}_j(\partial_t^{1/p})\psi_{n-j}(t) \quad (n = 1, \dots, m-1)$$

for some $\psi(t) \in \mathcal{O}_{1/p}(S(d, \delta))$, which has the exponential growth of order at most s as $t \rightarrow \infty$.

Interchanging the roles of coordinates (t, z) and repeating the proof of Lemma 3 we obtain

$$(27) \quad v(t, z) = \sum_{j=1}^l \tilde{v}_j(t, z),$$

where

$$\begin{aligned} \tilde{v}_j(t, z) &= \frac{1}{2p\pi i} \oint_{|s|=\varepsilon}^p \psi(s) \times \\ &\times \int_{\tau_0}^{\infty(\tilde{\theta}_j)} \sum_{k=0}^{m_j-1} \tilde{c}_{jk}(\tau^{1/p}) r^k(z, \partial_z^{1/p}) e_{1/p}(z \lambda_j^{-p}(\tau^{1/p})) e_{1/p}(t\tau) e^{-s\tau} d\tau ds. \end{aligned}$$

Moreover, since

$$d - \arg(\lim_{\tau \rightarrow \infty} \lambda_j^{-p}(\tau^{1/p})/\tau) = d - \arg \lambda_j^{-p} = d + p \arg \lambda_j,$$

we conclude that $\tilde{v}_j(t, z) \in \mathcal{O}_{1/p}(D(\tilde{r}) \times S(d + p \arg \lambda_j, \tilde{\delta}))$ is of exponential growth of order at most s as $z \rightarrow \infty$.

By Lemmas 4 and 6, $\tilde{v}_j(t, z)$ satisfies the formula

$$(28) \quad (\partial_t^{1/p} - \lambda_j(\partial_z^{1/p}))^{m_j} \tilde{v}_j(t, z) = 0 \quad \text{for } j = 1, \dots, l.$$

In a similar way to [4] we define for $j = 1, \dots, l$

$$P_j(\partial_t^{1/p}, \partial_z^{1/p}) := (\partial_t^{1/p} - \lambda_j(\partial_z^{1/p}))^{m_j-1} \prod_{k=1, k \neq j}^l (\partial_t^{1/p} - \lambda_k(\partial_z^{1/p}))^{m_k}$$

and

$$\bar{v}_j(t, z) := P_j(\partial_t^{1/p}, \partial_z^{1/p}) v(t, z).$$

Observe that by (27) and (28)

$$\bar{v}_j(t, z) = P_j(\partial_t^{1/p}, \partial_z^{1/p}) \tilde{v}_j(t, z)$$

and, in consequence, $\bar{v}_j(t, z) \in \mathcal{O}_{1/p}(D(\tilde{r}) \times S(d + p \arg \lambda_j, \tilde{\delta}))$ is of exponential growth of order s as $z \rightarrow \infty$.

Without loss of generality we may assume that

$$(\partial_t^{1/p})^n v(0, z) = 0 \text{ for } n < m - 1, \quad (\partial_t^{1/p})^{m-1} v(0, z) = \varphi(z).$$

Hence $\bar{v}_j(0, z) = \varphi(z)$ and we conclude that $\varphi(z) \in \mathcal{O}_{1/p}(S(d + p \arg \lambda_j, \tilde{\delta}))$ has also the exponential growth of order at most s for $j = 1, \dots, l$. \square

7. SUMMABILITY OF SOLUTIONS TO LINEAR PDES

In this section we apply Theorem 1 to show the Borel summability properties for formal solutions of linear PDEs with constant coefficients given by (2).

By Theorem 1 we have

Proposition 6. *Let $v(t, z)$ be a solution of the initial value problem (7) and let $s = q/(q-p)$. Then $v(t, z)$ satisfies the properties a)–c) in Proposition 3 if and only if the Cauchy data $\varphi_n(z)$ are analytically continued to $S((d + \arg \alpha_j + 2k\pi)p/q, \tilde{\delta})$*

(for some $\tilde{\delta} > 0$) and are of exponential growth of order at most s for $n = 0, \dots, p-1$, $j = 1, \dots, l$, $k = 0, \dots, q-1$, where $\{\alpha_1, \dots, \alpha_l\}$ is the set of the characteristic roots of

$$(29) \quad \alpha^p - \sum_{j=1}^p \alpha^{p-j} \tilde{p}_j = 0 \quad \text{with} \quad \tilde{p}_j := \lim_{\zeta \rightarrow \infty} P_{jq/p}(\zeta) / \zeta^{jq/p}.$$

