Phragmén-Lindelöf theorems for equations with nonstandard growth

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Abstract

The Phragmén-Lindelöf theorem on unbounded domains is studied for subsolutions of variable exponent $p(\cdot)$-Laplace equations of homogeneous and nonhomogeneous types. The discussion is illustrated by number of examples of unbounded domains such as half space, angular domains and domains narrowing at infinity. Our approach gives some new results also in the setting of p-Laplacian and harmonic operator.

Keywords: nonstandard growth equation, p-Laplace, p-harmonic, $p(\cdot)$-harmonic, $p(\cdot)$-subsolution, Phragmén-Lindelöf theorem, variable exponent.

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1 Introduction

In this paper we study the growth of $p(\cdot)$-harmonic subsolutions on unbounded domains in $\mathbb{R}^n$. Let u be a local weak subsolution in an unbounded domain Ω of either

$$\operatorname{div}(|\nabla u|^{p(\cdot)-2}\nabla u) = 0 \tag{1.1}$$

or

$$\operatorname{div}(|\nabla u|^{p(\cdot)-2}\nabla u) = f(x,u,\nabla u),$$

under suitable assumptions on function f. For solutions of such equations we investigate the asymptotic behavior of u in $\Omega\cap B_R$ for large radii R, where B_R denotes the ball of radius R centered at the origin. The prototype for our studies is the following classical Phragmén-Lindelöf theorem in the plane [26].

Let u be a subharmonic in the upper half plane and let $\lim_{z\to\mathbb{R}^+} u(z)\leq 0$. Then either $u\leq 0$ in the whole upper plane or it holds that

$$\liminf_{R\to\infty}\frac{\sup_{|z|=R}u(z)}{R}>0.$$

This result was extended to the setting of elliptic equations of second order in [11, 29], has been studied for elliptic equations in general domains [30], fully nonlinear equations [4, 5], as well as in the context of Riemmanian manifolds [24], see also [16, 17] for some further generalizations of the Phragmén-Lindelöf alternative. As for relation to applied sciences let us mention that the Phragmén-Lindelöf principle is connected to the so-called Saint-Venants Principle in elasticity theory (for more details see e.g. [15]).

One of the most fundamental equations of nonlinear analysis is the p-harmonic equation:

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \qquad 1\leq p\leq\infty.$$

The importance of this equation comes among others from the fact that it is a natural nonlinear generalization of harmonic functions ($p=2$), has variational characterization in terms of p-Dirichlet energy; also appears in numerous areas of pure and applied mathematics to mention for example differential geometry, viscosity solutions (especially the case $p=\infty$), relation to quasiregular mappings, nonlinear eigenvalue problems. One also studies generalizations of p-harmonic functions on metric spaces. As for applied sciences p-Laplace equation is used as a model equation in nonlinear elasticity theory, glaciology, stellar dynamics, description of flows through porous medias.

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Another recently blooming area in nonlinear analysis is the theory of PDEs with nonstandard growth (variable exponent analysis) and related energy functionals. Equation (1.1) serves as the model example. Here p is a measurable function $p:\Omega\to[1,\infty]$ called variable exponent while solutions naturally belong to appropriate Musielak-Orlicz space (see Preliminaries). Apart from interesting theoretical considerations such equations naturally arise, for instance, as a model for thermistor [32], in fluid dynamics [8], in the study of image processing [6] and electro-rheological fluids [1]; see [13] for a recent survey and further references, see also the monograph [28], where the role of (non)homogeneous $p(\cdot)$-Laplace equations in applications is discussed in more details. Despite the symbolic similarity to the constant exponent equations, various unexpected phenomena may occur when the exponent is a function, for instance the minimum of the $p(\cdot)$-Dirichlet energy may not exist even in the one-dimensional case for smooth functions p; also smooth functions need not to be dense in the corresponding variable exponent Sobolev spaces.

Several features of equation (1.1) have been studied, for example the regularity theory, potential theory, Harnack type estimates and boundary regularity to mention just few (see [13] and references therein). Such an equation has, however, many disadvantages comparing to the $p=const$ case, for instance: lack of scalability of solutions, nonhomogeneous Harnack inequality with constant depending on solution. These often make the analysis of nonstandard growth equation difficult and lead to technical and nontrivial estimates (nevertheless, see [2, 3] and Remark 3.4 below for a variant of equation (1.1) that overcomes some of the described difficulties, the so-called strong $p(\cdot)$-harmonic equation).

We would like now to discuss the state of art for the problem in the case of the Phragmén-Lindelöf principle for p-Laplacian and explain some difficulties arising when extending known approaches to the variable exponent setting. Lindqvist in [21] proved the principle for special domains of type $\mathbb{R}^n\setminus H^q$, where H^q is a q-dimensional hyperplane. This approach relies on n-harmonic measures and the comparison principle. Unfortunately, the same technique cannot be applied in our setting due to the lack of scalability for $p(\cdot)$-harmonic equation and lack of similar relations between n-harmonic measures and $p(\cdot)$-harmonic operator. Nevertheless, by using our approach, in Corollary 3.5 we retrieve part of Theorem 4.6 in [21] as a special case of one of our main results, Theorem 3.3. Another interesting approach toward the Phragmén-Lindelöf principle was taken by Granlund [12] and is based on de Giorgi type estimates and their iterations. The corresponding estimates for the $p(\cdot)$-harmonic operator are non-homogeneous and their iterations do not lead to the desired result as in [12]. Results by Jin and Lancaster discussed in [16], although applicable to wide class of quasilinear elliptic equations with C^2 solutions, cannot be directly used in our setting as the $p(\cdot)$-harmonic functions are, in general, $C^{1,\alpha}$ regular (cf. [9]). As for p-harmonic equations with nontrivial right-hand side we mention work of Kurta [20], where the Phragmén-Lindelöf theorem is proven for $|\nabla u|$ together with existence results for nontrivial solutions (see also [22]).

Organization of the paper

In Section 2 we recall basic facts and properties of variable exponent spaces, variational capacities and $p(\cdot)$-harmonic functions.

Section 3 is devoted to studying the main result of the paper, namely the Phragmén-Lindelöf theorem for subsolutions of homogeneous $p(\cdot)$-harmonic equation. Our approach is based on developing an energy estimate for the norm of the gradient of $p(\cdot)$-harmonic subsolution. Such estimate carries information about: (a) impact of the rate of growth of variable exponent $p(\cdot)$; (b) size of the underlying domain expressed in terms of capacity; (c) porosity of the domain. Under growth assumptions on the exponent we provide a general condition implying the assertion of theorem and illustrate discussion by a number of corollaries for domains typically appearing in the context of the Phragmén-Lindelöf alternative: a half space, an angular sector, a domain narrowing at infinity.

In Section 4 we present the corresponding results for nonhomogeneous $p(\cdot)$-harmonic equation. Our approach gives some new results also in the setting of p-Laplacian and harmonic functions, see Corollaries 4.4 and 4.5.

To the best of our knowledge the Phragmén-Lindelöf theorem has not been studied before in the context of $p(\cdot)$-harmonic functions. With our work we hope to start the studies of the maximum principles in unbounded domains for equations with nonstandard growth.

2 Preliminaries

Let Ω be an unbounded open set in $\mathbb{R}^n$. Denote $B_r=B(0,r)$ an open ball in $\mathbb{R}^n$ centered at the origin with radius $r>0$. Furthermore, let dx stand for the n-dimensional Lebesgue measure, $\lambda_{n-1}(A)$ denotes the $(n-1)$-dimensional measure of a set A, while by f_A we denote the integral average of function f over set A, that is

$$f_A := \fint_A f dx = \frac{1}{|A|}\int_A f dx.$$

A measurable function $p\colon \Omega \to [1,\infty]$ is called a *variable exponent*, and we denote

$$p_A^+ := \operatorname{ess\,sup}_{x\in A} p(x), \quad p_A^- := \operatorname{ess\,inf}_{x\in A} p(x), \quad p^+ := p_\Omega^+ \quad \text{and} \quad p^- := p_\Omega^-$$

for $A \subset \Omega$. If $A = \Omega$ or if the underlying domain is fixed, we will often skip the index and set $p_A = p_\Omega = p$. For background on variable exponent function spaces we refer to the monograph [7].

