

Remarks on time map for quasilinear equations

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Abstract

We present two different generalizations of R. Schaaf's [18] time map formula to quasilinear equations, including the case of p -Laplacian. We give conditions for monotonicity and for convexity of the time map, which imply uniqueness or multiplicity results for the corresponding Dirichlet boundary value problem. Our time map formulas can be also used for effective computations of the global solution curves.

Key words: Multiplicity of solutions, time maps, quasilinear equations, p -Laplacian.

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1 Introduction

The p -Laplace operator plays the fundamental role in nonlinear analysis. It serves as a model quasilinear equation both in pure mathematics and in various areas of the applied sciences (see e.g. [1, 4, 5, 9, 11, 13]). One of the mainstreams of the p -harmonic theory, with plethora of papers published in recent years, is the so-called *nonlinear eigenvalue problem* (see e.g. [2, 10, 16] and references therein). In the simplest form it reads

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda|u|^{p-2}u, \quad \lambda \in \mathbb{R}.$$

For $p = 2$ we retrieve familiar harmonic eigenvalue problem. In the more general case one considers on the right hand side functions $f = f(u)$ and investigates the solvability of arising boundary problems, as well as the multiplicity of solutions, existence of positive solutions, etc.

In this note we deal with one-dimensional quasilinear equations of p -harmonic type. The need for study of such equations arises for instance in relation to radial solutions of PDEs (see e.g. [8, 12]).

Let $u = u(t)$ be the unique solution of the initial value problem,

$$\varphi(u')' + f(u) = 0, \quad u(0) = 0, \quad u'(0) = r > 0. \quad (1.1)$$

Our main example will be the p -Laplacian case when $\varphi(s) = s|s|^{p-2}$, with a constant $p > 1$. Using ballistic analogy, we can interpret the initial value problem (1.1) as “shooting” from the ground level, at an angle $r > 0$. Let $T/2$ denote the time it takes for the projectile to reach its maximum amplitude α , $\alpha = \alpha(r)$. By symmetry of positive solutions, $T = T(r)$ is then the time

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when the projectile falls back to the ground, the *time map*. Then $u(t)$ is a positive solution of the Dirichlet problem (here $u = u(t)$)

$$\varphi(u')' + f(u) = 0 \quad \text{for } 0 < t < T, \quad u(0) = u(T) = 0. \quad (1.2)$$

The time map can be used to study uniqueness and multiplicity of solutions for the boundary value problem (1.2). Indeed, if the time map is monotone, then the problem (1.2) has at most one solution. If the time map is convex, or concave, then the problem (1.2) has at most two solutions. While the time map is uniquely defined, there are several different ways to express it, see e.g. P. Korman [14] for a discussion in case $p = 2$. Some derivations of the time map $T(r)$ begin with concept of energy, as in I. Addou and S.-H. Wang [3], J. Cheng [6], or H. Pan [17]. Another one, due to R. Schaaf [18], uses a change of variables to globally linearize the equation (see e.g. [19] for more discussion of Schaaf's ideas). For the case of p -Laplacian, it is known that the role of a linear equation is played by a certain quasilinear problem, which we refer to as *p-linear problem*. Similarly to R. Schaaf [18], we develop a formula for time map by globally p -linearizing the equation. We show that in addition to $A - B$ conditions (see [18, Chapter 1.4]), an extra condition is needed in case $p \neq 2$ to prove convexity of the time map. The extension is not straightforward, as we needed to find the right auxiliary function, satisfying the appropriate differential inequality.

By a completely different technique, involving parametric integration of the equation, we derive another formula for the time map, which is valid for general quasilinear equations, including in particular the case of p -Laplacian. In addition to proving multiplicity results, both formulas are suitable for computation of bifurcation diagrams.

Despite considering the one-dimensional case only, the analysis of problem (1.2) leads to tedious computations and conditions involving complicated differential expressions. However, we would like to emphasize that these conditions are not of technical nature, but arise as a consequence of nonlinearity of functions ϕ and f (see also Remark 4).