Proof. (\implies) Observe that $v(t, z) \in \mathcal{O}_{1/p}(S((d + 2k\pi)p/q, \delta) \times D(r))$ (for some $\delta, r > 0$) is of exponential growth of order at most s as $t \rightarrow \infty$. According to Theorem 1, $\varphi_n(z)$ are $1/p$ -analytic on $S((d + 2k\pi)p/q + p \arg \lambda_j, \tilde{\delta})$ (for some $\tilde{\delta}$) and are of exponential growth of order at most s for $n = 0, \dots, p-1$, $k = 0, \dots, q-1$ and $j = 1, \dots, \tilde{l}$, where $\{\lambda_1, \dots, \lambda_{\tilde{l}}\}$ is the set of the roots of the characteristic equation

$$(30) \quad \lambda^{pq} - \sum_{j=1}^p \lambda^{(p-j)q} \tilde{p}_j = 0.$$

Note that λ is the root of (30) if and only if $\alpha = \lambda^q$ satisfies (29). It means that $\varphi_n(z) \in \mathcal{O}_{1/p}(S((d + \arg \alpha_j + 2\pi k)p/q, \tilde{\delta}))$ is of exponential growth of order at most s there. Moreover, since $\varphi_n(z) \in \mathcal{O}(D(\tilde{r}))$ (for some $\tilde{r} > 0$), we have also $\varphi_n(z) \in \mathcal{O}(S((d + \arg \alpha_j + 2\pi k)p/q, \tilde{\delta}))$ for $n = 0, \dots, p-1$, $k = 0, \dots, q-1$ and $j = 1, \dots, l$.

(\impliedby) If $\varphi_n(z) \in \mathcal{O}_{1/p}(S((d + 2\pi k + \arg \alpha_j)p/q, \tilde{\delta}))$ are of exponential growth of order at most s then also $\varphi_n(z) \in \mathcal{O}_{1/p}(S((d + 2\pi k)p/q + p \arg \lambda_j, \tilde{\delta}))$ ($n = 0, \dots, p-1$, $k = 0, \dots, q-1$, $j = 1, \dots, \tilde{l}$), where $\{\lambda_1, \dots, \lambda_{\tilde{l}}\}$ is the set of the characteristic roots of (30). Therefore, by Theorem 1, $v(t, z) \in \mathcal{O}_{1/p}(S((d + 2\pi k)p/q, \delta) \times D(r))$ is of exponential growth of order s as $t \rightarrow \infty$. Finally, since $\varphi_n(z) \in \mathcal{O}(D(\tilde{r}))$ and the Cauchy data in (7) are non-vanishing only for multiplicity of $(\partial_t^{1/p})^q$ -derivative of $v(0, z)$, and since the linear operator $P(\partial_t^{1/p}, \partial_z^{1/p})$ in (7) depends only on $(\partial_t^{1/p})^q$ and $(\partial_z^{1/p})^p$, we obtain the desired conclusion. \square

Combining Propositions 3, 5 and 6 we obtain

Theorem 2. *Let $\hat{u}(t, z)$ be a formal power series solution of the initial value problem*

$$\partial_t^p u(t, z) = \sum_{j=1}^p \partial_t^{p-j} P_{jq/p}(\partial_z) u(t, z), \quad \partial_t^n u(0, z) = \varphi_n(z) \quad \text{for } n = 0, \dots, p-1,$$

where $t, z \in \mathbb{C}$, $p, q \in \mathbb{N}$, $p < q$, $\deg P_{jq/p}(\xi) \leq jq/p$ ($j = 1, \dots, p$), $\deg P_q(\xi) = q$ and $\varphi_n(z)$ are analytic in a complex neighbourhood of the origin.

Then the formal series $\hat{u}(t, z)$ is $p/(q-p)$ -summable in a direction d if and only if the Cauchy data $\varphi_n(z)$ are analytically continued to the set $S((d + \arg \alpha_j + 2\pi k)p/q, \tilde{\delta})$ (with some $\tilde{\delta} > 0$) for $k = 0, \dots, q-1$, $j = 1, \dots, l$ and this analytic continuation is of exponential growth of order $q/(q-p)$ as $z \rightarrow \infty$, where $\{\alpha_1, \dots, \alpha_l\}$ is the set of the roots satisfying the characteristic equation (29).

