HARNACK'S INEQUALITY AND THE STRONG $p(\cdot)$-LAPLACIAN

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ABSTRACT. We study solutions of the strong $p(\cdot)$-Laplace equation. We show that, in contrast to $p(\cdot)$-Laplace solutions, these solutions satisfy the ordinary, scale-invariant Harnack inequality. As consequences we derive the strong maximum principle and global integrability of solutions.

1. INTRODUCTION

During the last decade, function spaces with variable exponent have attracted a lot of interest, as can be seen from the surveys [11, 37], the monograph [10] or the recent papers [5, 12, 21, 29]. The impetus for these studies was both natural theoretical developments and applications to electrorheological fluids [1, 36] and image processing [8, 9, 24].

Partial differential equations related to variable exponent Sobolev spaces have also been investigated by several researchers, see the surveys [15, 34] or papers [2, 6, 13, 14, 19, 23, 33, 40]. The usual way of generalizing the p-Laplacian to the setting of variable exponents is to start with the minimization problem

$$\inf\left\{ \int_\Omega |\nabla u|^p \, dx \,:\, u \in u_0 + W_0^{1,p}(\Omega) \right\},$$

or the weak form of the differential equation $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$. In the variable exponent case these lead to non-equivalent, although closely related, problems, namely,

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla \varphi \, dx = 0 \quad \text{or} \quad \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla \varphi \, dx = 0.$$

Recently, we introduced a new variant of the $p(\cdot)$-Laplacian in [3]. It is based on the strong form of the p-Laplace equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$, i.e.

$$\Delta_p u := |\nabla u|^{p-4}\big[(p-2)\,\Delta_\infty u + |\nabla u|^2 \,\Delta\, u\big] = 0,$$

where

$$\Delta_\infty u := \sum_{i,j} u_{x_i} u_{x_j} u_{x_i,x_j}$$

and u_{x_i} denotes the partial derivative. If p is replaced by $p(x)$, we arrive at yet another generalization of the p-Laplace equation, the *strong $p(\cdot)$-Laplacian*

$$\widetilde{\Delta}_{p(\cdot)} u := |\nabla u|^{p(\cdot)-4}\big[(p(\cdot)-2)\,\Delta_\infty u + |\nabla u|^2 \,\Delta\, u\big].$$

In order to state our problem in a weak form, we note that

$$\widetilde{\Delta}_{p(\cdot)} u = \operatorname{div}(|\nabla u|^{p(\cdot)-2}\nabla u) - |\nabla u|^{p(\cdot)-2} \log(|\nabla u|)\nabla u \cdot \nabla p$$

when $u \in C^2(\Omega)$. The weak formulation of $\widetilde{\Delta}_{p(\cdot)} u = 0$ then requires that $u \in W^{1,p(\cdot)}_{\text{loc}}(\Omega)$ satisfies

$$(\star) \qquad \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx + \int_{\Omega} |\nabla u|^{p(x)-2} \log(|\nabla u|) \nabla u \cdot \nabla p \, \varphi \, dx = 0$$

for all $\varphi \in W^{1,p(\cdot)}_0(\Omega)$. This equation, like the previous two generalizations, reduces to the ordinary p-Laplace equation when p is constant. At first sight the strong $p(\cdot)$-Laplacian seems to have a distinct disadvantage over the earlier introduced versions. For instance, we need to assume that $\nabla p \in L^n \log L^n(\Omega)$ for the second term to make sense. (This can be weakened to $\nabla p \in L^{p(\cdot)} \log L^{p(\cdot)}(\Omega)$ if we only test with $\varphi \in C^\infty_0(\Omega)$.)

However, we found in [3] that solutions of $(\star)$ possesses some advantages over $p(\cdot)$-solutions:

- *scalability*: if u is a solution, then so is λu;
- *geometric regularity*: if u is a solution in a planar domain, then the gradient ∇u is a mapping of finite distortion $K_p(x)$, with

$$K_p(x) = \frac{1}{2}\left(p(x) - 1 + \frac{1}{p(x) - 1}\right).$$

The former property is trivial for the case of constant p, while the later is a generalization of results by Bojarski and Iwaniec [7] ($p > 2$) and Manfredi [30] ($1 < p < \infty$). These results for the strong $p(\cdot)$-Laplacian are noteworthy since neither of them hold for the $p(\cdot)$-Laplacian (see [3, Example 3.1]).

The scalability of Equation $(\star)$ is a very useful feature. A reflection of the nonscalability of the (ordinary) $p(\cdot)$-Laplacian is that the constant c in the Harnack inequality

$$(1.1) \qquad \operatorname*{ess\,sup}_{x \in B} |u(x)| \leqslant c \left(\operatorname*{ess\,inf}_{x \in B} |u(x)| + |B|^{\frac{1}{n}} \right)$$

cannot be chosen independent of the non-negative solution u of the $\Delta_{p(\cdot)}$-equation [17, Example 3.10]. In this paper we show that Equation $(\star)$ is better than the $p(\cdot)$-Laplacian also in this respect by establishing a Harnack inequality with constant independent of u, and the term $|B|^{1/n}$ omitted.

Theorem 1.2 (The Harnack Inequality). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy either*

(1) $1 < p^- \leqslant p^+ < n$ *and* $\nabla p \in L^n \log L^n(\Omega)$; *or*

(2) $1 < p^- \leqslant p^+ < \infty$ *and* $\nabla p \in L^{q(\cdot)}(\Omega)$, *where* $q \geqslant \max\{p, n\} + \delta$ *for some* $\delta > 0$.

If u is a non-negative solution of $(\star)$, then

$$\operatorname*{ess\,sup}_{x \in B} u(x) \leqslant c \operatorname*{ess\,inf}_{x \in B} u(x),$$

for balls B with $2B \Subset \Omega$. The constant is independent of the function u.

Note that the assumptions of the theorem holds e.g. if p is Lipschitz with $1 < p^- \leqslant p^+ < \infty$. Further note that it suffices to assume that $\|\nabla p\|_{L^n \log L^n(B)} \leqslant c$ for every ball B with $\operatorname{diam} B < \tau \operatorname{dist}(B, \partial\Omega)$, $\tau > 0$, see Remark 4.4.

Our proof is based on Moser iteration. For the weak $p(\cdot)$-Laplacian equation this method was first used by Alkhutov [4]. The difficulty came from the use of test functions of the type $u^\gamma \eta^{p^+}$ with constant exponents. Now the exponent of the test function will not exactly match the exponent of the equation, so one needs to take care of the error term. In this paper, we use test functions more similar to the classical constant exponent case, e.g. $u^{1-(1+\gamma)p(\cdot)}\eta^{p(\cdot)}$. Thus we avoid the error terms from exponent mismatch, which led to the dependence of the constant on u in (1.1). However, compared to the classical case we end up with several extra

terms involving the gradient of p. Dealing with these terms is the major difficulty in the proofs of the main lemmas.

The main difficulty with these additional terms is that they have "supercritical" growth: they are of order $t^{p(\cdot)-1} \log t$ while the "main term" has order only $t^{p(\cdot)-1}$. The term with the logarithm in ($\star$) has the same supercritical order of growth and the proofs rely on combining all of these terms and using the scalability in a suitable way. In fact, the special nature of our equation, especially the scalability, is crucial for us: we would not be able to handle the equation

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx + \int_\Omega |\nabla u|^{p(x)-2} \log |\nabla u| \, \nabla u \cdot \xi \, \varphi \, dx = 0$$

for arbitrary C^1 vector field ξ even when p is constant.