2 The p -linear problem

The purpose of this section is to solve the basic p -linear problem in order to illustrate the differences between the linear and nonlinear problems and shed some light on the difficulties arising in the nonlinear case. The discussion of the more general case will be provided in further sections.

In the Laplacian case, $p = 2$, the easily solvable linear problem is, of course, when the function $f(u)$ is linear. For the p -Laplacian case the corresponding easy problem arises when $f(u)$ is a multiple of $\varphi(u)$. Indeed, let us solve the equation (for $u = u(t)$)

$$\varphi(u'(t))' + (p - 1)\varphi(u(t)) = 0. \quad (2.1)$$

Let us denote $\Phi(u) = \int_0^u \varphi(s) ds = \frac{1}{p}|u|^p$. (The function $\Phi(u)$ is even, because $\varphi(s)$ is odd, so compute it first for $u > 0$). We have $\varphi'(s) = (p - 1)|s|^{p-2}$ and so

$$s\varphi'(s) = (p - 1)\varphi(s). \quad (2.2)$$

Multiplying the equation (2.1) by u' , we have by (2.2)

$$\begin{aligned} u'(t)\varphi'(u'(t))u''(t) + (p - 1)u'(t)\varphi(u(t)) &= 0, \\ (p - 1)\varphi(u'(t))u''(t) + (p - 1)u'(t)\varphi(u(t)) &= 0, \end{aligned}$$

$$\begin{aligned}\frac{d}{dt} [\Phi(u'(t)) + \Phi(u(t))] &= 0, \\ \Phi(u'(t)) + \Phi(u(t)) &= \text{constant}.\end{aligned}\tag{2.3}$$

Let now $\sin_p t$ denote the solution of the initial value problem

$$\varphi(u'(t))' + (p-1)\varphi(u(t)) = 0 \quad u(0) = 0, \quad u'(0) = 1.\tag{2.4}$$

Observe that $\sin_p t$ is a generalization of $\sin t = \sin_2 t$, as was noticed by P. Lindqvist [15], see also Y. Cheng [7]. For $u = \sin_p t$ the ‘‘energy integral’’ (2.3) becomes

$$\Phi(u'(t)) + \Phi(u(t)) = \frac{1}{p}.\tag{2.5}$$

We see that at its critical points $\sin_p t = \pm 1$. Let $\frac{\pi_p}{2}$ denote the first point of maximum of $\sin_p t$ (i.e. $\sin_p \frac{\pi_p}{2} = 1$). It is easy to see from the equation (2.1) that $\sin_p t$ is symmetric with respect to $\frac{\pi_p}{2}$, and hence $\sin_p \pi_p = 0$, with π_p being the first root of $\sin_p t$. Also from (2.5)

$$u'(t) = \sqrt[p]{1 - u^p(t)}, \quad \text{for } t \in (0, \frac{\pi_p}{2}),$$

which allows us to express $\sin_p t$ as an elliptic integral. In particular,

$$\frac{\pi_p}{2} = \int_0^1 \frac{du}{\sqrt[p]{1 - u^p}} = \frac{\pi}{p \sin \frac{\pi}{p}}.$$

(To compute the last integral using *Mathematica*, enter the command: `Integrate[$\frac{1}{\sqrt[p]{1-u^p}}$, {u, 0, 1}], Assumptions $\rightarrow p > 1$].) Observe that $\pi_2 = \pi$, as expected.`

3 Time map through global p -linearization

We are interested in positive solutions of the two point problem for $u = u(t)$

$$\varphi(u'(t))' + f(u(t)) = 0, \quad 0 < t < T, \quad u(0) = u(T) = 0,\tag{3.1}$$

where $\varphi(s) = s|s|^{p-2}$, with a constant $p > 1$. We do not consider the end point T to be fixed, but rather depending on $r = u'(0)$. I.e., we wish to solve the initial value problem

$$\varphi(u'(t))' + f(u(t)) = 0 \quad u(0) = 0, \quad u'(0) = r,\tag{3.2}$$

and calculate the first root T . We transform (3.2) into the system form

$$\begin{aligned}u' &= y \\ \varphi(y)' &= -f(u) \\ u(0) &= 0, \quad y(0) = r.\end{aligned}\tag{3.3}$$