8. SUMMABILITY OF SOLUTION TO FRACTIONAL LINEAR PDES

In this section we consider the non-Kowalevskian fractional linear PDEs in two complex variables with constant coefficients. As in the previous section, we apply Theorem 1 to the study of the summability property for formal solutions of these

equations. First, we extend the definition of summability to a wider class of formal power series.

Definition 7. Let $\gamma > 0$. The Banach space of γ -analytic functions on $D(r)$, continuous on its closure and equipped with the norm $\|\varphi\|_r := \max_{|z| \leq r} |\varphi(z)|$ is denoted by $\mathbb{E}_\gamma(r)$.

The space of formal power series

$$\hat{u}(t, z) = \sum_{j=0}^{\infty} u_j(z) t^j \quad \text{with} \quad u_j(z) \in \mathbb{E}_\gamma(r)$$

is denoted by $\mathbb{E}_\gamma(r)[[t]]$. Moreover, we set $\mathbb{E}_\gamma[[t]] := \bigcup_{r>0} \mathbb{E}_\gamma(r)[[t]]$.

Definition 8. Let $\alpha, \gamma > 0$, $k > 0$ and $d \in \mathbb{R}$. We say that $\hat{u}(t, z) \in \mathbb{E}_\gamma[[t^\alpha]]$ is k -summable in directions $d + 2j\pi/\alpha$ ($j \in \mathbb{Z}$) if and only if the formal power series $\hat{w}(t, z) := \hat{u}(t^{1/\alpha}, z)$ is k/α -summable with respect to t in a direction αd .

Let us suppose that $\hat{u}(t, z) = \sum_{j=0}^{\infty} \frac{u_j(z)}{\Gamma(1+\alpha j)} t^{\alpha j}$. Then $\hat{w}(t, z) = \sum_{j=0}^{\infty} \frac{u_j(z)}{\Gamma(1+\alpha j)} t^j$. Using k/α -Borel transform of $\hat{w}(t, z)$ we obtain the series $\sum_{j=0}^{\infty} \frac{u_j(z)}{\Gamma(1+\alpha j)\Gamma(1+\alpha j/k)} t^j$. By the theory of moment summability, we may replace this transform by the following α -modified k/α -Borel transform of $\hat{w}(t, z)$, which is defined by

$$\mathcal{B}_\alpha^{k/\alpha} \hat{w}(t, z) := \sum_{j=0}^{\infty} \frac{u_j(z)}{\Gamma(1 + j\alpha(1 + 1/k))} t^j.$$

Observe that this modified transform is connected with the operator $B^{\alpha, \alpha(1+1/k)}$ by the formula

$$\mathcal{B}_\alpha^{k/\alpha} \hat{w}(t, z) = B^{\alpha, \alpha(1+1/k)} \hat{u}(t^{k/(\alpha(k+1))}, z).$$

It means that we have the following characterisation of k -summability

Proposition 7. *The formal series $\hat{u}(t, z) \in \mathbb{E}_\gamma[[t^\alpha]]$ is k -summable in directions $d + 2j\pi/\alpha$ ($j \in \mathbb{Z}$) if and only if the function $v(t, z) := B^{\alpha, \alpha(1+1/k)} \hat{u}(t, z)$ satisfies the following conditions:*

- a) $z \mapsto v(t, z)$ is γ -analytic in some complex neighbourhood of the origin in \mathbb{C} ,
- b) $t \mapsto v(t, z)$ is $\alpha(1 + 1/k)$ -analytic in some complex neighbourhood of the origin in \mathbb{C} and can be $\alpha(1 + 1/k)$ -analytically continued to some sectors in directions $\frac{d+2j\pi/\alpha}{1+1/k}$ ($j \in \mathbb{Z}$) and has the exponential growth of order at most $k + 1$ there.

Now, we prove

Proposition 8. *Let $\alpha \in (0, 1)$, $m \in \mathbb{N}$ and $\hat{u}(t, z)$ be a formal power series solution of the Cauchy problem*

$$(\partial_t^\alpha)^m u(t, z) = \sum_{j=1}^m (\partial_t^\alpha)^{m-j} P_j(\partial_z) u(t, z), \quad (\partial_t^\alpha)^n u(0, z) = \varphi_n(z) \quad n = 0, \dots, m-1,$$

where $\deg P_j(\xi) \leq j$, $\deg P_m(\xi) = m$ and $\varphi_n(z)$ are analytic in some complex neighbourhood of the origin. Then $\hat{u}(t, z)$ is $\alpha/(1 - \alpha)$ -summable in directions $d + 2j\pi/\alpha$ ($j \in \mathbb{Z}$) if and only if $\varphi_n(z)$ can be analytically continued to some sectors in

directions $\alpha d + \arg \lambda$ and are of exponential growth of order at most $1/(1 - \alpha)$ there for every λ satisfying the characteristic equation

$$(31) \quad \lambda^m - \sum_{j=1}^m \lambda^{m-j} p_j = 0 \quad \text{with} \quad p_j := \lim_{\xi \rightarrow \infty} P_j(\xi)/\xi^j.$$