On the other hand, once we do have a proper Harnack inequality, we immediately obtain several corollaries by well-known paths. First, we have the following strong minimum principle:

Corollary 1.3. *Let Ω and p be as in the previous theorem. If u is a non-negative solution of ($\star$), then either $u > 0$ or $u \equiv 0$.*

Again it is worth noting that this conclusion does not follow from the weaker kinds of Harnack inequality (1.1) available for the $p(\cdot)$-Laplacian (but see also [16] for a new approach to this problem). With the Weak Harnack Inequality (Theorem 4.6), we can actually prove the strong minimum principle for supersolutions.

As usual, it is possible to iterate the Harnack inequality in order to prove Hölder continuity, see, e.g., [18, Theorem 6.6]. In this case the procedure works also for the weaker Harnack inequality (1.1) of the $p(\cdot)$-Laplacian, although the constants in that case will depend on u.

Corollary 1.4. *Let Ω and p be as in the previous theorem. If u is a solution of ($\star$), then u is Hölder continuous in $D \Subset \Omega$. The Hölder constant and exponent depend only on n, p and $\mathrm{dist}(D, \partial\Omega)$.*

Continuing to contrast with the $p(\cdot)$-Laplacian case, we obtain a global integrability result:

Theorem 1.5. *Let Ω be a Hölder domain and let p be as in the previous theorem. If u is a non-negative supersolution of ($\star$), then there exists $q > 0$, depending only on n, p and $\mathrm{diam}\,\Omega$, such that*

$$\int_\Omega u^q \, dx < \infty.$$

Remark 1.6. The assumption $\nabla p \in L^n \log L^n(\Omega)$ implies that p has modulus of continuity $(\log \frac{1}{t})^{1/(n-1)}$. If $n = 2$, this implies the log-Hölder continuity. In higher dimensions the assumption $\nabla p \in L^n \log L^{2(n-1)}(\Omega)$ would suffice for this.

Remark 1.7. In the last two years, several authors [26, 31, 32, 35] have considered $p_i(\cdot)$-Laplacian type equations when $p_i \to \infty$. This leads to equations somewhat similar to our, namely

$$-p(\cdot)\,\Delta_\infty u - |\nabla u|^2 \log |\nabla u| \, \nabla u \cdot \nabla p = 0.$$

Also in this context, the Harnack inequality holds in the weaker form similar to $p(\cdot)$-Laplacian [26], as is to be expected due to the lack of scalability of the equation.

2. Preliminaries

By $\Omega \subset \mathbb{R}^n$ we denote a bounded open set. By c we denote a generic constant, whose value may change between appearances even within a single line. By f_A we denote the integral average of f over A. For a ball $B \subset \mathbb{R}^n$, we denote by cB the c-fold dilate with the same center.

For background on variable exponent function spaces we refer to the surveys [11, 37] or the (forthcoming) monograph [10]. Most of the results in this section were proved in [22]. The variable exponent Lebesgue space is a special case of a Musielak–Orlicz space. For a constant function p, it coincides with the standard Lebesgue space. Often it is assumed that p is bounded, since this condition implies many desirable features for $L^{p(\cdot)}(\Omega)$.

A measurable function $p \colon \Omega \to [1, \infty]$ is called a *variable exponent*, and we denote

$$p_A^+ := \operatorname*{ess\,sup}_{x \in A} p(x), \quad p_A^- := \operatorname*{ess\,inf}_{x \in A} p(x), \quad p^+ := p_\Omega^+ \quad \text{and} \quad p^- := p_\Omega^-$$

for $A \subset \Omega$. We define a *(semi)modular* on the set of measurable functions by setting

$$\varrho_{L^{p(\cdot)}(\Omega)}(u) := \int_\Omega |u(x)|^{p(x)} \, dx;$$

here we use the convention $t^\infty = \infty \chi_{(1,\infty]}(t)$ in order to get a left-continuous modular, see [10, Chapter 3] for details. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $u \colon \Omega \to \mathbb{R}$ for which the modular $\varrho_{L^{p(\cdot)}(\Omega)}(u/\lambda)$ is finite for some $\lambda > 0$. The Luxemburg norm on this space is defined as

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \varrho_{L^{p(\cdot)}(\Omega)}\left(\tfrac{u}{\lambda}\right) \leqslant 1 \right\}.$$

Equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a Banach space. There is no functional relationship between norm and modular, but we do have the following useful inequality:

$$\min \left\{ \varrho_{L^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^-}}, \varrho_{L^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^+}} \right\} \leqslant \|f\|_{L^{p(\cdot)}(\Omega)} \leqslant \max \left\{ \varrho_{L^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^-}}, \varrho_{L^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^+}} \right\}.$$

In particular, the norm equals one if and only if the modular equals one.

If E is a measurable set of finite measure, and $p \geqslant q$ are variable exponents, then $L^{p(\cdot)}(E)$ embeds continuously into $L^{q(\cdot)}(E)$. In particular, every function $u \in L^{p(\cdot)}(\Omega)$ also belongs to $L^{p^-}(\Omega)$. The variable exponent Hölder inequality takes the form

$$\int_\Omega fg \, dx \leqslant 2 \, \|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{p'(\cdot)}(\Omega)},$$

where p' is the point-wise *conjugate exponent*, $1/p(x) + 1/p'(x) \equiv 1$.

The function α defined in a bounded domain Ω is said to be log-*Hölder continuous* if there is constant $L > 0$ such that

$$|\alpha(x) - \alpha(y)| \leqslant \frac{L}{\log(e + 1/|x - y|)}$$

for all $x, y \in \Omega$. We write $p \in \mathcal{P}^{\log}(\Omega)$ if $1/p$ is log-Hölder continuous; the smallest constant for which $\frac{1}{p}$ is log-Hölder continuous is denoted by $c_{\log}(p)$. If $p \in \mathcal{P}^{\log}(\Omega)$, then

$$|B|^{p_B^+} \approx |B|^{p_B^-} \approx |B|^{p(x)} \approx |B|^{p_B}$$

for every ball $B \subset \Omega$ and $x \in B$ [10, Lemma 5.1.6]; here p_B is the harmonic average,

$$\frac{1}{p_B} := \fint_B \frac{1}{p(x)} \, dx.$$

(Note that this is a special convention for the exponent, otherwise, f_A denotes the usual, arithmetic average.) The constants in the equivalences depend on $c_{\log}(p)$ and $\operatorname{diam} \Omega$.

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u belongs to $L^{p(\cdot)}(\Omega)$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

In general, smooth functions are not dense in the variable exponent Sobolev space [41], but the log-Hölder condition suffices to guarantee that they are [10, Section 9.1]. In this case, we define *the Sobolev space with zero boundary values*, $W_0^{1,p(\cdot)}(\Omega)$, as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

The Sobolev conjugate exponent is also defined point-wise, $p^*(x) := \frac{np(x)}{n-p(x)}$ for $p^+ < n$. If p is log-Hölder continuous, then the Sobolev–Poincaré inequality

$$\|u - u_\Omega\|_{L^{p^*(\cdot)}(\Omega)} \leqslant c \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

holds when Ω is a nice domain, for instance convex or John [10, Section 8.2]. If $u \in W_0^{1,p(\cdot)}(\Omega)$, then $\|u\|_{L^{p^*(\cdot)}(\Omega)} \leqslant c \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ in any open set Ω.