In this paper we deal with bounded variable exponent functions, that is we assume that

$$1 < p^- \le p(x) \le p^+ < \infty \qquad \text{for almost every } x \in \Omega.$$

The set of all such exponents in Ω will be denoted $\mathcal{P}(\Omega)$. We define a *(semi)modular* on the set of measurable functions by setting

$$\varrho_{L^{p(\cdot)}(\Omega)}(u) := \int_\Omega |u(x)|^{p(x)}\,dx;$$

here we use the convention $t^\infty = \infty\chi_{(1,\infty]}(t)$ in order to get a left-continuous modular, see [7, Chapter 2] for details. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $u\colon \Omega \to \mathbb{R}$ for which the modular $\varrho_{L^{p(\cdot)}(\Omega)}(u/\mu)$ is finite for some $\mu > 0$. The Luxemburg norm on this space is defined as

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf\Big\{\mu > 0 : \ \varrho_{L^{p(\cdot)}(\Omega)}\big(\tfrac{u}{\mu}\big) \le 1\Big\}.$$

Equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a Banach space. The variable exponent Lebesgue space is a special case of Musielak-Orlicz space, cf. [19]. For a constant function p it coincides with the standard Lebesgue space. Often it is assumed that p is bounded, since this condition is known to imply many desirable features for $L^{p(\cdot)}(\Omega)$.

There is no functional relationship between norm and modular, but we do have the following useful inequality:

$$\min\Big\{\varrho_{L^{p(\cdot)}(\Omega)}(u)^{\frac{1}{p^-}}, \varrho_{L^{p(\cdot)}(\Omega)}(u)^{\frac{1}{p^+}}\Big\} \le \|u\|_{L^{p(\cdot)}(\Omega)} \le \max\Big\{\varrho_{L^{p(\cdot)}(\Omega)}(u)^{\frac{1}{p^-}}, \varrho_{L^{p(\cdot)}(\Omega)}(u)^{\frac{1}{p^+}}\Big\}. \tag{2.1}$$

If E is a measurable set of finite measure and p and q are variable exponents satisfying $q \le p$, then $L^{p(\cdot)}(E)$ embeds continuously into $L^{q(\cdot)}(E)$. In particular, every function $u \in L^{p(\cdot)}(\Omega)$ also belongs to $L^{p_\Omega^-}(\Omega)$. The variable exponent Hölder inequality takes the form

$$\int_\Omega uv\,dx \le 2\,\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}, \tag{2.2}$$

where p' is the point-wise *conjugate exponent*, $1/p(x) + 1/p'(x) \equiv 1$.

For the sake of completeness of discussion let us also observe that both the pointwise Young inequality and its parameter variant hold also in the variable exponent setting for a fixed $\epsilon \in (0,1]$:

$$\int_\Omega u(x)v(x)\,dx \le \int_\Omega \tfrac{\epsilon^{p(x)}}{p(x)}u(x)^{p(x)}\,dx + \int_\Omega \tfrac{\epsilon^{-p'(x)}}{p'(x)}v(x)^{p'(x)}\,dx. \tag{2.3}$$

A function α defined in a domain Ω is said to be *locally* log-*Hölder continuous* if there is constant $c_1 > 0$ such that

$$|\alpha(x) - \alpha(y)| \le \frac{c_1}{\log(e + 1/|x-y|)}$$

for all $x, y \in \Omega$. We also assume that α satisfies *log-Hölder decay condition* if there exist constants α_∞ and $c_2 > 0$ such that

$$|\alpha(x) - \alpha_\infty| \le \frac{c_2}{\log(e + |x|)}$$

for all $x \in \Omega$. We say that α is *globally* log-*Hölder continuous* if it is both locally log-Hölder continuous and satisfies the decay condition. The maximum $\max\{c_1, c_2\}$ is called *the* log-*Hölder constant*. In what follows for the sake of simplicity we will omit word globally and use term log-*Hölder continuous* instead.

We denote $p \in \mathcal{P}^{\log}(\Omega)$ if $1/p$ is log-Hölder continuous and the log-Hölder constant is denoted by $c_{\log}(p)$. By [7, Remark 4.1.5] we know that, since variable exponent p is assumed to be bounded, $p \in \mathcal{P}^{\log}(\Omega)$ if and only if p is log-Hölder continuous.

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u belongs to $L^{p(\cdot)}(\Omega)$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

In general, smooth functions are not dense in the variable exponent Sobolev space [7, Section 9.2], but the log-Hölder condition suffices to guarantee that they are [7, Section 8.1]. In this case, we define *the Sobolev space with zero boundary values*, $W_0^{1,p(x)}(\Omega)$, as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

Below we will also use the notion of *variational capacity (capacity of condenser)* of a subset considered with respect to a surrounding set. We refer to [14, Chapter 2] for a comprehensive discussion of capacity in the constant exponent case and to [7, Chapter 10] for a corresponding presentation of capacities in the variable exponent setting.

Suppose that K is a compact subset of Ω. We denote

$$W(K,\Omega) := \{u \in C_0^\infty(\Omega) : \, u \geq 1 \text{ on } K\}.$$

Let $p = const$. We define

$$\operatorname{cap}_p(K,\Omega) := \inf_{u \in W(K,\Omega)} \int_\Omega |\nabla u|^p \, dx. \tag{2.4}$$

Among properties of such a capacity let us mention that it is a monotone and subadditive set function. Also a set of zero capacity has zero measure. However, the opposite need not to hold and hence capacity provides us with a finer tool to discuss the measure theoretic properties of Sobolev functions than the Lebesgue measure. In Section 3 we will use capacity estimates for annuli in $\mathbb{R}^n$. For the readers convenience we recall them now (cf. 2.11 in [14]).

Let $0 < r < R < \infty$. Then for $x_0 \in \mathbb{R}^n$ it holds that

$$\operatorname{cap}_p(\overline{B(x_0,r)}, B(x_0,R)) \leq \begin{cases} \omega_{n-1}(\frac{|n-p|}{p-1})^{p-1} |R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}}|^{1-p} & p \neq n \\ \omega_{n-1}(\log \frac{R}{r})^{1-n} & p = n. \end{cases}$$

In particular for $R = 2r$ we have that

$$\operatorname{cap}_p(\overline{B(x_0,r)}, B(x_0,2r)) \leq c_1(n,p) r^{n-p} \tag{2.5}$$

for all $p > 1$.

The main differential operator studied in this paper is the so-called *$p(\cdot)$-Laplacian ($p(\cdot)$-harmonic operator)* $\Delta_{p(\cdot)}$.

Definition 2.1. We say that a function $u : \Omega \to \mathbb{R}$ such that $u \in W^{1,p(\cdot)}_{loc}(\Omega)$ is *$p(\cdot)$-harmonic* if it satisfies $p(\cdot)$-harmonic equation

$$-\Delta_{p(\cdot)} u := -\operatorname{div}(|\nabla u|^{p(\cdot)-2} \nabla u) \;=\; 0 \quad \text{ in } \Omega$$

in the weak sense, i.e.

$$\int_\Omega |\nabla u(x)|^{p(x)-2} \langle \nabla u(x), \nabla \phi(x) \rangle \, dx \;=\; 0, \tag{2.6}$$

for all $\phi \in C_0^\infty(\Omega)$.

For a survey of results for $p(\cdot)$-harmonic equation and for further references see e.g. [13].

In a similar way we define a *weak subsolution of $p(\cdot)$-harmonic equation* (called for short *$p(\cdot)$-subsolution*) as satisfying

$$\int_\Omega |\nabla u(x)|^{p(x)-2} \langle \nabla u(x), \nabla \phi(x) \rangle \, dx \;\leq\; 0, \tag{2.7}$$

for all $\phi \in C_0^\infty(\Omega)$ such that $\phi \geq 0$ in Ω.