With $F(u) = \int_0^u f(t) dt$, define the function $g(x)$, by the relation

$$F(g(x)) = \frac{p-1}{p}|x|^p, \quad \text{sgn } g(x) = \text{sgn } x.\tag{3.4}$$

Observe that $g(0) = 0$. Differentiate (3.4)

$$f(g(x))g'(x) = (p-1)\varphi(x).\tag{3.5}$$

In (3.3) we let $u = g(x)$, then multiply the second equation by $g'(x)$, and use (3.5)

$$\begin{aligned} g'(x)x' &= y, \\ g'(x)\varphi(y)' &= -f(g(x))g'(x) = -(p-1)\varphi(x), \\ x(0) &= 0, \quad y(0) = r. \end{aligned} \tag{3.6}$$

We now change the independent variable in (3.6), $t \rightarrow \theta$, by solving

$$\frac{dt}{d\theta} = g'(x(t)), \quad t(0) = 0. \tag{3.7}$$

Then the chain rule allows us to transform system (3.6) into

$$\begin{aligned} \frac{dx}{d\theta} &= y \\ \frac{d}{d\theta}\varphi(y) &= -(p-1)\varphi(x) \\ x(0) &= 0, \quad y(0) = r, \end{aligned}$$

or equivalently

$$\frac{d}{d\theta}\varphi\left(\frac{dx}{d\theta}\right) + (p-1)\varphi(x) = 0, \quad x(0) = 0, \quad x'(0) = r. \tag{3.8}$$

The problem has been p -linearized! Solution of (3.8) is

$$x = r \sin_p \theta.$$

The first positive root of x occurs at $\theta = \pi_p$. Using this relation in (3.7), and integrating, we have the formula for the time map

$$T = \int_0^{\pi_p} g'(r \sin_p \theta) d\theta. \tag{3.9}$$

In case $p = 2$ this formula was derived by R. Schaaf [18] (recall that $\pi_2 = \pi$ and $\sin_2 \theta = \sin \theta$).

What are the conditions on $f(u)$? Depends on whether one wants to study positive or sign-changing solutions of (3.1). For positive solutions, that we study in this paper, we shall assume that $f(u) \in C^2([0, a])$, for some $a > 0$, and it satisfies

$$f(u) > 0 \quad \text{for } u \in [0, a]; \tag{3.10}$$

$$\text{either } f(0) > 0, \quad \text{or } f(0) = 0 \text{ and } f'(0) > 0. \tag{3.11}$$

Under condition (3.10), the inverse function F^{-1} is defined, and

$$g(x) = F^{-1}\left(\frac{p-1}{p}|x|^p\right) = F^{-1}\left(\frac{p-1}{p}x^p\right),$$

since $x \geq 0$ for positive solutions.

Remark 1. When solving for $\theta = \theta(t)$ in (3.7), we have $\theta(t) = \int_0^t \frac{ds}{g'(x(s))}$, which for $f(0) > 0$ becomes an improper integral at $s = 0$. Nevertheless, this integral converges. Indeed, with $u(t)$ denoting a solution of (3.1), we have for small t , $u(t) \sim u'(0)t$, $F(u(t)) \sim f(0)u(t) \sim f(0)u'(0)t$. Then denoting by c_i various positive constants, we have

$$x(s) = \left[\frac{p}{p-1} F(g(x(s))) \right]^{1/p} = c_1 F(u(s))^{1/p} \sim c_2 s^{1/p},$$

for s small enough. Therefore

$$g'(x(s)) \sim \frac{p-1}{f(0)} \varphi(x(s)) = c_3 s^{\frac{p-1}{p}} \quad \text{for } s \text{ small,}$$

and the integral converges.