The Zygmund space $L^p \log L^q(\Omega)$ is defined for $p \geqslant 1$ and $q \in \mathbb{R}$ by the modular

$$\varrho_{L^p \log L^q(\Omega)}(f) := \int_\Omega |f|^p \big(\log(e + |f|)\big)^q \, dx.$$

The Luxemburg norm is defined from this as before. In Zygmund spaces we have the following Hölder inequality

$$(2.1) \qquad \int_\Omega \prod_{i=1}^k f_i \, dx \leqslant \prod_{i=1}^k \big\|f_i\big\|_{L^{p_i} \log L^{q_i}(\Omega)},$$

where

$$\sum_{i=1}^k \frac{1}{p_i} = 1 \quad \text{and} \quad \sum_{i=1}^k \frac{q_i}{p_i} = 0$$

[20, (4.92), p. 77]. The inequality is based on point-wise Young-type inequalities, and directly generalizes to the variable exponent case.

3. THE INFIMUM AND SUPREMUM ESTIMATES

In this section we prove two components of the Harnack inequality, namely, we estimate the essential supremum of a subsolution by the γ integral average from above, and the essential infimum of a superpolution by the $-\gamma$ integral average from below, $\gamma > 0$. Let us recall the definition of these terms.

Definition 3.1. We say that $u \in W_{\text{loc}}^{1,p(\cdot)}(\Omega)$ is a *supersolution* (*of* ($\star$)) if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx + \int_\Omega |\nabla u|^{p(x)-2} \log(|\nabla u|) \nabla u \cdot \nabla p \, \varphi \, dx \geqslant 0$$

for all non-negative $\varphi \in W^{1,p(\cdot)}(\Omega)$ with compact support. It is a *subsolution*, if $-u$ is a supersoluton; and a *solution* if it is both a sub- and a supersolution.

We start with the infimum-estimate; it is based on a Caccioppoli estimate, which comes in two versions.

Lemma 3.2 (Caccioppoli estimate)**.** *Let* $\nabla p \in L^n \log L^n(\Omega)$ *with* $1 < p^- \leqslant p^+ < n$, *and let* u *be a non-negative supersolution of* ($\star$)*. Then for every* $\gamma > 0$ *there exists* c, *depending only on* p^- *and* p^+, *such that*

$$\left\|\frac{\eta \nabla u}{u^{1+\gamma}}\right\|_{L^{p(\cdot)}(\Omega)} \leqslant c \left\|\frac{\nabla \eta}{u^\gamma}\right\|_{L^{p(\cdot)}(\Omega)} + c \left\|\frac{\eta}{u^\gamma}\right\|_{L^{p^*(\cdot)}(\Omega)} \|\nabla p\|_{L^n \log L^n(\text{supp}\,\eta)}$$

for every non-negative Lipschitz function $\eta \in C_0(\Omega)$.

Proof. We consider first the function $u_\delta := u + \delta$. It is still a supersolution, and $u_\delta > \delta > 0$. For simplicity, we drop the subscript δ from the notation.

Let $\eta \in C_0(\Omega)$ be a non-negative Lipschitz function and set $\varphi := u^{1-(1+\gamma)p(\cdot)}\eta^{p(\cdot)}$. Since $u > \delta$, φ is bounded; since $\eta \in C_0(\Omega)$, φ has compact support. Denoting $\eta u^{-(1+\gamma)}$ by f, we find that

$$\nabla\varphi = -((1+\gamma)p(\cdot) - 1)f^{p(\cdot)}\nabla u + p(\cdot)u^{-\gamma}f^{p(\cdot)-1}\nabla\eta + f^{p(\cdot)}u\log f\,\nabla p.$$

Now f and $u^{-\gamma}$ are bounded, and $f^{p(\cdot)}u\log f = \eta u^{-\gamma}f^{p(\cdot)-1}\log f$ is thus also bounded; hence $\nabla\varphi \in L^{p(\cdot)}(\Omega)$. Therefore, $\varphi \in W_0^{1,p(\cdot)}(\Omega)$, and can be used as a test function.

Using φ as a test function in Equation $(\star)$, we obtain

$$((1+\gamma)p^- - 1)\int_\Omega |\nabla u|^{p(x)}f^{p(x)}\,dx$$

$$\leq \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot [p(x)u^{-\gamma}f^{p(x)-1}\nabla\eta + f^{p(x)}u\log f\,\nabla p]$$

$$+ |\nabla u|^{p(x)-2}\log|\nabla u|\nabla u \cdot \nabla p\,f^{p(x)}u\,dx$$

$$\leq \int_\Omega |\nabla u|^{p(x)-1}u^{-\gamma}f^{p(x)-1}\Big[p^+\,|\nabla\eta| + \eta\,\big|\log(f\,|\nabla u|)\big|\,|\nabla p|\Big]\,dx.$$

Denoting further $g := f\,|\nabla u|$, we rewrite this as:

$$(3.3)\qquad \int_\Omega g^{p(x)}\,dx \leq c\int_\Omega g^{p(x)-1}u^{-\gamma}\,[|\nabla\eta| + \eta\,|\log g|\,|\nabla p|]\,dx.$$

Here we used also that $(1+\gamma)p^- - 1 \geq p^- - 1 > 0$.

Let us assume for the moment that $\|g\|_{p(\cdot)} = 1$. Then

$$\big\|g^{p(\cdot)-1}\big\|_{L^{p'(\cdot)}(\Omega)} = 1 \quad\text{and}\quad \big\|g^{p(\cdot)-1}\log g\big\|_{L^{p'(\cdot)}\log L^{-p'(\cdot)}(\Omega)} \leq c$$

with constant depending on p^- and p^+; to establish the latter we calculate

$$\varrho_{L^{p'(\cdot)}\log L^{-p'(\cdot)}(\Omega)}(g^{p(\cdot)-1}\log g) = \int_\Omega (g^{p(x)-1}|\log g|)^{p(x)'}\Big(\log\big(e + g^{p(x)-1}|\log g|\big)\Big)^{-p(x)'}\,dx$$

$$= \int_\Omega g^{p(x)}\Big(\frac{|\log g|}{\log(e + g^{p(x)-1}|\log g|)}\Big)^{p(x)'}\,dx$$

$$\leq c\,\varrho_{p(\cdot)}(g) + c = c,$$

where we used that $p^- > 1$ and that the modular of g equals 1. Using that the norm equals one if and only if the modular equals one and Hölder's inequality for Zygmund spaces (2.1) in (3.3), we find that

$$\|g\|_{L^{p(\cdot)}(\Omega)} = \int_\Omega g^{p(x)}\,dx \leq c\,\big\|g^{p(\cdot)-1}\big\|_{L^{p'(\cdot)}(\Omega)}\Big\|\frac{\nabla\eta}{u^\gamma}\Big\|_{L^{p(\cdot)}(\Omega)}$$

$$+ c\,\big\|g^{p(\cdot)-1}\log g\big\|_{L^{p'(\cdot)}\log L^{-p'(\cdot)}(\Omega)}\Big\|\frac{\eta}{u^\gamma}\Big\|_{L^{p^*(\cdot)}(\Omega)}\|\nabla p\|_{L^n\log L^n(\mathrm{supp}\,\eta)}$$

$$\leq c\,\Big\|\frac{\nabla\eta}{u^\gamma}\Big\|_{L^{p(\cdot)}(\Omega)} + c\,\Big\|\frac{\eta}{u^\gamma}\Big\|_{L^{p^*(\cdot)}(\Omega)}\|\nabla p\|_{L^n\log L^n(\mathrm{supp}\,\eta)}.$$

This proves the claim for the case $\|g\|_{p(\cdot)} = 1$. Since u is a non-negative supersolution if and only if λu is a non-negative supersolution, and since the claim we are proving is homogeneous (of order $-\gamma < 0$), we obtain from this the general case by scaling.