Throughout the paper by c and C we denote generic constants, whose values may change between appearances even within a single line.

3 A Phragmén-Lindelöf theorem for $p(\cdot)$-harmonic subsolutions

The purpose of this section is to discuss one of the main results of the paper, a Phragmén-Lindelöf type theorem for $p(\cdot)$-subsolutions on unbounded domains (see Remark 3.4 for a variant of this result for the strong $p(\cdot)$-Laplacian). Upon showing theorem we discuss number of its applications including domains previously studied in the literature. We also compare our results to these known in the case $p = const$. Our results are new in the variable exponent case and generalize these for the constant exponent case. Furthermore, for some domains discussed previously we partially retrieve assertions known for $p = const$ and obtain results not known even in that case, see Section 3.3 and Remark 3.8.

3.1 Function τ

The growth of a solution to PDE considered on an unbounded set depends among others on the geometry of this set. For instance, we expect that there should be a difference in the behavior of solution depending on how porous a domain Ω is or on the ratio of volume of Ω to the volume of ball B_R for large radii R. Below we propose a candidate for a function that captures some density properties. Such a function will be used in constructing test functions in the proof of Phragmén-Lindelöf theorem for $p(\cdot)$-harmonic equation.

Fix $R > 1$ and $c > 0$. We define

$$\tau_R^c(|x|) := R^{-\fint_0^{|x|} \frac{\lambda_{n-1}(\Omega\cap S_t)}{\lambda_{n-1}(S_t)}dt} \quad \text{for } |x| \geq c \tag{3.1}$$

and set $\tau_R^c(|x|) := \tau_R^c(c)$ for $|x| < c$. In what follows R and c will be often fixed or their values will be clear from the context of the presentation, and so we will omit them in the definition of τ and write $\tau_R^c = \tau$. Next, denote

$$\rho_\Omega(t) := \frac{\lambda_{n-1}(\Omega \cap S_t)}{\lambda_{n-1}(S_t)},$$
$$\rho_\Omega^-(|x|) := \operatorname{ess\,inf}_{0\leq t\leq |x|} \frac{\lambda_{n-1}(\Omega \cap S_t)}{\lambda_{n-1}(S_t)} \quad \text{and} \quad \rho_\Omega^+(|x|) := \operatorname{ess\,sup}_{0\leq t\leq |x|} \frac{\lambda_{n-1}(\Omega \cap S_t)}{\lambda_{n-1}(S_t)}.$$

One can interpret $\rho_\Omega^-(|x|)$ and $\rho_\Omega^+(|x|)$ as, respectively, lower and upper density functions of $(n-1)$-dimensional cross-cuts of Ω by the sphere of radius t.

In our studies we will be mainly interested in behavior of solutions for large radii. Therefore, without much loss of generality we may assume that domain Ω satisfies that

$$\rho_\Omega^-(c) = \rho_\Omega^+(c) = \rho_\Omega(c), \tag{3.2}$$

which holds if $\rho_\Omega(t)$ is constant for $t \leq c$ (see Example 3.2).

Since $\frac{\lambda_{n-1}(\Omega\cap S_t)}{\lambda_{n-1}(S_t)} \in [0, 1]$ for all $t > 0$, it holds that

$$\frac{1}{R} \leq \tau(|x|) < 1 \quad \text{for all } x \in \Omega. \tag{3.3}$$

Remark 3.1. The definition of τ was inspired by a work of Miklyukov, who also proved a variant of the Phragmén-Lindelöf principle for a class of A-harmonic equation, see [23].

Let us illustrate the presentation by computing τ for some domains.

Example 3.2. Denote $x = (x_1, \ldots, x_n)$ a point in $\mathbb{R}^n$.
(1) Let Ω be a half-space in $\mathbb{R}^n$, i.e. $\Omega = \{x \in \mathbb{R}^n : x_1 \geq 0\}$. Then $\frac{\lambda_{n-1}(\Omega\cap S_t)}{\lambda_{n-1}(S_t)} = \frac{1}{2}$ for all $t > 0$. Hence $\tau(|x|) = R^{-\frac{1}{2}}$ for all $x \in \Omega$ and $c = 0$.
(2) Let Ω be an angular sector in $\mathbb{R}^n$ with angle α, then $\tau(|x|) = R^{-\alpha}$.
(3) If Ω is a cone in $\mathbb{R}^n$ with angle $0 < \alpha \leq \pi$, then $\tau(|x|) = R^{\frac{1}{2}(1-\cos\frac{\alpha}{2})}$.
(4) Let Ω be an infinitely long strip in $\mathbb{R}^2$ of width $h \geq 1$, $\Omega = \{x \in \mathbb{R}^2 : 0 \leq x_2 \leq h\}$. Then

$$\tau_R^h(|x|) = \tau(|x|) = \begin{cases} R^{-\frac{1}{2}}, & 0 \leq |x| \leq h, \\ R^{-\frac{1}{2}\frac{h}{|x|} - \frac{1}{|x|}\int_h^{|x|} \frac{1}{\pi}\arcsin\frac{h}{t}dt}, & |x| > h. \end{cases}$$

In the proof of Theorem 3.3 below we will use $\nabla\tau$ and an estimate for $|\nabla\tau|$. If $|x| \geq c$, then

$$\nabla\tau(|x|) = \tau(|x|) \ln R \left[\frac{1}{|x|} \fint_0^{|x|} \frac{\lambda_{n-1}(\Omega \cap S_t)}{\lambda_{n-1}(S_t)}dt - \frac{1}{|x|} \frac{\lambda_{n-1}(\Omega \cap S_{|x|})}{\lambda_{n-1}(S_{|x|})} \right] \frac{x}{|x|}. \tag{3.4}$$

With the above notation we have that

$$\frac{|\nabla\tau(|x|)|}{\tau(|x|)} \leq \max\{\rho_\Omega^+(|x|) - \rho_\Omega(|x|), \rho_\Omega(|x|) - \rho_\Omega^-(|x|)\} \frac{\ln R}{|x|} \leq \left(\rho_\Omega^+(|x|) - \rho_\Omega^-(|x|)\right) \frac{\ln R}{|x|} \quad \text{for } |x| \geq c. \tag{3.5}$$

By (3.3) and (3.5) we have that $\tau, \nabla\tau \in L^\infty_{loc}(\Omega)$. By (3.2) and (3.5) we see that $\lim_{|x|\to c^+} |\nabla\tau(|x|)| = 0$. Note also, that under assumption (3.2) it holds that $|\nabla\tau(|x|)| \equiv 0$ for $|x| < c$. Hence, we conclude that $\tau \in W^{1,\infty}_{loc}(\Omega)$.

Estimate in (3.5) can be improved if we additionally use the fact that $\ln R \leq R^\alpha$ for some $0 < \alpha < 1$ and R large enough (in fact for $R \geq \alpha^{-\frac{1}{\alpha}}$). In what follows we will take $\alpha = \frac{1}{2}$. Hence (3.5) becomes

$$\frac{|\nabla\tau|}{\tau} \leq \frac{1}{|x|} R^{\alpha(\rho_\Omega^+(|x|) - \rho_\Omega^-(|x|))} = \frac{1}{|x|} R^{\frac{1}{2}\left(\rho_\Omega^+(|x|) - \rho_\Omega^-(|x|)\right)}. \tag{3.6}$$

3.2 The main theorem

In this section we prove the main result of this paper, namely the Phragmén-Lindelöf theorem for $p(\cdot)$-subsolutions in an unbounded domain.