Proof. By Proposition 7, $\hat{u}(t, z) \in \mathbb{E}[[t^\alpha]]$ is $\alpha/(1 - \alpha)$ -summable in directions $d + 2j\pi/\alpha$ if and only if the function $v(t, z) := B^{\alpha,1}\hat{u}(t, z)$ is analytic on the set $S(\alpha d, \delta) \times D(r)$ (for some $\delta > 0$ and $r > 0$) and is of exponential growth of order at most $1/(1 - \alpha)$ as $t \rightarrow \infty$. By Proposition 4, $v(t, z)$ is a solution of the Cauchy problem

$$\partial_t^m v(t, z) = \sum_{j=1}^m \partial_t^{m-j} P_j(\partial_z) v(t, z), \quad \partial_t^n v(0, z) = \varphi_n(z) \quad n = 0, \dots, m-1.$$

By Theorem 1, $v(t, z) \in \mathcal{O}(S(\alpha d, \delta) \times D(r))$ (for some $\delta > 0$ and $r > 0$) is of exponential growth of order at most $1/(1 - \alpha)$ as $t \rightarrow \infty$ if and only if $\varphi_n(z) \in \mathcal{O}(S(\alpha d + \arg \lambda_j, \tilde{\delta}))$ (with some $\tilde{\delta} > 0$) for every $n = 0, \dots, m-1$ and $j = 1, \dots, l$, where $\{\lambda_1, \dots, \lambda_l\}$ is the set of the characteristic roots of (31) and $\varphi_n(z)$ are of exponential growth of order at most $1/(1 - \alpha)$. \square

Corollary 1. *Let $\beta \in (0, 2)$. The formal solution $\hat{u}(t, z)$ of the fractional wave-diffusion equation*

$$\partial_t^\beta u(t, z) = \partial_z^2 u(t, z), \quad u(0, z) = \varphi(z) \in \mathcal{O}(D(r)) \quad \text{with some } r > 0$$

is $\beta/(2 - \beta)$ -summable in directions $d + 2j\pi/\beta$ ($j \in \mathbb{Z}$) if and only if $\varphi(z) \in \mathcal{O}(S(\beta d/2, \delta) \cup S(\beta d/2 + \pi, \delta))$ (for some $\delta > 0$) is of exponential growth of order at most $2/(2 - \beta)$ there.

Proof. It is sufficient to apply Proposition 8 with $\alpha = \beta/2$, $m = 2$, $P_1(\xi) = 0$, $P_2(\xi) = \xi^2$, $\varphi_0(z) = \varphi(z)$ and $\varphi_1(z) = 0$. Observe that $\lambda_1 = 1$ and $\lambda_2 = -1$ are the roots of the characteristic equation. \square

Moreover, in a similar way to [7] one can generalise the above result to the n -dimensional diffusion-wave equation ($n > 1$) as follows

Corollary 2. *Let $\beta \in (0, 2)$ and $\Delta_z := \partial_{z_1}^2 + \dots + \partial_{z_n}^2$. The formal solution $\hat{u}(t, z)$ of the fractional wave-diffusion equation*

$$\partial_t^\beta u(t, z) = \Delta_z u(t, z), \quad u(0, z) = \varphi(z) \in \mathcal{O}(D^n(r)) \quad \text{with some } r > 0$$

is $\beta/(2 - \beta)$ -summable in directions $d + 2j\pi/\beta$ ($j \in \mathbb{Z}$) if and only if the function

$$\Phi_n(t, z) := \begin{cases} \int_{\partial B^n(1)} \varphi(z + tx) dS(x) & \text{if } n \text{ is odd} \\ \int_{B^n(1)} \frac{\varphi(z+tx) dx}{\sqrt{1-|x|^2}} & \text{if } n \text{ is even} \end{cases}$$

is analytically continued to infinity in some sectors in directions $\beta d/2$ and $\beta d/2 + \pi$ (with respect to t) and to some ball with a centre at the origin (with respect to z) and this continuation is of exponential growth of order at most $2/(2 - \beta)$ as $t \rightarrow \infty$.