Similarly to R. Schaaf [18], one could also study sign-changing solutions under the assumption that $f(u) \in C^2(R)$ satisfies $f(0) = 0$, and

$$uf(u) > 0 \quad \text{for all } u \in R, \text{ and } f'(0) > 0.$$

4 Multiplicity of positive solutions of the p -Laplace equation

Derivatives of the time map are easy to compute:

$$T'(r) = \int_0^{\pi_p} g''(r \sin_p \theta) \sin_p \theta \, d\theta; \quad (4.1)$$

$$T''(r) = \int_0^{\pi_p} g'''(r \sin_p \theta) \sin_p^2 \theta \, d\theta. \quad (4.2)$$

Since we study positive solutions, we shall assume that $f(u)$ satisfies the conditions (3.10) and (3.11). We need to compute the derivatives of function $g(x)$, defined in (3.4). The first order derivative was already computed in (3.5)

$$g'(x) = (p-1) \frac{\varphi(x)}{f(g(x))}. \quad (4.3)$$

Observe that

$$g'(x) > 0, \quad \text{for } x > 0. \quad (4.4)$$

Using (4.3), we compute

$$g''(x) = (p-1) \frac{\varphi'(x) f^2(g(x)) - (p-1) \varphi^2(x) f'(g(x))}{f^3(g(x))}. \quad (4.5)$$

In order to make the next derivative $g'''(x)$ manageable, we use the idea of R. Schaaf [18]: rewrite $g''(x)$ as a function of $u = g(x)$. Starting with $\frac{p-1}{p} x^p = F(g(x)) = F(u)$, we express $x = \left[\frac{p}{p-1} F(u) \right]^{1/p}$, and then

$$\varphi'(x) = a F(u)^{\frac{p-2}{p}}, \quad \text{with } a \equiv (p-1) \left(\frac{p}{p-1} \right)^{\frac{p-2}{p}},$$

$$\varphi^2(x) = b F(u)^{\frac{2p-2}{p}}, \quad \text{with } b \equiv \left(\frac{p}{p-1} \right)^{\frac{2p-2}{p}}.$$

We rewrite

$$g''(x) = (p-1) \frac{a F(u)^{\frac{p-2}{p}} f^2(u) - (p-1) b F(u)^{\frac{2p-2}{p}} f'(u)}{f^3(u)} \quad \text{with } u = g(x), \quad (4.6)$$

and then compute

$$g'''(x) = \left(\frac{p}{p-1} \right)^{-\frac{2}{p}} \frac{v(u)}{f^4(u) F(u)^{2/p}} g'(x), \quad (4.7)$$

where

$$v(u) = (2 - 3p + p^2) f^4(u) - 3(p - 1) p f^2(u) F(u) f'(u) + 3p^2 F^2(u) f'^2(u) - p^2 f(u) F^2(u) f''(u). \quad (4.8)$$

The sign of $g'''(x)$ is the same as that of $v(u)$, in view of (4.4). So, if $v(u)$ is of one sign, the same is true $T''(r)$. We shall study the function $v(u)$ in the next two sections.

We have the following simple uniqueness result.

Proposition 1. *Assume that the function $(p - 1)f^2(u) - pF(u)f'(u)$ is of one sign for all $u \in (0, \infty)$. Then the problem (3.1) has at most one positive solution, for any $T > 0$.*

Proof: We see from (4.6) that $g''(x)$ is of one sign, and then it follows from (4.1) that the time map $T(r)$ is monotone. \diamond

5 The sign of the function $v(u)$ for small $u > 0$

We see from (4.2) and (4.7) that $T''(r)$ is of one sign, provided the same is true for $v(u)$. In case of $A - B$ functions (which we shall recall below) the strategy in R. Schaaf in [18] involved showing that $v(u) > 0$ for small $u > 0$, and that $v(u)$ cannot vanish. This implies that $v(u) > 0$ for $u > 0$. For C functions she showed that $v(u) < 0$ for small $u > 0$, and that $v(u)$ cannot vanish, giving that $v(u) < 0$ for $u > 0$. We shall proceed similarly. Compute

$$v(0) = (p - 1)(p - 2)f^4(0). \quad (5.1)$$

The repeated differentiation of $v(u)$ gives us the following lemma, crucial for the discussion in Section 6.