Let $p : \Omega \to (1, \infty)$ be a bounded log-Hölder continuous variable exponent function such that $p^- \neq n-1$. Furthermore, suppose that $p \equiv const$ on B_c, where c is a constant in (3.1), and ∇p satisfies the following decay condition for some given $c_p > 0$ and $0 \leq \alpha_p < \infty$ such that $\alpha_p \neq \frac{n}{p^-}$:

$$|\nabla p(x)| \leq c_p |x|^{-\alpha_p} \quad \text{for all } |x| > c, x \in \Omega. \tag{3.7}$$

Theorem 3.3. *Let Ω be an unbounded domain in $\mathbb{R}^n$ and u be a $p(\cdot)$-subsolution as in (2.7). Moreover, let p satisfy conditions (3.7) and suppose that $\lim_{\Omega \ni x \to \partial\Omega} u(x) \leq 0$. Denote*

$$m_R := \operatorname{ess\,sup}_{x \in \Omega \cap B_R} \tau(|x|)(u_+(x))^{p(x)}.$$

Then either $u \leq 0$ in Ω or

$$\liminf_{R \to \infty} \frac{m_R}{R^\gamma} > 0, \tag{3.8}$$

for any γ such that $\lim_{R \to \infty} \Gamma(R) = 0$, where

$$\begin{aligned} \Gamma(R) := R^{1+\gamma} \Big(& R^{n - p^- \alpha_p} + R^{\frac{1}{2} p^+ \left(\rho_\Omega^+(2R) - \rho_\Omega^-(2R) \right) + n - p^-} \\ & + \max \left\{ \operatorname{cap}_{p^+}^{p^-/p^+}(\overline{\Omega \cap B_R}, \Omega \cap B_{2R}), \operatorname{cap}_{p^+}(\overline{\Omega \cap B_R}, \Omega \cap B_{2R}) \right\} R^{n - n\frac{p^-}{p^+}} \Big). \end{aligned} \tag{3.9}$$

Proof. Suppose that $u(x_0) > 0$ for some $x_0 \in \Omega$. By the maximum principle for $p(\cdot)$-Laplacian (cf. Theorem 3.4 in [10]) and the assumption that $\lim_{\Omega \ni x \to \partial\Omega} u(x) \leq 0$, we know that there exists an unbounded component of Ω containing x_0 such that $u > 0$ inside. For the sake of simplicity of notation denote this component Ω. Define a test function

$$\phi(x) = \eta(x)^{p(x)} \tau(|x|) u_+(x),$$

where:

1. $\eta \in C_0^\infty(\Omega \cap B_{2R})$;
2. $0 \leq \eta \leq 1$, and $\eta \equiv 1$ on $B_R \cap \operatorname{supp} \eta$.

It is easy to see, that

$$\nabla \phi = u_+ \tau \eta^{p(\cdot)} \ln \eta \nabla p + p(\cdot) u_+ \tau \eta^{p(\cdot)-1} \nabla \eta + \eta^{p(\cdot)} \tau \nabla u_+ + u_+ \eta^{p(\cdot)} \nabla \tau. \tag{3.10}$$

Using ϕ as a test function in (2.7) we obtain the following inequality:

$$\begin{aligned} \int_{\Omega \cap B_{2R}} |\nabla u_+|^{p(x)} \tau(|x|) \eta^{p(x)} \leq & \int_{\Omega \cap B_{2R}} |\nabla u_+|^{p(x)-1} |\eta^{p(x)} \ln \eta| u_+ |\nabla p| \tau(|x|) \\ & + \int_{\Omega \cap B_{2R}} p(x) |\nabla u_+|^{p(x)-1} \eta^{p(x)-1} |\nabla \eta| u_+ \tau(|x|) \\ & + \int_{\Omega \cap B_{2R}} |\nabla u_+|^{p(x)-1} \eta^{p(x)-1} u_+ |\nabla \tau| \eta. \end{aligned} \tag{3.11}$$

Denote integrals on the right-hand side of (3.11) by I_0, I_1 and I_2 respectively. Using the Young inequality (2.3) for some $\epsilon \in (0,1)$, whose value will be determined later in the proof, we easily estimate I_0 as follows.

$$\begin{aligned} I_0 & \leq \int_{\Omega \cap B_{2R}} \left(\epsilon |\nabla u_+|^{p(x)-1} \eta^{p(x)-1} \tau^{\frac{p(x)-1}{p(x)}} \right) \left(\epsilon^{-1} \eta^{\frac{1}{2}} u_+ |\nabla p| \tau^{\frac{1}{p(x)}} \right) \\ & \leq \int_{\Omega \cap B_{2R}} \epsilon^{\frac{p(x)}{p(x)-1}} \frac{p(x)}{p(x)-1} |\nabla u_+|^{p(x)} \tau(|x|) \eta^{p(x)} + \int_{\Omega \cap B_{2R}} \frac{\epsilon^{-p(x)}}{p(x)} u_+^{p(x)} |\nabla p|^{p(x)} \tau \eta^{\frac{p(x)}{2}}. \end{aligned} \tag{3.12}$$

In order to estimate integral I_1 we use again the Young inequality.

$$\begin{aligned} I_1 & \leq \int_{\Omega \cap B_{2R}} p(x) \left(|\nabla u_+|^{p(x)-1} \eta^{p(x)-1} \tau^{\frac{p(x)-1}{p(x)}} \right) \left(u_+ \tau^{\frac{1}{p(x)}} |\nabla \eta| \right) \\ & \leq \int_{\Omega \cap B_{2R}} \delta^{\frac{p(x)}{p(x)-1}} (p(x)-1) |\nabla u_+|^{p(x)} \tau \eta^{p(x)} + \int_{\Omega \cap B_{2R}} \delta^{-p(x)} u_+^{p(x)} \tau |\nabla \eta|^{p(x)}. \end{aligned} \tag{3.13}$$

Similarly, we obtain that

$$
\begin{aligned}
I_2 &\le \int_{\Omega\cap B_{2R}} \big(|\nabla u_+|^{p(x)-1}\eta^{p(x)-1}\tau^{\frac{p(x)-1}{p(x)}}\big)\big(u_+|\nabla\tau|\tau^{\frac{1-p(x)}{p(x)}}\eta\big) \\
&\le \int_{\Omega\cap B_{2R}} \sigma^{\frac{p(x)}{p(x)-1}}\frac{p(x)-1}{p(x)}|\nabla u_+|^{p(x)}\tau\eta^{p(x)} + \int_{\Omega\cap B_{2R}} \frac{\sigma^{-p(x)}}{p(x)}u_+^{p(x)}\eta^{p(x)}|\nabla\tau|^{p(x)}\tau^{1-p(x)}.
\end{aligned} \tag{3.14}
$$

We use inequalities (3.12)-(3.14) in estimate (3.11) and choose $\epsilon,\delta,\sigma\in(0,1)$ so that integrals with $|\nabla u_+|^{p(x)}$ can be included into the left-hand side of (3.11). Hence, we arrive at the following inequality:

$$
\begin{aligned}
c(p^-,p^+)\int_{\Omega\cap B_{2R}} |\nabla u_+|^{p(x)}\tau\eta^{p(x)} \le &\int_{\Omega\cap B_{2R}} u_+^{p(x)}|\nabla p|^{p(x)}\tau\eta^{\frac{p(x)}{2}} \\
&+\int_{\Omega\cap B_{2R}} u_+^{p(x)}\tau|\nabla\eta|^{p(x)} \\
&+\int_{\Omega\cap B_{2R}} u_+^{p(x)}\tau\Big(\frac{|\nabla\tau|}{\tau}\Big)^{p(x)}\eta^{p(x)}.
\end{aligned} \tag{3.15}
$$

Next, we estimate three integrals in (3.15). Each of them corresponds to a different feature of the discussed problem: the first integral captures impact of the rate of growth of variable exponent, the second integral describes the size of set Ω expressed in terms of capacity (see below). Finally, the third integral carries information about change in amount of Ω contained in ball B_{2R} expressed in terms of function τ.