This completes the proof for $u > \delta$. Since the constants do not depend on δ, we may replace u_δ by u on the right hand side of the inequality, possibly obtaining infinite norms (in which case the claim is trivially true). Since $u_\delta \searrow u$ and u appears with negative powers, we obtain the final claim from this by monotone convergence as $\delta \searrow 0$. □

To deal with the case $p^+ \geqslant n$, we use the following variant of the Caccioppoli estimate. The derivation is the same as above, except we use the Hölder inequality with exponent $(p - \delta')^*$ in the last stage of the proof.

Lemma 3.4 (Caccioppoli estimate 2). *Let $\nabla p \in L^{q(\cdot)}(\Omega)$ with $1 < p^- \leqslant p^+ < n + \delta'$, where $q \geqslant \max\{p, n\} + \delta$ for some $\delta > 0$. Here $\delta' > 0$ depends on δ, p^- and n. Let u be a non-negative supersolution of $(\star)$. Then there exists c, depending on p^- and p^+, such that*

$$\left\| \frac{\eta \nabla u}{u^{1+\gamma}} \right\|_{L^{p(\cdot)}(\Omega)} \leqslant c \left\| \frac{\nabla \eta}{u^\gamma} \right\|_{L^{p(\cdot)}(\Omega)} + c \left\| \frac{\eta}{u^\gamma} \right\|_{L^{(p-\delta')^*(\cdot)}(\Omega)} \|\nabla p\|_{L^{q(\cdot)}(\operatorname{supp}\eta)}$$

for every non-negative Lipschitz function $\eta \in C_0(\Omega)$ and $\gamma > 0$.

Now we can prove the first part of the Harnack estimate, by the usual Moser iteration scheme.

Theorem 3.5 (The ess inf-estimate). *Let $p \in \mathcal{P}^{\log}(\Omega)$ be as in Theorem 1.2 and let u be a non-negative supersolution of $(\star)$. Then for every $\alpha > 0$ there exist $c, c' > 0$ depending on p^-, p^+, n, Ω and $c_{\log}(p)$ such that*

$$\left(\fint_{2B} u^{-\alpha} \, dx \right)^{-\frac{1}{\alpha}} \leqslant c \operatorname*{ess\,inf}_{x \in B} u(x),$$

for balls B with $2B \Subset \Omega$ so small that $\|\nabla p\|_{L^n \log L^n(2B)} < c'$ (if $p^+ < n$) or $\|\nabla p\|_{L^{q(\cdot)}(2B)} < c'$ (if $p^+ < \infty$).

Proof. We consider first the case $p^+ < n$ and $\|\nabla p\|_{L^n \log L^n(2B)} < c'$. Let $\gamma > 0$. Then by Lemma 3.2 we conclude that

$$(3.6) \quad \begin{aligned} \left\| \nabla(u^{-\gamma}\eta) \right\|_{L^{p(\cdot)}(\Omega)} &\leqslant \left\| \frac{\eta \nabla u}{u^{1+\gamma}} \right\|_{L^{p(\cdot)}(\Omega)} + \left\| \frac{\nabla \eta}{u^\gamma} \right\|_{L^{p(\cdot)}(\Omega)} \\ &\leqslant c \left\| \frac{\nabla \eta}{u^\gamma} \right\|_{L^{p(\cdot)}(\Omega)} + c \left\| \frac{\eta}{u^\gamma} \right\|_{L^{p^*(\cdot)}(\Omega)} \|\nabla p\|_{L^n \log L^n(\operatorname{supp}\eta)}. \end{aligned}$$

We choose η with support in $rB' \subset 2B$, $\eta|_{\varrho B'} = 1$ and $|\nabla \eta| \leqslant 4/R(r - \varrho)$, where $1 \leqslant \varrho < r \leqslant 3$ and $R := \operatorname{diam} B'$. Using the Sobolev inequality [10, Theorem 8.3.1] for the first inequality, we obtain

$$\left\| \frac{\eta}{u^\gamma} \right\|_{L^{p^*(\cdot)}(2B)} \leqslant c \left\| \nabla(u^{-\gamma}\eta) \right\|_{L^{p(\cdot)}(\Omega)} \leqslant c \left\| \frac{\nabla \eta}{u^\gamma} \right\|_{L^{p(\cdot)}(2B)} + c_1 \left\| \frac{\eta}{u^\gamma} \right\|_{L^{p^*(\cdot)}(2B)} \|\nabla p\|_{L^n \log L^n(2B)}.$$

Assuming now that $c_1 c' < \frac{1}{2}$, we can absorb the second term on the right hand side into the left, obtaining $\|\eta \, u^{-\gamma}\|_{L^{p^*(\cdot)}(2B)} \leqslant c \|\nabla \eta \, u^{-\gamma}\|_{L^{p(\cdot)}(2B)}$. Using this for the third inequality, we find that

$$\left\| u^{-\gamma} \right\|_{L^{n'p(\cdot)}(\varrho B')} \leqslant c \left\| u^{-\gamma} \right\|_{L^{p^*(\cdot)}(\varrho B')} \leqslant c \left\| \frac{\eta}{u^\gamma} \right\|_{L^{p^*(\cdot)}(\Omega)} \leqslant c \left\| \frac{\nabla \eta}{u^\gamma} \right\|_{L^{p(\cdot)}(\Omega)} \leqslant \frac{c}{R(r - \varrho)} \left\| u^{-\gamma} \right\|_{L^{p(\cdot)}(rB')}.$$

We are now in a position to apply the iteration scheme. Let $r_j := \varrho + 2^{-j}(r - \varrho)$ and $\xi_j := (n')^j$ for $j = 0, 1, 2, \ldots$ We apply the inequality with the balls $r_{j+1}B'$ and $r_j B'$ and $\gamma = \xi_j$. This gives

$$\left\| u^{-\xi_{j+1}} \right\|_{L^{p(\cdot)}(r_{j+1}B')}^{\frac{1}{n'}} = \left\| u^{-\xi_j} \right\|_{L^{n'p(\cdot)}(r_{j+1}B')} \leqslant \frac{c}{R(r_j - r_{j+1})} \left\| u^{-\xi_j} \right\|_{L^{p(\cdot)}(r_j B')}.$$

We then multiply this inequality by R^{1-n}, raise it to the power of $\frac{1}{\xi_j}$ and use the definition of r_j:

$$\left\| R^{-n} u^{-\xi_{j+1}} \right\|_{L^{p(\cdot)}(r_{j+1}B')}^{\frac{1}{\xi_{j+1}}} = \left\| R^{1-n} u^{-\xi_j} \right\|_{L^{n'p(\cdot)}(r_{j+1}B')}^{\frac{1}{\xi_j}} \leqslant \left(\frac{2^j c}{r - \varrho} \right)^{\frac{1}{\xi_j}} \left\| R^{-n} u^{-\xi_j} \right\|_{L^{p(\cdot)}(r_j B')}^{\frac{1}{\xi_j}}.$$

Therefore

$$(3.7) \quad \lim_{j \to \infty} \left\| R^{-n} u^{-\xi_j} \right\|_{L^{p(\cdot)}(r_j B')}^{\frac{1}{\xi_j}} \leqslant \prod_{j=0}^{\infty} \left(\frac{2^j c}{r - \varrho} \right)^{\frac{1}{\xi_j}} \left\| R^{-n} u^{-\xi_0} \right\|_{L^{p(\cdot)}(r_0 B')}^{\frac{1}{\xi_0}} = c \left\| R^{-n} u^{-1} \right\|_{L^{p(\cdot)}(r B')}.$$