Lemma 5.1. *Assume that $f(0) = 0$. Then*

$$v(0) = v'(0) = v''(0) = v'''(0) = 0, \quad \text{and} \quad v''''(0) = 6(p - 2)(p - 4)f'^4(0). \quad (5.2)$$

Lemma 5.2. *Assume that any one of the following three conditions holds:*

- (i) $f(0) > 0$, and $p > 2$;
- (ii) $f(0) = 0$, and $f'(0) > 0$, and additionally $p > 4$;
- (iii) $f(0) = 0$, and $f'(0) > 0$, and additionally $1 < p < 2$.

Then $v(u) > 0$ for small $u > 0$.

Proof: In case (i) we use (5.1). In the cases (ii) and (iii), $v''''(0) > 0$, and by (5.2) the positivity of $v(u)$ follows. \diamond

6 Branches of positive solutions with at most one turn

Below we discuss the convexity of the time map. Such property allows us to give the upper estimate for the multiplicity of the solutions to (3.1).

Let us recall the notions of $A-$, $B-$ and $A - B$ functions introduced by R. Schaaf in [18].

Definition 6.1. *Let I be some interval and let $f : I \rightarrow \mathbb{R}$ be of class $C^2(I)$. Then*

(1) f is called an A -function on I if and only if

$$f' f''' - \frac{5}{3}(f'')^2 < 0 \quad \text{on } I.$$

(2) f is called an B -function on I if and only if

$$f f'' - 3(f')^2 \leq 0 \quad \text{on } I.$$

(3) f is called an $A - B$ -function on I if and only if

- a) f' has only simple zeros on I ,
- b) f is an A -function on any subinterval of I on which $f' > 0$,
- c) f is an B -function on any subinterval of I on which $f' < 0$.

Theorem 6.1. Let $f \in C^2[0, a]$ satisfy any one of the three conditions of Lemma 5.2. Assume also that

$$f'(u) > 0 \quad \text{for } u \in [0, a],$$

and f is an A -function on $[0, a]$. Assume finally that the following condition holds on $(0, a)$

$$(p-2) \left[-\frac{2}{p} f^2 f' + F \left((f')^2 - \frac{5}{3} f f'' \right) \right] \geq 0, \quad (6.1)$$

Then $T''(r) > 0$ on $(0, a)$.

Remark 2. For $p = 2$ Theorem 6.1 reduces to [18, Theorem 1.4.2].

Proof: The time map formula (3.9) immediately gives us that

$$T''(r) = \int_0^{\pi_p} (\sin_p \theta)^2 g'''(r \sin_p \theta) d\theta. \quad (6.2)$$

It is enough to prove that g''' is positive on the interval (b^-, b^+) , for

$$b^- = 0, \quad b^+ = \left(\frac{p}{p-1} F(a) \right)^{\frac{1}{p}}.$$

By (4.4) and (4.7), suffices to show that $v(u) > 0$ for $v(u)$ as in (4.8). The trick is to consider the function

$$h(u) = F^{-\frac{2}{p}} \left(\frac{1}{p}(p-1)(p-2)f^4 + pF^2(3(f')^2 - f f'') \right) - 3(p-1)F^{1-\frac{2}{p}} f^2 f', \quad (6.3)$$

which is positive on $(0, a)$ if and only if $v(u) > 0$ ($h(u) = \frac{1}{p} F^{-\frac{2}{p}} v(u)$). So,

$$g''' > 0 \quad \text{on } (0, b^+) \Leftrightarrow h > 0 \quad \text{on } (0, a).$$