Definition of m_R results in the following inequality:

$$
\begin{aligned}
&\int_{\Omega\cap B_{2R}} u_+^{p(x)}|\nabla p|^{p(x)}\tau\eta^{\frac{p(x)}{2}} \\
&\le \mathrm{m}_{2R}\int_{\Omega\cap B_{2R}} |\nabla p|^{p(x)}\,dx \\
&\le \mathrm{m}_{2R}\,c_p\int_c^{2R}\lambda_{n-1}(\Omega\cap S_t^{n-1})t^{-p^-\alpha_p}dt,
\end{aligned} \tag{3.16}
$$

where in the last estimate we have used spherical coordinates and the growth condition (3.7) for ∇p. Simple integration gives us that (3.16) is bounded by the following expression.

$$
\int_{\Omega\cap B_{2R}} u_+^{p(x)}|\nabla p|^{p(x)}\tau\eta^{\frac{p(x)}{2}} \le \begin{cases} c\,\mathrm{m}_{2R}\,R^{n-p^-\alpha_p} & \text{if } \alpha_p\neq\frac{n}{p^-} \\ c\,\mathrm{m}_{2R}\,\ln R & \text{if } \alpha_p=\frac{n}{p^-}. \end{cases} \tag{3.17}
$$

Here, constant c depends on c_p, p^-, α_p, n and ω_{n-1}, the measure of the unit $(n-1)$-dimensional sphere.

In order to estimate the second integral we employ variational capacity (see Preliminaries (2.4)). By the variable exponent Hölder inequality (2.2) and the unit ball property (2.1), applied to norms of $\nabla\eta$ and a unit constant function, we have that

$$
\begin{aligned}
\int_{\Omega\cap B_{2R}} |\nabla\eta|^{p(x)} &\le 2\|\nabla\eta\|_{L^{\frac{p^+}{p(\cdot)}}(\Omega\cap B_{2R})}\|1\|_{L^{\frac{p^+}{p^+-p(\cdot)}}(\Omega\cap B_{2R})} \\
&\le 2\max\Big\{\Big(\int_{\Omega\cap B_{2R}}|\nabla\eta|^{p^+}\Big)^{p^-/p^+}, \int_{\Omega\cap B_{2R}}|\nabla\eta|^{p^+}\Big\}\max\big\{1,|\Omega\cap B_{2R}|^{\frac{p^+-p^-}{p^+}}\big\}.
\end{aligned} \tag{3.18}
$$

Here we have also used the convention that $\frac{1}{\infty}:=0$. Since this estimate holds for any test function η defined in the beginning of this proof it holds also if we take infimum over all such η. Thus for $R\ge 1$ we obtain:

$$
\int_{\Omega\cap B_{2R}} |\nabla\eta|^{p(x)} \le 2c\max\big\{\mathrm{cap}_{p^+}^{p^-/p^+}(\overline{\Omega\cap B_R},\Omega\cap B_{2R}),\mathrm{cap}_{p^+}(\overline{\Omega\cap B_R},\Omega\cap B_{2R})\big\}R^{n(1-\frac{p^-}{p^+})}. \tag{3.19}
$$

Notice that a priori we cannot improve the above capacity estimate, since we do not have any additional information about how substantial is an amount of Ω contained in large balls. For this reason we will leave estimate (3.19) in that form and instead be more specific in the illustrative results (see Section 3.3). In order to improve the clarity of the presentation we denote the expression on the right hand side of inequality (3.19) by $\mathrm{cap_R}$.

Finally we estimate the third integral in (3.15). Using properties (3.2) and (3.6) of τ we have that

$$
\int_{\Omega\cap B_{2R}} u_+^{p(x)}\tau\frac{|\nabla\tau|^{p(x)}}{\tau^{p(x)}}\eta^{p(x)} \le \mathrm{m}_{2R}\int_{\Omega\cap B_{2R}\setminus(\Omega\cap B_c)}\Big(\frac{1}{|x|}R^{\frac{1}{2}\left(\rho_\Omega^+(|x|)-\rho_\Omega^-(|x|)\right)}\Big)^{p(x)}. \tag{3.20}
$$

We integrate (3.20) using spherical coordinates to obtain that

$$\begin{aligned}\int_{\Omega\cap B_{2R}} u_+^{p(x)}\tau\eta^{p(x)}\frac{|\nabla\tau|^{p(x)}}{\tau^{p(x)}} &\le \mathrm{m}_{2R}\int_{\Omega\cap B_{2R}\setminus(\Omega\cap B_c)} R^{\frac{p(x)}{2}(\rho^+_\Omega(|x|)-\rho^-_\Omega(|x|))}|x|^{-p(x)}\\ &\le \mathrm{m}_{2R}\, R^{\frac12 p^+\left(\rho^+_\Omega(2R)-\rho^-_\Omega(2R)\right)}\int_c^{2R}\lambda_{n-1}(\Omega\cap S_t^{n-1})t^{-p^-}\,dt\\ &\le \begin{cases} C\mathrm{m}_{2R}R^{\frac12 p^+\left(\rho^+_\Omega(2R)-\rho^-_\Omega(2R)\right)+n-p^-} & p^-\neq n-1\\ C\mathrm{m}_{2R}R^{\frac12 p^+\left(\rho^+_\Omega(2R)-\rho^-_\Omega(2R)\right)}\ln R & p^-=n-1.\end{cases}\end{aligned}\tag{3.21}$$

Here, constant C depends only on p^+, p^-, n and the measure of the unit sphere in $\mathbb{R}^n$. Note that by the properties of τ and η we have that

$$\frac{1}{R}\int_{\Omega\cap B_R}|\nabla u_+|^{p(x)} \le \int_{\Omega\cap B_{2R}}|\nabla u_+|^{p(x)}\tau(|x|)\eta^{p(x)}.$$

Combining this observation together with (3.15), (3.17), (3.19) and (3.21) we arrive at the following inequality.

$$\int_{\Omega\cap B_R}|\nabla u_+|^{p(x)} \le c\frac{\mathrm{m}_{2R}}{R^\gamma}\,\Gamma(R),\tag{3.22}$$

where

$$\Gamma(R) = R^{1+\gamma}\left(R^{n-p^-\alpha_p} + R^{\frac12 p^+\left(\rho^+_\Omega(2R)-\rho^-_\Omega(2R)\right)+n-p^-} + \mathrm{cap_R}\right)\tag{3.23}$$

for $\alpha_p \neq \frac{n}{p^-}$ and $p^- \neq n-1$;

$$\Gamma(R) = R^{1+\gamma}\left(\ln R + R^{\frac12 p^+\left(\rho^+_\Omega(2R)-\rho^-_\Omega(2R)\right)+n-p^-} + \mathrm{cap_R}\right)$$

for $\alpha_p = \frac{n}{p^-}$ and $p^- \neq n-1$;

$$\Gamma(R) = R^{1+\gamma}\left(R^{n-p^-\alpha_p} + R^{\frac12 p^+\left(\rho^+_\Omega(2R)-\rho^-_\Omega(2R)\right)}\ln R + \mathrm{cap_R}\right)$$

for $\alpha_p \neq \frac{n}{p^-}$ and $p^- = n-1$;

$$\Gamma(R) = R^{1+\gamma}\left(\ln R + R^{\frac12 p^+\left(\rho^+_\Omega(2R)-\rho^-_\Omega(2R)\right)}\ln R + \mathrm{cap_R}\right)$$

for $\alpha_p = \frac{n}{p^-}$ and $p^- = n-1$.

Recall that in the beginning of the proof we have assumed that $u(x_0) > 0$ for some $x_0 \in \Omega$. First, let us consider the case $\alpha_p \neq \frac{n}{p^-}$ and $p^- \neq n-1$. Suppose now, on the contrary to assertion of theorem that $\liminf_{R\to\infty}\frac{\mathrm{m}_{2R}}{R^\gamma} = 0$ and denote $\{R_i\}_{i=1}^\infty$ a sequence of radii along which the liminf is attained. Then for γ and $\Gamma(R)$ as in assumption (3.9), and hence (3.23), we have that $\Gamma(R_i)\to 0$ for $R_i\to\infty$ which gives the contradiction, since then by (3.22) we get that $\lim_{R_i\to\infty}\int_{\Omega\cap B_{R_i}}|\nabla u_+|^{p(x)} = 0$. Hence $\nabla u_+ \equiv 0$ in Ω and so $u_+ \equiv \mathit{const}$. In a consequence we obtain that $u \le 0$ in Ω (u is locally continuous and our standing assumption is that the limit of u at the boundary of the underlying domain is non-positive). This contradicts our initial assumption and completes the proof of theorem in this case.