The inequality $\| \cdot \|_1 \leqslant c \| \cdot \|_{p(\cdot)}$ and a trivial estimate give

$$\frac{1}{c} \left(\fint_{\varrho B'} u^{-\xi_j} \, dx \right)^{\frac{1}{\xi_j}} \leqslant \left\| R^{-n} u^{-\xi_j} \right\|_{L^{p(\cdot)}(\varrho B')}^{\frac{1}{\xi_j}} \leqslant c \left\| R^{-n} \right\|_{L^{p(\cdot)}(2B')}^{\frac{1}{\xi_j}} \left(\operatorname*{ess\,inf}_{x \in r_j B'} u(x) \right)^{-1}.$$

Since the left and right hand sides of the inequality tend to $(\operatorname{ess\,inf}_{x \in \varrho B'} u(x))^{-1}$ as $\xi_j \to \infty$, we see that the left hand side of (3.7) equals $(\operatorname{ess\,inf}_{x \in \varrho B'} u(x))^{-1}$. Thus

$$\left(\operatorname*{ess\,inf}_{x \in \varrho B'} u(x) \right)^{-1} \leqslant c \left\| R^{-n} u^{-1} \right\|_{L^{p(\cdot)}(rB')} \leqslant c \left\| R^{-n} u^{-1} \right\|_{L^{p^+}(rB')} \leqslant c \, |rB'|^{1 - \frac{1}{p^+}} \left(\fint_{rB'} u^{-p^+} \, dx \right)^{\frac{1}{p^+}},$$

which is the claim of the theorem with $\alpha = p^+$, since $|rB'|^{1 - \frac{1}{p^+}} \leqslant |\Omega|^{1 - \frac{1}{p^+}}$ can be absorbed in the constant. From this the case of general $\alpha > 0$ is obtained in the usual way, see, e.g., [28, Corollary 3.10]. This completes the proof in the case $p^+ < n$.

We consider then the case $p^+ < \infty$ when the gradient satisfies the stronger assumption $\nabla p \in L^{q(\cdot)}(\Omega)$, where $q \geqslant \max\{p, n\} + \delta$ for some $\delta > 0$. If $2B$ is a ball in which $p < n + \delta'$ (δ' is from Lemma 3.4), then the same argument as before works when we use Lemma 3.4 instead of Lemma 3.2 and the Sobolev inequality $\|u^{-\gamma} \eta\|_{L^{(p - \delta')^*(\cdot)}(\Omega)} \leqslant c \|\nabla(u^{-\gamma} \eta)\|_{L^{p(\cdot)}(\Omega)}$.

If $p > n + \frac{\delta'}{2}$ in $2B$, then we estimate in (3.6) instead

$$\begin{aligned}
\left\| \frac{\eta}{u^{\gamma}} \right\|_{L^{\infty}(\Omega)} &\leqslant c \left\| \nabla(u^{-\gamma} \eta) \right\|_{L^{n + \frac{\delta'}{2}}(\Omega)} \\
&\leqslant c \left\| \nabla(u^{-\gamma} \eta) \right\|_{L^{p(\cdot)}(\Omega)} \\
&\leqslant c \left\| \frac{\nabla \eta}{u^{\gamma}} \right\|_{L^{p(\cdot)}(\Omega)} + c \left\| \frac{\eta}{u^{\gamma}} \right\|_{L^{\infty}(\Omega)} \|\nabla p\|_{L^{q(\cdot)}(\operatorname{supp} \eta)};
\end{aligned}$$

here we used Morrey's embedding theorem. As before, we absorb the second term on the right hand side into the left. Then we raise both sides to the power $\frac{1}{\gamma}$. In this case the claim follows directly with $\gamma = \alpha$, without the need for any iteration.

Finally, we consider the case when $2B$ satisfies neither of the conditions in the previous two paragraphs. By the log-Hölder condition, the sets $\{p > n + \delta'\}$ and $\{p < n + \frac{\delta'}{2}\}$ are a positive distance R apart, and this distance is determined only by $c_{\log}(p)$. We cover $2B$ by balls B_i of diameter $\frac{1}{3}R$. The number of such balls needed depends only on R and n. By the preceding argument, the claim holds in each small ball $2B_i$. Thus

$$\begin{aligned}
\left(\fint_{3B} u^{-\alpha} \, dx \right)^{-\frac{1}{\alpha}} &\leqslant c \left(\sum_i \frac{|2B_i|}{|2B|} \fint_{2B_i} u^{-\alpha} \, dx \right)^{-\frac{1}{\alpha}} \\
&\leqslant c \left(\sum_i \frac{|2B_i|}{|2B|} (\operatorname*{ess\,inf}_{x \in B} u(x))^{-\alpha} \right)^{-\frac{1}{\alpha}} = c \operatorname*{ess\,inf}_{x \in B} u(x).
\end{aligned}$$

This is the claim with the ball $2B$ replaced by $3B$. By carrying out the preceding steps for $\frac{4}{3}B$ instead of $2B$, we obtain the claim for $2B$ as stated in the theorem. $\quad \square$

We then prove an estimate for the essential supremum. The proof follows [28, Theorem 3.11].

Theorem 3.8 (The ess sup-estimate)**.** *Let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy the conditions of Theorem 3.5 and let u be a non-negative subsolution of $(\star)$. Then for every $\alpha > 0$ there exist $c, c' > 0$ depending only on p^-, p^+, n and $c_{\log}(p)$ such that*

$$\operatorname*{ess\,sup}_{x \in B} u(x) \leqslant c \left(\fint_{2B} u^\alpha \, dx \right)^{\frac{1}{\alpha}}$$

for balls B with $2B \Subset \Omega$ so small that $\|\nabla p\|_{L^n \log L^n(2B)} < c'$ or $\|\nabla p\|_{L^{q(\cdot)}(2B)} < c'$.

Proof. We consider first the case $p^+ < n$. Let $\gamma \geqslant 1$, $l > 1$ and define

$$G_l(t) := \begin{cases} \frac{1}{\gamma} t^\gamma, & t \in [0, l), \\ l^{\gamma-1} t - (1 - \frac{1}{\gamma}) l^\gamma, & t \geqslant l. \end{cases}$$

Note that $G_l \in C^1([0, \infty))$ and $G'_l(t) = \min\{t, l\}^{\gamma-1}$. We further define

$$H_l(x, \xi) = \int_0^\xi G'_l(t)^{p(x)} \, dt.$$

Let $\eta \in C_0^\infty(\Omega)$ and define

$$\varphi := H_l(\cdot, u) \eta^{p(\cdot)}.$$

Since $H_l(x, \xi) \leqslant l^{(\gamma-1)p(x)} \xi$, it is clear that $\varphi \in L^{p(\cdot)}(\Omega)$. For the gradient we obtain

$$\nabla \varphi = H_l(\cdot, u) p(\cdot) \eta^{p(\cdot)-1} \nabla \eta + H_l(\cdot, u) \eta^{p(\cdot)} \log \eta \, \nabla p$$

$$+ \eta^{p(\cdot)} G'_l(u)^{p(\cdot)} \nabla u + \eta^{p(\cdot)} \int_0^{u(x)} G'_l(t)^{p(x)} \log G'_l(t) \, dt \, \nabla p$$

$$= H_l(\cdot, u) \eta^{p(\cdot)-1} \big[p(\cdot) \nabla \eta + \eta \log(\eta G'_l(u)) \nabla p \big]$$

$$+ \eta^{p(\cdot)} G'_l(u)^{p(\cdot)} \nabla u + \eta^{p(\cdot)} \int_0^{u(x)} G'_l(t)^{p(x)} \log \frac{G'_l(t)}{G'_l(u(x))} \, dt \, \nabla p.$$