By Lemma 5.2, $h(u)$ is positive for small u , and we shall show that $h(u)$ cannot vanish. Observe that h and h' contain the same expression:

$$\begin{aligned} h(u) &= F^{-\frac{2}{p}} \frac{1}{p}(p-1)(p-2)f^4 - p f F^{2-\frac{2}{p}} f'' \\ &\quad + \frac{1}{p} F^{-\frac{2}{p}} 3 f' \left[p^2 F^2 f' - (p-1) p f^2 F \right] \end{aligned} \quad (6.4)$$

$$\begin{aligned} h'(u) &= -2(p-2)(p-1)f^5 + p(p-1)(p-2)f^3 F f' \\ &\quad + F p \left(5 f'' \left[p^2 F^2 f' - (p-1) p f^2 F \right] - p^2 f F f''' \right). \end{aligned} \quad (6.5)$$

From this we find

$$f'h' = \frac{5}{3}f''h + pF^{2-\frac{2}{p}}f \left(\frac{5}{3}(f'')^2 - f'f''' \right) + \frac{1}{p}(p-2)(p-1)f^3F^{-\frac{p+2}{p}} \left[-\frac{2}{p}f^2f' + F \left((f')^2 - \frac{5}{3}ff'' \right) \right]. \quad (6.6)$$

Note that for $p = 2$ we recover R. Schaaf's [18, Formula (1-4-8)]:

$$f'h' = \frac{5}{3}f''h + 2Ff \left(\frac{5}{3}(f'')^2 - f'f''' \right).$$

If we assume that $h(u_0) = 0$ at some $u_0 \in (0, a)$, then the left hand side of (6.6) is less or equal to zero, while on the right hand side the first term is zero, the second one is positive (since f is A -function). This leads to contradiction, proving that $h(u) > 0$ for all $u > 0$. \diamond

Example 1. Let $f(u) = e^u - 1$, and $p = \frac{3}{2}$. Then $f(u)$ is an increasing A -function, while a computation shows that

$$-\frac{2}{p}f^2f' + F \left((f')^2 - \frac{5}{3}ff'' \right) < 0, \text{ for all } u > 0,$$

which implies that (6.1) holds. We conclude that the branch of positive solutions of (3.1) has at most one turn.

Theorem 6.2. Let $f \in C^2[0, a]$. Assume that either condition **(i)**, or condition **(ii)** of Lemma 5.2 holds. Also, let f be an $A - B$ -function on $[0, a]$. Assume finally that the following condition holds on $(0, a)$

$$-\frac{2}{p}f^2f' + F \left((f')^2 - \frac{5}{3}ff'' \right) \geq 0. \quad (6.7)$$

Then $T''(r) > 0$ on $(0, a)$.

Proof: Observe that both of our assumptions require that $p > 2$, and so the condition (6.7) is the same as (6.1). By Lemma 5.2, $h(u)$ is positive for small u . As in the preceding theorem, we see that $h(u)$ cannot vanish on any subinterval where it is an increasing A -function. Observe that $h(u)$ cannot vanish on any subinterval where it is a decreasing B function, since we see directly from (6.3) that the positivity of $h(u)$ follows from the assumptions that $p > 2$, $f' < 0$, and that f is a B -function. \diamond

R. Schaaf [18] has given a detailed study of $A - B$ -functions. In particular, an $A - B$ -function can have at most one critical point on the interval $(0, a)$, and it must be a strict maximum. The class of $A - B$ -functions includes all polynomials with simple roots. For example, $f(u) = u(1 - u^2)$ is an $A - B$ -function for $u \in (0, 1)$, and it is easy to see that the condition (6.7) holds (with a strict inequality) for large p , say for $p = 5$, and the last theorem applies.

Remark 3. In her book, R. Schaaf [18] has developed the concept of C functions. For C functions she was able to prove that $T''(r) < 0$ on $(0, a)$. It appears that no extension of this concept to $p \neq 2$ case is possible. Indeed, if $2 \leq p < 4$, then similarly to Lemma 5.2, we can prove that $h(u) < 0$ for small u . But then to achieve a contradiction at a possible root of $h(u)$, we need to require that $1 < p \leq 2$, i.e. $p = 2$ is the only case that this approach works.