In order to discuss cases when $\alpha_p = \frac{n}{p^-}$ or $p^- = n-1$, let us start with observation that using definitions of m_{2R} and τ we may further estimate (3.22) as follows:

$$\begin{aligned}\frac{\mathrm{m}_{2R}}{R^\gamma}\,\Gamma(R) &\le \frac{\operatorname{ess\,sup}_{x\in\Omega\cap B_{2R}}\tau(|x|)(u_+(x))^{p(x)}}{R^\gamma}\,\Gamma(R)\\ &\le \frac{\operatorname{ess\,sup}_{x\in\Omega\cap B_{2R}}(u_+(x))^{p(x)}}{R^{\gamma+\rho^-_\Omega(2R)}}\,\Gamma(R).\end{aligned}\tag{3.24}$$

If $\gamma+\rho^-_\Omega(2R)$ is negative for all sufficiently large radii R, then clearly assertion (3.8) is trivially true since $u_+ > 0$ in Ω. Hence, in order to obtain nontrivial assertion we need to have that $\gamma > -\sup_R \rho^-_\Omega(2R)$ which is necessary greater than -1, as $0 \le \rho_\Omega(t) \le 1$ for all t.

However, observe that if $\alpha_p = \frac{n}{p^-}$ or $p^- = n-1$ in one of the three above expressions for $\Gamma(R)$, then conditions $\Gamma(R) \to 0$ and $\gamma > -1$ cannot both be satisfied. Indeed, having

$$\lim_{R\to\infty} R^{1+\gamma} \ln R = 0 \quad \text{ or } \quad \lim_{R\to\infty} R^{1+\gamma+\frac{p^+}{2}\left(\rho_\Omega^+(2R)-\rho_\Omega^-(2R)\right)} \ln R = 0$$

requires $\gamma < -1$. This observation results in the trivial assertion in both cases $\alpha_p = \frac{n}{p^-}$ or $p^- = n-1$. Therefore, theorem holds only for $\Gamma(R)$ as in (3.23) and so the proof of Theorem 3.3 is completed. □

Remark 3.4. In [2] the authors introduced the so-called *strong $p(\cdot)$-Laplacian* denoted $\widetilde{\Delta}_{\mathrm{p}(\cdot)}$ and studied the related homogeneous equation and its weak solutions

$$\begin{aligned} \widetilde{\Delta}_{p(\cdot)}\, u &:= \operatorname{div}(|\nabla u|^{p(\cdot)-2}\nabla u) - |\nabla u|^{p(\cdot)-2} \ln|\nabla u| \langle \nabla u, \nabla p\rangle \\ &= |\nabla u|^{p(\cdot)-4}\left[(p(\cdot)-2)\,\Delta_\infty u + |\nabla u|^2\, \Delta\, u\right] = 0, \end{aligned} \tag{3.25}$$

where Δ_∞ stands for the infinity Laplacian. The weak formulation of $\widetilde{\Delta}_{p(\cdot)}\, u = 0$ then requires that

$$\int_\Omega |\nabla u|^{p(x)-2} \langle \nabla u, \nabla \phi\rangle\, dx + \int_\Omega |\nabla u|^{p(x)-2} \log(|\nabla u|) \langle \nabla u, \nabla p\rangle\, \phi\, dx = 0 \tag{3.26}$$

for all $\phi \in W_0^{1,p(\cdot)}(\Omega)$. This equation, like the $p(\cdot)$-harmonic, reduces to the ordinary p-Laplace equation when p is constant.

Such an equation has many advantages comparing to the $p(\cdot)$-harmonic one studied here: scalability of solutions, homogeneous Harnack inequality with constant independent on solution [3], relations to quasiregular mappings in the plane, the infinity Laplacian and viscosity solutions [2] (see also [18], [25] and [31] for some further studies on this equation).

The theorem of Phragmén-Lindelöf type can also be proven for subsolutions of the strong $p(\cdot)$-Laplace equation $-\widetilde{\Delta}_{\mathrm{p}(\cdot)} \leq 0$. The formulation and the proof are similar to these of Theorem 3.3 and, therefore, we will not present the details. Instead, we will comment the major changes one has to introduce in the proof of Theorem 3.3.

Consider test function ϕ as defined in the proof of Theorem 3.3 with $\nabla\phi$ as in (3.10). The weak formulation of $-\widetilde{\Delta}_{\mathrm{p}(\cdot)} \leq 0$ for ϕ reads

$$\int_{\Omega\cap B_{2R}} |\nabla u|^{p(x)-2}\langle \nabla u, \nabla\phi\rangle \leq -\int_{\Omega\cap B_{2R}} |\nabla u|^{p(x)-2} \ln|\nabla u| \langle \nabla u, \nabla p\rangle \eta^{p(x)} u_+ \tau. \tag{3.27}$$

Hence, the counterpart of inequality (3.11) has one additional term

$$\int_{\Omega\cap B_{2R}} |\nabla u_+|^{p(x)-1+\epsilon} |\nabla p| \eta^{p(x)} u_+ \tau. \tag{3.28}$$

Here we have estimated $\ln|\nabla u| \leq |\nabla u|^\epsilon$ for some fixed $0 < \epsilon \leq 1$. Applying the Young inequality we obtain the following estimate of (3.28)

$$\delta \int_{\Omega\cap B_{2R}} |\nabla u|^{p(x)} \tau \eta^{p(x)} + \delta^{-\frac{p^+}{1-\epsilon}} \int_{\Omega\cap B_{2R}} (u-k)_+^{\frac{p(x)}{1-\epsilon}} |\nabla p|^{\frac{p(x)}{1-\epsilon}} \tau \eta^{p(x)} \tag{3.29}$$

for some $0 < \delta < 1$. The resulting estimates of (3.29) are then similar to these for integral I_0, cf. (3.16) and (3.17).

3.3 Applications of Theorem 3.3

In the previous section we formulate and prove Theorem 3.3, a variable exponent analog of the classical Phragmén-Lindelöf theorem, where the growth condition at infinity for a $p(\cdot)$-harmonic function u on an unbounded domain Ω has been expressed in terms of a general condition (3.9). The purpose of this section is to illustrate Theorem 3.3 with examples of domains typically appearing in the context of Phragmén-Lindelöf theorem, such as half-space, sectors and domains narrowing at infinity.

3.3.1 The Phragmén-Lindelöf principle for a half-space

Denote $\mathbb{R}^n_+$ the half-space in $\mathbb{R}^n$, i.e. set $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \,:\, x_1 \geq 0\}$ (see also Example 3.2).

Corollary 3.5. *Let u be a $p(\cdot)$-subsolution of (2.7) in $\mathbb{R}^n_+$. Suppose that variable exponent p, ∇p and α_p satisfy set of assumptions (3.7), $u(x_0) > 0$ for some $x_0 \in \Omega$ and $\lim_{\Omega \ni x \to \partial\Omega} u(x) \leq 0$.*
If $p^+ \geq n$ and α_p satisfies

$$0 \leq \alpha_p \leq \tfrac{p^+}{p^-} + \tfrac{n}{p^-}\big(\tfrac{p^-}{p^+} - 1\big) \tag{3.30}$$

then

$$\liminf_{R\to\infty} \frac{\operatorname{ess\,sup}_{x\in\Omega\cap B_{2R}} (u_+(x))^{p(x)}}{R^\gamma} > 0, \tag{3.31}$$

for any $\gamma \leq p^-\alpha_p - n - \frac{1}{2}$. A necessary condition for $\gamma > 0$ is that $\alpha_p > \frac{2n+1}{2p^-}$. In particular, if $\alpha_p > \frac{2n+3}{2p^-}$, then one can take $\gamma = 1$.
If $p^+ < n$ and (3.30) is satisfied, then theorem holds under additional assumption that $1 \leq \frac{p^-}{p^+} + \frac{p^+}{n}$.
Otherwise, if

$$\alpha_p > \tfrac{p^+}{p^-} + \tfrac{n}{p^-}\big(\tfrac{p^-}{p^+} - 1\big), \tag{3.32}$$

then the assertion of theorem holds for $\gamma < p^+ - \frac{1}{2} + n(\frac{p^-}{p^+} - 2)$. A necessary condition for $\gamma > 0$ is that $\frac{p^-}{p^+} > 2 + \frac{1}{2n} - \frac{p^+}{n}$.