Since G'_l is bounded, we see that $|\nabla \varphi| \leqslant cu |\nabla p| + c |\nabla u|$, so that $\nabla \varphi \in L^{p(\cdot)}(\Omega)$. Let us denote the integral on the right hand side of the previous equality by $L_l(x, u(x))$. Testing with φ in Equation $(\star)$, we obtain that

$$0 \geqslant \int_\Omega H_l(x, u) \eta^{p(x)-1} |\nabla u|^{p(x)-2} \big[p(x) \nabla u \cdot \nabla \eta + \eta \log(\eta G'_l(u)) \nabla u \cdot \nabla p \big]$$

$$+ \eta^{p(x)} G'_l(u)^{p(x)} |\nabla u|^{p(x)} + \eta^{p(x)} L_l(x, u(x)) |\nabla u|^{p(x)-2} \nabla u \cdot \nabla p$$

$$+ |\nabla u|^{p(x)-2} \log |\nabla u| \, \nabla u \cdot \nabla p \, H_l(x, u) \, \eta^{p(x)} \, dx.$$

From this we conclude that

(3.9)
$$\int_\Omega G'_l(u)^{p(x)} |\nabla u|^{p(x)} \eta^{p(x)} \, dx$$

$$\leqslant \int_\Omega H_l(x, u) |\nabla u|^{p(x)-1} \eta^{p(x)-1} \Big[p^+ |\nabla \eta| + \eta |\nabla p| \big| \log \big(G'_l(u) |\nabla u| \eta \big) \big| \Big]$$

$$+ c |\nabla u|^{p(x)-1} \eta^{p(x)} |\nabla p| \, |L_l(x, u)| \, dx.$$

A somewhat lengthy but elementary computation shows that

$$\frac{H_l(x, \xi)}{G'_l(\xi)^{p(x)-1}} = \begin{cases} \frac{1}{a} \xi^\gamma, & \xi < l, \\ l^{\gamma-1} \xi - (1 - \frac{1}{a}) l^\gamma, & \xi \geqslant l, \end{cases}$$

where $a := (\gamma - 1)p(x) + 1$. The right hand side of the previous inequality is a decreasing expression in a, and hence we see that it is less than or equal to $G_l(\xi)$ since $\gamma \leqslant a$. Thus

$$H_l(x, \xi) \leqslant G'_l(\xi)^{p(x)-1} G_l(\xi).$$

Similarly, using the expression for G_l' and the substitution $s = \frac{t}{\xi}$, we derive that

$$|L_l(x,\xi)| = \int_0^\xi G_l'(t)^{p(x)} \log \frac{G_l'(\xi)}{G_l'(t)}\, dt = c_{p(x),\gamma} \max\{\xi, l\}^{(\gamma-1)p(x)+1},$$

where

$$c_{p(x),\gamma} := \int_0^1 s^{(\gamma-1)p(x)} \log \frac{1}{s^{\gamma-1}}\, ds.$$

Since $p^- > 1$, we see that the integrand is a bounded function, so that $c_{p(x),\gamma} \leqslant c$; on the other hand with the change of variables $\sigma = s^{\gamma-1}$ we conclude $c_{p(x),\gamma} \leqslant \frac{c}{\gamma-1}$. Combined, these estimates yield $c_{p(x),\gamma} \leqslant \frac{c}{\gamma}$. Hence

$$|L_l(x,\xi)| \leqslant \frac{c}{\gamma} \max\{\xi, l\}^\gamma\, G_l'(\xi)^{p(x)-1} \leqslant c\, G_l'(\xi)^{p(x)-1} G_l(\xi).$$

Using these estimates in (3.9), we find that

$$\int_\Omega G_l'(u)^{p(x)} |\nabla u|^{p(x)} \eta^{p(x)}\, dx \leqslant c \int_\Omega G_l'(u)^{p(x)-1} G_l(u) |\nabla u|^{p(x)-1} \eta^{p(x)-1} \cdot$$
$$\cdot \Big[|\nabla \eta| + \eta |\nabla p| \big| \log\big(G_l'(u)|\nabla u|\,\eta\big) \big| + \eta |\nabla p| \Big]\, dx.$$

Next we denote $v := G_l(u)$ and note that $\nabla v = G_l'(u) \nabla u$ by the chain rule. Hence the previous inequality can be written as

$$\int_\Omega |\nabla v|^{p(x)} \eta^{p(x)}\, dx \leqslant c \int_\Omega v |\nabla v|^{p(x)-1} \eta^{p(x)-1} \Big[|\nabla \eta| + \eta |\nabla p| \big(1 + \big| \log(|\nabla v|\,\eta) \big| \big) \Big]\, dx.$$

This inequality is analogous to (3.3), albeit slightly more complicated because G_l ruins the (immediate) possibility of scaling. Assume first that $\|g\|_{p(\cdot)} \in (\frac{1}{2}, 1]$, where $g := \eta |\nabla v|$. Then

$$\| g^{p(\cdot)-1} \|_{L^{p'(\cdot)}} \leqslant 1 \quad \text{and} \quad \| g^{p(\cdot)-1}(1 + \log g) \|_{L^{p'(\cdot)} \log L^{-p'(\cdot)}} \leqslant c,$$

by the same reasoning as in Theorem 3.5. Hence it follows by Hölder's inequality that

$$\int_\Omega g^{p(x)}\, dx \leqslant c \int_\Omega v g^{p(x)-1} \Big[|\nabla \eta| + \eta |\nabla p| \big(1 + \big| \log g \big|\big) \Big]\, dx$$
$$\leqslant c\, \| g^{p(\cdot)-1} \|_{L^{p'(\cdot)}} \| v \nabla \eta \|_{p(\cdot)}$$
$$+ c\, \| g^{p(\cdot)-1}(1 + \log g) \|_{L^{p'(\cdot)} \log L^{-p'(\cdot)}} \| v\eta \|_{L^{p^*(\cdot)}} \| \nabla p \|_{L^n \log L^n(\operatorname{supp}\eta)}$$
$$\leqslant c\, \| v \nabla \eta \|_{p(\cdot)} + c\, \| v\eta \|_{L^{p^*(\cdot)}} \| \nabla p \|_{L^n \log L^n(\operatorname{supp}\eta)}.$$

Since $\|g\|_{p(\cdot)} \in (\frac{1}{2}, 1]$, the left hand side is estimated from below by $c \|g\|_{p(\cdot)}$.

Taking into account also that $\big\| \nabla(\eta v) \big\|_{L^{p(\cdot)}(\Omega)} \leqslant \big\| \eta \nabla v \big\|_{L^{p(\cdot)}(\Omega)} + \big\| v \nabla \eta \big\|_{L^{p(\cdot)}(\Omega)}$, we further obtain

$$\big\| \nabla(\eta v) \big\|_{L^{p(\cdot)}(\Omega)} \leqslant c \big\| v \nabla \eta \big\|_{L^{p(\cdot)}(\Omega)} + c \big\| v\eta \big\|_{L^{p^*(\cdot)}(\Omega)} \big\| \nabla p \big\|_{L^n \log L^n(\operatorname{supp}\eta)}.$$

By the Sobolev inequality [10, Theorem 8.3.1],

$$\big\| v\eta \big\|_{L^{p^*(\cdot)}(\Omega)} \leqslant c \big\| \nabla(\eta v) \big\|_{L^{p(\cdot)}(\Omega)} \leqslant c \big\| v \nabla \eta \big\|_{L^{p(\cdot)}(\Omega)} + c \big\| v\eta \big\|_{L^{p^*(\cdot)}(\Omega)} \big\| \nabla p \big\|_{L^n \log L^n(\operatorname{supp}\eta)}.$$

If $\| \nabla p \|_{L^n \log L^n(\operatorname{supp}\eta)}$ is sufficiently small, the last term on the right hand side can be absorbed in the left hand side, and we obtain

$$\big\| G_l(u)\eta \big\|_{L^{p^*(\cdot)}(\Omega)} = \big\| v\eta \big\|_{L^{p^*(\cdot)}(\Omega)} \leqslant c \big\| v \nabla \eta \big\|_{L^{p(\cdot)}(\Omega)} = c \big\| G_l(u)\nabla \eta \big\|_{L^{p(\cdot)}(\Omega)}$$

in the case $\|g\|_{p(\cdot)} \in (\frac{1}{2}, 1]$.