Proof. By Proposition 7, $\hat{u}(t, z) \in \mathbb{E}[[t^\beta]]$ is $\beta/(2 - \beta)$ -summable in directions $d + 2j\pi/\beta$ ($j \in \mathbb{Z}$) if and only if the function $v(t, z) := B^{\beta,2}\hat{u}(t, z)$ can be analytically continued to $(S(\beta d/2, \delta) \cup S(\beta d/2 + \pi, \delta)) \times D^n(\tilde{r})$ (for some $\delta, \tilde{r} > 0$) and is of exponential growth of order at most $2/(2 - \beta)$ as $t \rightarrow \infty$.

On the other hand, $v(t, z)$ is a solution of the wave equation

$$\partial_t^2 v(t, z) = \Delta_z v(t, z), \quad v(0, z) = \varphi(z), \quad v_t(0, z) = 0.$$

The assertion follows by the generalisation of the Kirchhoff and Poisson formula for the solution of the wave equation. \square

We also apply the similar methods to the fractional equations with respect to z . Namely, we have

Proposition 9. *Let $m, p, q \in \mathbb{N}$, $p > q$ and $\hat{u}(t, z)$ be a formal power series solution of the Cauchy problem*

$$\begin{aligned} (\partial_t^{1/p})^m u(t, z) &= \sum_{j=1}^m (\partial_t^{1/p})^{m-j} P_j(\partial_z^{1/q}) u(t, z), \\ (\partial_t^{1/p})^n u(0, z) &= \varphi_n(z) \quad \text{for } n = 0, \dots, m-1, \end{aligned}$$

where $\deg P_j(\xi) \leq j$, $\deg P_m(\xi) = m$ and $\varphi_n(z)$ are $1/q$ -analytic in some complex neighbourhood of the origin. Then $\hat{u}(t, z)$ is $q/(p-q)$ -summable in a direction d if and only if $\varphi_n(z)$ can be $1/q$ -analytically continued to some sectors in directions $q(d/p + \arg \lambda)$ and are of exponential growth of order at most $p/(p-q)$ there for every λ satisfying the characteristic equation (31).

Proof. We repeat the proof of Proposition 8 with $v(t, z) := B^{1/p, 1/q} \hat{u}(t, z)$. \square

Proposition 10. *Let $m, q \in \mathbb{N}$, $\alpha \in (0, 1/q)$ and $\hat{u}(t, z)$ be a formal power series solution of the Cauchy problem*

$$\begin{aligned} (\partial_t^\alpha)^m u(t, z) &= \sum_{j=1}^m (\partial_t^\alpha)^{m-j} P_j(\partial_z^{1/q}) u(t, z), \\ (\partial_t^\alpha)^n u(0, z) &= \varphi_n(z) \quad \text{for } n = 0, \dots, m-1, \end{aligned}$$

where $\deg P_j(\xi) \leq j$, $\deg P_m(\xi) = m$ and $\varphi_n(z)$ are $1/q$ -analytic in some complex neighbourhood of the origin. Then $\hat{u}(t, z)$ is $q\alpha/(1-q\alpha)$ -summable in directions $d + 2j\pi/\alpha$ ($j \in \mathbb{Z}$) if and only if $\varphi_n(z)$ can be $1/q$ -analytically continued to some sectors in directions $q(\alpha d + \arg \lambda)$ and are of exponential growth of order at most $1/(1-q\alpha)$ there for every λ satisfying the characteristic equation (31).

Proof. As in a previous case, we repeat the proof of Proposition 8 with $v(t, z) := B^{\alpha, 1/q} \hat{u}(t, z)$. \square

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FACULTY OF MATHEMATICS AND NATURAL SCIENCES, COLLEGE OF SCIENCE, CARDINAL STEFAN WYSZYŃSKI UNIVERSITY, WÓYCICKIEGO 1/3, 01-938 WARSZAWA, POLAND

INSTITUTE OF MATHEMATICS POLISH ACADEMY OF SCIENCES, P.O. BOX 137, ŚNIADECKICH 8 00-950 WARSZAWA, POLAND

E-mail address: s.michalik@uksw.edu.pl

URL: www.impan.gov.pl/~slawek