Remark 4. Our numerical computations show that the condition (6.7) is not technical, i.e. the time map $T(r)$ need not be convex if this condition fails. Indeed, we take $f(u) = u + u^2$ and $p = 5$. Our computations show that the condition (6.7) is then violated, and the function $v(u)$ changes sign. (In the case $f(u) = u + u^2$ and $p = 2$, $v(u)$ is positive, in accordance with R. Schaaf [18].)

7 Time map by parametric integration

We wish to solve the initial value problem (here $u = u(x)$)

$$\varphi(u'(x))' + f(u(x)) = 0 \quad u(0) = 0, \quad u'(0) = r > 0, \quad (7.1)$$

and calculate the first root T , i.e. $u(T) = 0$. The map $T = T(r)$ is commonly referred to as the *time map*. We assume that the function $\varphi(t) \in C^1(\mathbb{R})$ is odd, and it satisfies

$$\varphi'(t) > 0 \quad \text{for all } t \in \mathbb{R}. \quad (7.2)$$

One example is the case of p -Laplacian, $\varphi(t) = t|t|^{p-2}$. Another example is $\varphi(t) = \frac{t}{\sqrt{1+t^2}}$ of prescribed mean curvature equation, see e.g. H. Pan [17]. We shall also assume that $f(u) \in C(\mathbb{R})$ satisfies

$$f(u) > 0 \quad \text{for all } u \geq 0. \quad (7.3)$$

It is easy to see that the solution $u(x)$ is symmetric with respect to the point of maximum. So that at $x = T/2$ the solution achieves its maximum α , $\alpha = u(T/2)$. We denote $\Psi(u) = \int_0^u t\varphi'(t) dt$, $\Psi'(u) = u\varphi'(u)$, and as usual $F(u) = \int_0^u f(t) dt$. We multiply the equation by u' , and transform it as follows

$$u'(x)\varphi'(u'(x))u''(x) + f(u(x))u'(x) = 0;$$

$$\frac{d}{dx} [\Psi(u') + F(u)] = 0;$$

$$\Psi(u'(x)) + F(u(x)) = \Psi(r) \quad \text{for all } x.$$

We now use parametric integration technique. Writing

$$F(u) = \Psi(r) - \Psi(u'), \quad (7.4)$$

we introduce a parameter t , by letting $u' = rt$. By (7.2) and (7.3), $u''(x) < 0$, and hence $u'(x)$ is monotone decreasing. When x varies over the interval $[0, T/2]$, t varies over $1 \geq t \geq 0$. Defining $G(u) = F^{-1}(u)$ (by (7.3) the inverse function is well defined for $u \geq 0$), we get from (7.4)

$$u = G(\Psi(r) - \Psi(rt)). \quad (7.5)$$

Then

$$dx = \frac{du}{u'(x)} = \frac{G'(\Psi(r) - \Psi(rt))(-rt\varphi'(rt)r)}{rt} dt = -G'(\Psi(r) - \Psi(rt))\varphi'(rt)r dt. \quad (7.6)$$

Integrating

$$T/2 = \int_0^1 G'(\Psi(r) - \Psi(rt))\varphi'(rt)r dt. \quad (7.7)$$

We have a formula for the time map. Moreover, one can easily get the solution of (7.1) in parametric form, from (7.5) and (7.6).