We then use the scalability of ($\star$) and the claim and thus may assume without loss of generality that $\| \eta \nabla G_\infty(u) \|_{p(\cdot)} = 1$ where $G_\infty(t) := \frac{1}{\gamma} t^\gamma$. Since $|\nabla G_l(u)| \nearrow |\nabla G_\infty(u)|$, it follows that there is a bound l_0 such that $\| \eta \nabla G_l(u) \|_{p(\cdot)} \in (\frac{1}{2}, 1]$ whenever $l \geqslant l_0$. Thus the previous inequality holds for this range of l. Since also $G_l \nearrow$

G_∞ point-wise, $\left\|G_l(u)\eta\right\|_{L^{p^*(\cdot)}(\Omega)} \leqslant c\left\|G_\infty(u)\nabla\eta\right\|_{L^{p(\cdot)}(\Omega)}$. Then it follows by monotone convergence that

$$\left\|\tfrac{1}{\gamma}u^\gamma\eta\right\|_{L^{p^*(\cdot)}(\Omega)} \leqslant c\left\|\tfrac{1}{\gamma}u^\gamma\nabla\eta\right\|_{L^{p(\cdot)}(\Omega)}.$$

On the left hand side we use $L^{p^*(\cdot)}(\Omega) \hookrightarrow L^{n'p(\cdot)}(\Omega)$:

$$\left\|u^\gamma\eta\right\|_{L^{n'p(\cdot)}(\Omega)} \leqslant c\left\|u^\gamma\nabla\eta\right\|_{L^{p(\cdot)}(\Omega)}.$$

We can now perform the iteration as in the proof of Theorem 3.5 to yield the claim for any $\alpha > 0$. Also the proof for the case $p^+ \geqslant n$ follows the same scheme as in that proof. □

4. Crossing zero

In this section we derive the remaining part of the Harnack inequality, i.e. we connect the α-integral averages for positive and negative α. For this we need yet another Caccioppoli estimate, which is based on the following Young-type inequality.

Lemma 4.1 (Young-type inequality). *For $s, t > 0$ and $q > 1$ we have*

$$t\, s^{q-1}\,|\log s| \leqslant 2s^q + (t\,|\log t|)^q + c_q.$$

Proof. If $s < 1$, then the left hand side is at most ct, and the claim is clear. If $t < 1$, then the claim follows from $s^{q-1}\log s \leqslant 2s^q + c$. So we may assume $s, t \geqslant 1$, and work with $\log$ instead of $|\log|$.

For $s, t \geqslant 1$ we prove that

$$t\, s^{q-1}\log s \leqslant 2s^q + (t\log t)^q.$$

We divide both sides by s^q and denote $z := t/s$:

$$z\log(t/z) \leqslant 2 + (z\log t)^q.$$

Denoting further $w := z\log t$, we rewrite this as

$$0 \leqslant 2 + z\log z + w^q - w.$$

Since $z\log z \geqslant -e^{-1}$ and $w^q - w \geqslant -1$, this inequality is clear, and the claim follows. □

Lemma 4.2 (Caccioppoli estimate 3). *Let $\nabla p \in L^{p(\cdot)}\log L^{p(\cdot)}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$, and let u be a non-negative supersolution of $(\star)$. Then*

$$\int_\Omega (|\nabla\log u|\,\eta)^{p(x)}\,dx \leqslant c\int_\Omega |\nabla\eta|^{p(x)} + (\eta\,|\nabla p|\,|\log(\eta\,|\nabla p|)|)^{p(x)} + \chi_{\{\eta>0\}}\,dx$$

for non-negative Lipschitz functions $\eta \in C_0(\Omega)$. The constant depends only on p^- and p^+.

Proof. The assertion follows from (3.3) with $\gamma = 0$ once we use Young's inequality and the previous lemma. □

Corollary 4.3. *Let $p \in \mathcal{P}^{\log}(\Omega)$ with $\nabla p \in L^{p(\cdot)}\log L^{p(\cdot)}(\Omega)$ and $1 < p^- \leqslant p^+ < \infty$; let also $2B \Subset \Omega$ be a ball with radius r. Assume further that $\varrho_{L^{p(\cdot)}(2B)}(\nabla p\log|\nabla p|) \leqslant c'r^{n-p_{2B}}$. If u is a non-negative supersolution of $(\star)$, then*

$$\fint_B |\log u - (\log u)_B|^{p(x)}\,dx \leqslant c.$$

The constant depends only on c', $\operatorname{diam}\Omega$, p^-, p^+ and $c_{\log}(p)$.

Proof. We choose in the previous lemma η with $0 \leqslant \eta \leqslant 1$, $\eta|_B = 1$, support in $2B$ and $|\nabla \eta| \leqslant 2/r$. Then

$$\int_\Omega (|\nabla \log u| \eta)^{p(x)} \, dx \leqslant c \, \varrho_{L^{p(\cdot)}(2B)}(2/r) + c \, \varrho_{L^{p(\cdot)}(2B)}(\nabla p \log |\nabla p|) + c \, r^n \leqslant c \, r^{n - p_{2B}},$$

since $r^{p(x)} \approx r^{p_{2B}}$ for $x \in 2B$. By the Poincaré inequality [10, Proposition 8.2.8],

$$\int_B \left(\frac{|v - v_B|}{r} \right)^{p(x)} dx \leqslant c \int_B |\nabla v|^{p(x)} \, dx + c \, |B|$$

for $v \in W^{1,p(\cdot)}(B)$. We apply this to the function $v := \log u$:

$$\fint_B \left(\frac{|\log u - (\log u)_B|}{r} \right)^{p(x)} dx \leqslant c \fint_B |\nabla \log u|^{p(x)} \, dx + c$$

$$\leqslant c \fint_\Omega (|\nabla \log u| \eta)^{p(x)} \, dx + c \leqslant c \, r^{-p_{2B}}.$$

Using again $r^{p(x)} \approx r^{p_{2B}}$ on the left hand side, we obtain the claim. □

Remark 4.4. Suppose that $p \in \mathcal{P}^{\log}(\Omega)$. Then we may use Young's inequality in the ball B of radius r:

$$\int_B |f|^{p(x)} \, dx \leqslant \int_B r^{\alpha \frac{n}{p(\cdot)}} |f|^n + (r^{-\alpha})^{\frac{n}{n - p(\cdot)}} \, dx \approx r^{\alpha \frac{n}{p_B}} \int_B |f|^n \, dx + r^{n - \alpha \frac{n}{n - p_B}}$$

Now we apply this with $f = |\nabla p| \log |\nabla p|$ and $\alpha = \frac{p_B}{n}(n - p_B)$. Then we see that $\varrho_{L^{p(\cdot)}(2B)}(\nabla p \log |\nabla p|) \leqslant c' r^{n - p_{2B}}$ if $\nabla p \in L^n \log L^n(\Omega)$, with constant depending on the $L^n \log L^n$ norm of ∇p. For simplicity we therefore move to the latter condition, although also the slightly more general condition $\varrho_{L^{p(\cdot)}(2B)}(\nabla p \log |\nabla p|) \leqslant c' r^{n - p_{2B}}$ would suffice.