In particular, if $n = 2$ and $p^- > \max\{\frac{7}{2\alpha_p}, \frac{p^+}{4}(11-2p^+)\}$, then $\gamma = 1$ and we retrieve the growth condition of classical Phragmén-Lindelöf theorem for harmonic functions in $\mathbb{R}^2_+$.

In the special case $p^- = p^+ = p = const$, we have that $\gamma = 1$ provided that $p > n + \frac{3}{2}$, obtaining part of the Phragmén-Lindelöf theorem for p-harmonic functions in $\mathbb{R}^n_+$ (Theorem 4.6 in [21]).

Proof. If $\Omega = \mathbb{R}^n_+$, then $\tau \equiv R^{-\frac{1}{2}}$ (cf. Example 3.2) and so $\nabla\tau \equiv 0$ and $\rho^+_\Omega(2R) = \rho^-_\Omega(2R) = R^{-\frac{1}{2}}$ for all positive R. Thus, integral (3.20) vanishes and estimate (3.15) consists of only two integrals (also $\Gamma(R)$ in (3.9) has only two terms). The capacity estimate for Euclidean annuli (cf. (2.5) and [14, Section 2.11]) reads:

$$\operatorname{cap}_{p^+}(\overline{\mathbb{R}^n_+ \cap B_R}, \mathbb{R}^n_+ \cap B_{2R}) \leq \begin{cases} cR^{n-p^+} & \text{for } p^+ \neq n, \\ (\log 2)^{1-n} & \text{for } p^+ = n. \end{cases} \tag{3.33}$$

Hence, condition (3.9) in Theorem 3.3 becomes as follows.

$$R^{-\frac{1}{2}}\Gamma(R) = R^{\frac{1}{2}+\gamma+n-p^-\alpha_p} + R^{\frac{1}{2}+\gamma+2n-p^+-n\frac{p^-}{p^+}} \to 0 \quad \text{for } R\to\infty \quad \text{and } p^+ \neq n,$$

$$R^{-\frac{1}{2}}\Gamma(R) = R^{\frac{1}{2}+\gamma+n-p^-\alpha_p} + R^{\frac{1}{2}+\gamma+n(1-\frac{p^-}{p^+})} \to 0 \quad \text{for } R\to\infty \quad \text{and } p^+ = n.$$

Therefore for $p^+ \neq n$ it holds that

$$\begin{cases} \gamma < -\frac{1}{2} - n + p^-\alpha_p \\ \gamma < -\frac{1}{2} - 2n + p^+ + n\frac{p^-}{p^+}. \end{cases} \tag{3.34}$$

Similarly, for $p^+ = n$ we have that

$$\begin{cases} \gamma < -\frac{1}{2} - n + p^-\alpha_p \\ \gamma < -\frac{1}{2} + n(\frac{p^-}{p^+} - 1). \end{cases} \tag{3.35}$$

The analysis of systems (3.34) and (3.35) leads to two cases (3.30) and (3.32). If $p^+ \geq n$, then the upper bound in (3.30) is always positive. If $p^+ < n$, then one additionally needs to assume that $1 \leq \frac{p^-}{p^+} + \frac{p^+}{n}$. Then, the assertion of theorem follows immediately from (3.34) by computations.

By taking $n = 2$ and $\gamma = 1$ in (3.34) we obtain the assertion in the planar case.

If $p^- = p^+ = p = const$, then $\nabla p \equiv 0$ and so integral (3.16) vanishes. In such a case the right hand side of (3.15) consists of one integral only (and hence (3.9) has only one term). As a result system of equations (3.34) reduces to the second inequality, which reads

$$\gamma < -\tfrac{1}{2} - n + p.$$

Requiring $p > n + \frac{3}{2}$ allows us to take $\gamma = 1$ and the classical p-harmonic case follows immediately. ∎

Remark 3.6. The above corollary can be given shorter and more compact formulation by the price of the lesser transparency of the mutual relations between p^+, p^-, n, α_p and γ.

Let u be a $p(\cdot)$-subsolution of (2.7) in $\mathbb{R}^n_+$. Suppose that variable exponent p, ∇p and α_p satisfy set of assumptions (3.7), $u(x_0) > 0$ for some $x_0 \in \Omega$ and $\lim_{\Omega \ni x \to \partial\Omega} u(x) \leq 0$. Then (3.31) holds for γ satisfying the following condition:

$$\gamma < \min\{-\tfrac{1}{2} - n + p^-\alpha_p\,,\, -\tfrac{1}{2} + p^+ + n(\tfrac{p^-}{p^+} - 2)\}. \tag{3.36}$$

In particular for $n = 2$ condition (3.36) reads: $\gamma < \min\{-\frac{5}{2} + p^-\alpha_p\,,\, -\frac{9}{2} + p^+ + 2\frac{p^-}{p^+}\}$, while for $p = const$ we obtain $\gamma < -\frac{1}{2} - n + p$.

3.3.2 The Phragmén-Lindelöf principle for an angular sector

In the next observation we discuss the case of an angular sector. For the sake of simplicity and clarity of the presentation we restrict our discussion to the planar case. Fix $0 < \alpha \leq 1$ and let

$$S_\alpha := \{re^{it} \in \mathbb{C} \,:\, 0 \leq t \leq \alpha\pi\}.$$

Corollary 3.7. *Let u be a $p(\cdot)$-subsolution of (2.7) in S_α. Suppose that variable exponent p and ∇p are as in (3.7), $u(x_0) > 0$ for some $x_0 \in \Omega$ and $\lim_{\Omega \ni x \to \partial\Omega} u(x) = 0$.*
If α_p satisfies

$$\tfrac{4-\alpha}{p^-} < \alpha_p \leq \tfrac{p^+}{p^-} + \tfrac{2}{p^-}\big(\tfrac{p^-}{p^+} - 1\big), \tag{3.37}$$

then assertion (3.31) holds for $\gamma < \alpha - 3 + p^-\alpha_p$. A necessary condition for $\gamma > 0$ is that $\alpha_p > \frac{4-\alpha}{p^-}$.
If instead $\alpha_p > \frac{p^+}{p^-} + \frac{2}{p^-}\big(\frac{p^-}{p^+} - 1\big)$, then (3.31) holds for a positive γ provided that

$$\gamma < p^+ + 2\tfrac{p^-}{p^+} - \tfrac{9}{2} \qquad \text{and} \qquad p^- > (\tfrac{9}{2} - p^+)\tfrac{p^+}{2}.$$

In particular, by (3.37) we have that if $\alpha_p > (3 - \alpha + \frac{1}{\alpha})\frac{1}{p^-}$, then $\gamma = \frac{1}{\alpha}$ and we retrieve the growth condition of classical Phragmén-Lindelöf theorem for harmonic functions in S_α, see e.g. [27, Theorem 18, Section 9].
If α_p satisfies (3.32), then we get $\gamma = \frac{1}{\alpha}$ provided $p^- > \frac{p^+}{2}(\frac{9}{2} + \frac{1}{\alpha} - p^+)$.
In the special case when $p^- = p^+ = p = const$, we have that $\gamma = \frac{1}{\alpha}$ provided that $p > 3 - \alpha + \frac{1}{\alpha}$.

Remark 3.8. The above corollary is new also in the constant exponent case.