In the case of p -Laplacian, $\varphi(t) = t|t|^{p-2}$ and the formula becomes

$$T/2 = (p-1)r^{p-1} \int_0^1 G' \left(\frac{p-1}{p}r^p - \frac{p-1}{p}r^p t^p \right) t^{p-2} dt. \quad (7.8)$$

When $p = 2$, it is easy to transform this formula into the well-known R. Schaaf's formula [18]. Indeed, in case $p = 2$, we have

$$T = 2 \int_0^1 G' \left(\frac{1}{2}r^2 - \frac{1}{2}r^2t^2 \right) r dt = 2 \int_0^{\pi/2} G' \left(\frac{1}{2}r^2 \sin^2 \theta \right) r \sin \theta d\theta, \quad (7.9)$$

after a change of variables, $t = \cos \theta$. R. Schaaf [18] defines a function $g(x)$ by the relation

$$g(x) = F^{-1} \left(\frac{1}{2}x^2 \right) = G \left(\frac{1}{2}x^2 \right).$$

Clearly,

$$g'(x) = G' \left(\frac{1}{2}x^2 \right) x,$$

and so (7.9) gives us

$$T = 2 \int_0^{\pi/2} g'(r \sin \theta) d\theta = \int_0^\pi g'(r \sin \theta) d\theta,$$

which is R. Schaaf's time map formula, [18] (see discussion of (3.9) above).

Let us now study positive solutions of a parameter dependent problem for p -Laplacian, with $\varphi(t) = t|t|^{p-2}$:

$$\varphi(u'(x))' + \lambda f(u(x)) = 0, \quad \text{for } x \in (-1, 1), \quad u(-1) = u(1) = 0. \quad (7.10)$$

It is known that positive solutions of (7.10) are even functions, with $xu'(x) < 0$ for $x \in (-1, 1) \setminus \{0\}$, see e.g. P. Korman [14]. Hence, the global maximum occurs at $x = 0$. Let $\alpha = u(0)$ denote the maximum value of solution. We wish to draw the bifurcation diagram for this problem in (λ, α) plane. Consider an initial value problem

$$\varphi(v'(\xi))' + f(v(\xi)) = 0, \quad v(0) = \alpha, \quad v'(0) = 0. \quad (7.11)$$

The change of variables $x = \frac{1}{\lambda^{1/p}}\xi$ changes the equation in (7.10) into the one in (7.11). If R is the first root of v , then $R = \lambda^{1/p}$, i.e.

$$\lambda = R^p = (T/2)^p, \quad (7.12)$$

since $2R$ is the time map. From the energy relation $F(\alpha) = \Psi(r)$ (see (7.4)), i.e.

$$r = \Psi^{-1}(F(\alpha)). \quad (7.13)$$

For the given α , we compute r by (7.13), then T by (7.8), and then λ from (7.12). Numerically, for a mesh of α_i 's, we compute corresponding λ_i 's, and then plot the points (λ_i, α_i) , to get the bifurcation diagram.

Example 2. Consider the case of p -Laplacian, and $f(u) = e^u$. Compute $F(u) = e^u - 1$, $G(u) = \ln(u + 1)$. The formula (7.13) becomes

$$r = \left[\frac{p}{p-1} (e^\alpha - 1) \right]^{1/p}.$$

The formula (7.8) takes the form

$$T/2 = (p-1)r^{p-1} \int_0^1 \frac{t^{p-2}}{\frac{p-1}{p}r^p - \frac{p-1}{p}r^p t^p + 1} dt.$$

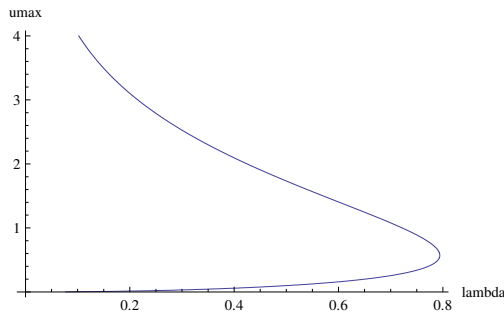


Figure 1: Bifurcation diagram for $\frac{3}{2}$ -Laplacian, and $f(u) = e^u$

Mathematica is able to compute this integral through hypergeometric functions for any $p > 1$. Combining these formulas with (7.12), we find the bifurcation diagram. In Figure 1 we present the computed bifurcation diagram for the case $p = \frac{3}{2}$. It is similar to the one in case $p = 2$, see e.g. P. Korman [14].

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