Theorem 4.5. *Let $p \in \mathcal{P}^{\log}(\Omega)$ with $\nabla p \in L^n \log L^n(\Omega)$ and $1 < p^- \leqslant p^+ < \infty$. If u is a non-negative supersolution of $(\star)$, then there exists $\alpha > 0$ such that*

$$\left(\fint_B u^\alpha \, dx \right)^{\frac{1}{\alpha}} \leqslant c \left(\fint_B u^{-\alpha} \, dx \right)^{-\frac{1}{\alpha}}$$

for every ball B with $3B \Subset \Omega$. The constants c and α depend only on $\mathrm{diam}\,\Omega$, p^-, p^+, $c_{\log}(p)$ and the norm of ∇p.

Proof. By Remark 4.4, the conditions of the previous corollary are satisfied, so we obtain

$$\fint_B \left| \log u - (\log u)_B \right| dx \leqslant \fint_B \left| \log u - (\log u)_B \right|^{p(x)} + 1 \, dx \leqslant c.$$

Therefore, $\log u \in BMO(B)$ uniformly whenever $3B \Subset \Omega$. Thus the standard proof applies. For completeness we include some details.

The measure theoretic John–Nirenberg Lemma (see for example [18, Corollary 19.10, p. 371 in Dover's edition] or [28, Theorem 1.66, p. 40]) implies that there exist positive constants α and c depending on the *BMO*-norm such that

$$\fint_B e^{\alpha |f - f_B|} \, dx \leqslant c,$$

where $f := \log u$. Using this we can conclude that

$$\fint_B e^{\alpha f} \, dx \fint_B e^{-\alpha f} \, dx = \fint_B e^{\alpha(f - f_B)} \, dx \fint_B e^{-\alpha(f - f_B)} \, dx \leqslant \left(\fint_B e^{\alpha |f - f_B|} \, dx \right)^2 \leqslant c,$$

which implies that

$$\left(\fint_B u^\alpha \, dx \right)^{1/\alpha} = \left(\fint_B e^{\alpha f} \, dx \right)^{1/\alpha} \leqslant c \left(\fint_B e^{-\alpha f} \, dx \right)^{-1/\alpha} = c \left(\fint_B u^{-\alpha} \, dx \right)^{-1/\alpha}. \qquad \square$$

Proof of the Harnack Inequality, Theorem 1.2. If B is so small that the conditions of Theorem 3.5 and Theorem 3.8 hold, then the claim follows immediately from these two results and Theorem 4.5. If this is not the case, B can nevertheless be covered by a finite number of balls in which the conditions hold. Moreover, the number of balls needed depends only on the given parameters, n, p^-, p^+ and $c_{\log}(p)$. Therefore we obtain the claim in this case by combining the claims over the small balls. □

If we only combine Theorems 3.5 and 4.5, then we arrive at the weak Harnack inequality for supersolutions:

Theorem 4.6 (The Weak Harnack Inequality). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain; let $p \in \mathcal{P}^{\log}(\Omega)$ be as in Theorem 1.2. Then there exists $\alpha > 0$ such that*

$$\left(\fint_{2B} u^\alpha \, dx \right)^{\frac{1}{\alpha}} \leqslant c \; \underset{x \in B}{\operatorname{ess\,inf}} \, u(x),$$

for balls B with $2B \Subset \Omega$ and non-negative supersolutions u of $(\star)$. The constant c is independent of the function u.

If $u(x_0) = 0$, then the Weak Harnack Inequality implies that $\fint_{2B} u^\alpha \, dx = 0$, hence $u \equiv 0$ in $2B$ since $u \geqslant 0$. Thus the set $\{u = 0\}$ is open. As in [17, Theorem 4.1], we can prove that the supersolution is lower semi-continuous, so that $\{u = 0\}$ is closed. This directly implies the strong minimum principle, Corollary 1.3 also for supersolutions.

5. Global integrability

Now we also have all the tools necessary to prove global integrability of non-negative supersolutions of $(\star)$. Recall that this result is not known for the $p(\cdot)$-Laplacian; it cannot be derived from (1.1), since the constant in this inequality depends on the norm of u in the first place.

In fact, all the arguments have already been laid out by Lindqvist in [25]. For completeness, we reiterate the most pertinent parts.

A *Hölder domain* Ω is a proper subdomain of $\mathbb{R}^n$ in which

$$k_\Omega(x, x_0) \leqslant c \, \log \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{dist}(x, \partial\Omega)} + c$$

for some $x_0 \in D$ and every $x \in D$. Here k_Ω denotes the *quasihyperbolic metric*,

$$k_\Omega(x, y) := \inf \int_\gamma \frac{ds(z)}{\operatorname{dist}(z, \partial\Omega)},$$

where the infimum is taken over rectifiable paths in Ω joining x and y and the integration is with respect to arc-length. We refer to [27] for an up-to-date overview of this class of domains; but note that for instance every John domain is a Hölder domain. The name comes from the fact that in the plane a simply connected domain is a Hölder domain if and only if its Riemann mapping is Hölder continuous.

For our purposes we only need a result by Lindqvist, which relies on a characterization of Hölder domains by Smith and Stegenga [38]. The BMO norm is defined for $f \in L^1_{\text{loc}}(\Omega)$ by

$$\|f\|_{*,\Omega} = \sup_{B \Subset \Omega} \fint_B |f - f_B| \, dx.$$

Lemma 5.1 (Lemma 3.7, [25]). *Suppose that $\Omega \subset \mathbb{R}^n$ is a Hölder domain. If $f \in L^1_{\text{loc}}(\Omega)$ and $\|f\|_{*,\Omega} < \infty$, then $f \in L^1(\Omega)$ and*

$$\fint_\Omega \exp\left(\alpha \frac{|f - f_\Omega|}{\|f\|_{*,\Omega}} \right) dx \leqslant 2$$

for some α depending on n and the Hölder domain constant.

Thus we are prepared to prove the global integrability.

Proof of Theorem 1.5. Following Lindqvist [25], we observe that it was shown in [39, Corollary 2.26] that

$$\|f\|_{*,\Omega} \leqslant c \sup_{2B \subset \Omega} \fint_B |f - f_B| \, dx$$

for some constant c depending on n. We saw in the proof of Theorem 4.5 that the right hand side of this inequality is bounded for $f = \log u$. Therefore $\|\log u\|_{*,\Omega} < Q < \infty$. Thus, by Lemma 5.1,

$$\fint_\Omega \exp\left(\tfrac{\alpha}{Q} |\log u - (\log u)_\Omega|\right) dx \leqslant 2.$$

In particular, it follows that $(\log u)_\Omega$ is finite. Thus we obtain

$$\fint_\Omega u^{\frac{\alpha}{Q}} \, dx = \fint_\Omega \exp\left(\tfrac{\alpha}{Q} \log u\right) dx \leqslant \fint_\Omega \exp\left(\tfrac{\alpha}{Q} |\log u - (\log u)_\Omega|\right) \exp\left(\tfrac{\alpha}{Q} |(\log u)_\Omega|\right) dx$$

$$\leqslant 2 \exp\left(\tfrac{\alpha}{Q} |(\log u)_\Omega|\right). \qquad \square$$

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