Proof of Corollary 3.7. Let S_α be a sector in $\mathbb{R}^2$. Then $\tau \equiv R^{-\alpha}$ and so $\nabla\tau \equiv 0$. The capacity estimate is the same as in (3.33) with constant c depending additionally on the angle α. The remaining details of the reasoning are similar to these in Corollary 3.5 and, therefore, are omitted. □

Remark 3.9. Similarly to Remark 3.6 we provide an alternative formulation of the above corollary.

Let u be a $p(\cdot)$-subsolution of (2.7) in $\mathbb{R}^n_+$. Suppose that variable exponent p, ∇p and α_p satisfy set of assumptions (3.7), $u(x_0) > 0$ for some $x_0 \in \Omega$ and $\lim_{\Omega \ni x \to \partial\Omega} u(x) \leq 0$. Then (3.31) holds for γ satisfying the following condition:

$$\gamma < \min\{\alpha - 3 + p^-\alpha_p\,,\, \alpha - 5 + p^+ + 2\tfrac{p^-}{p^+}\}. \tag{3.38}$$

In particular for $p = const$ we have that $\gamma < \alpha - 3 + p$.

3.3.3 The Phragmén-Lindelöf principle for domains narrowing at infinity

In this section we illustrate Theorem 3.3 by discussing the case of a set narrow at infinity. It will turn out that for such a set impact of capacity term (3.19) on the growth of u for large x is negligible.
Let

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x \geq 0\,,\, -e^{-x} \leq y \leq e^{-x}\}.$$

If α denotes angle between the line segment joining the intersection point of the circle S_t with the curve $y = e^{-x}$ and x-axis, then $\tan\alpha = \frac{e^{-t}}{t}$ and hence

$$\begin{aligned}
\rho_\Omega(t) &= \frac{\lambda_{n-1}(\Omega \cap S_t)}{\lambda_{n-1}(S_t)} = 2\arctan e^{-t}t^{-1}, \\
\rho_\Omega^+(2R) &= 2\arctan e^{-1}, \\
\rho_\Omega^-(2R) &= 2\arctan e^{-2R}(2R)^{-1}.
\end{aligned} \tag{3.39}$$

The Phragmén-Lindelöf theorem for Ω takes the following form.

Corollary 3.10. *Let u be a $p(\cdot)$-subsolution of (2.7) in Ω. Suppose that variable exponent p and ∇p are as in (3.7), $u(x_0) > 0$ for some $x_0 \in \Omega$ and $\lim_{\Omega \ni x \to \partial\Omega} u(x) = 0$.*
If α_p satisfies

$$0 \le \alpha_p \le 1 - \frac{p^+}{p^-} \arctan \frac{1}{e} \tag{3.40}$$

then claim (3.31) holds for $\gamma < p^-\alpha_p - 3$. A necessary condition for $\gamma > 0$ is that $\alpha_p > \frac{3}{p^-}$.

If $\alpha_p > 1 - \frac{p^+}{p^-} \arctan \frac{1}{e}$, then (3.31) holds for $\gamma < p^- - 3 - p^+ \arctan \frac{1}{e}$. A necessary condition for $\gamma > 0$ is that $p^+ < \frac{p^- - 3}{\arctan \frac{1}{e}}$.

Proof. Using the definition of capacity (2.4) we obtain the following estimate:

$$\mathrm{cap}_{p^+}(\overline{\Omega \cap B_R}, \Omega \cap B_{2R}) \le cR^{p^+} e^{-R} \to 0 \quad \text{ for } R \to \infty$$

and therefore condition (3.9) in Theorem 3.3 simplifies significantly. Indeed, since for all γ, p^-, p^+ the capacity term in (3.9) approaches 0 when $R \to \infty$, it holds that

$$R^{-\rho_\Omega^-(2R)}\Gamma(R) \;=\; R^{-2\arctan e^{-2R}(2R)^{-1}}\Gamma(R) \le R^{1+\gamma+n-p^-\alpha_p} + R^{1+\gamma+n-p^-+p^+\arctan\frac{1}{e}}.$$

By (3.39) and (3.5) we know that $\tau \le R^{-2\arctan\frac{1}{e}}$ and $\frac{|\nabla \tau|}{\tau} \le 2(\arctan \frac{1}{e})\frac{\ln R}{|x|}$. These observations altogether imply the assertion of the corollary. □

4 The Phragmén-Lindelöf theorem for nonhomogeneous equations

The purpose of this section is to study the Phragmén-Lindelöf type theorems for a class of nonhomogeneous $p(\cdot)$-harmonic equations. Similar results for the growth of $|\nabla u|$ in the constant exponent case were obtained by Kurta [20]. In the setting of variable exponent our results are new. Moreover, we obtain some results new also in the setting of p-Laplacian and harmonic functions, see Corollaries 4.4 and 4.5.

4.1 A class of nonhomogeneous equations with nonstandard growth

Let us introduce a class of $p(\cdot)$-harmonic type equations, which will be the subject of our investigations.

Let $f = f(x,t,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ be a nontrivial function, i.e. $f \not\equiv 0$ such that $f(\cdot,t,\xi) \in L^1_{loc}(\Omega)$ for all $(t,\xi) \in R \times \mathbb{R}^n$. Consider equation

$$\Delta_{\mathrm{p}(\cdot)}(u) = f(x,u,\nabla u). \tag{4.1}$$

Similarly to the discussion for homogeneous $p(\cdot)$-harmonic equation weak solutions to equation (4.1) will be called, within this section, $p(\cdot)$-harmonic functions.

Definition 4.1. Let $\Omega \subset \mathbb{R}^n$ be open. We say that $u \in W^{1,p(\cdot)}_{loc}(\Omega)$ is a $p(\cdot)$-*subsolution* if u satisfies the following equation in the weak sense:

$$\Delta_{\mathrm{p}(\cdot)}(u) \ge f(x,u,\nabla u).$$

That is

$$-\int_\Omega |\nabla u|^{p(x)-2}\langle \nabla u, \nabla \phi\rangle dx \;\ge\; \int_\Omega f(x,u,\nabla u)\phi\, dx, \tag{4.2}$$

for all nonnegative functions $\phi \in C_0^\infty(\Omega)$. As for f we further assume that there exists $q > n$, a constant $c > 1$ and an exponent function $\alpha : \Omega \to [1,\infty)$ such that the following inequality holds pointwise for almost all $x \in \Omega$

$$u(x)f(x,u(x),\nabla u(x)) \ge c|u(x)|^{\alpha(x)}\left(1 + |\nabla u(x)|^{p^+\frac{p^- - \alpha^+}{\alpha^+}\frac{q}{n-1}}\right). \tag{4.3}$$

If a variable exponent Sobolev function v satisfies the opposite inequality in (4.2), then we call v a $p(\cdot)$-*supersolution*. A function which is both a $p(\cdot)$-supersolution and $p(\cdot)$-subsolution is $p(\cdot)$-harmonic.

Remark 4.2. We would like to point that the similar nonhomogeneous equation can be studied if instead of $\Delta_{p(\cdot)}$ one takes the strong $p(\cdot)$-Laplacian $\widetilde{\Delta}_{p(\cdot)}$ (see Remark 3.4 or [2] for the definition of strong $p(\cdot)$-Laplacian). The same applies to discussion in the next section, namely the growth rate estimate at infinity can be as well studied for the modified $p(\cdot)$-Laplacian. We leave such investigations for a future project.

4.2 The main theorem

Theorem 4.3. *Let Ω be an unbounded domain in $\mathbb{R}^n$ and u be a $p(\cdot)$-subsolution as in (4.2) with f satisfying the growth condition (4.3) with $\alpha^+ \leq p^-$. Suppose that $u \equiv 0$ on $\partial\Omega$. Denote*

$$m_R := \operatorname{ess\,sup}_{x\in\Omega\cap B_R}(1+|u(x)|)$$

and choose $1 \leq \delta < \frac{p^-}{\alpha^+}$.
Then either $u \equiv 0$ in Ω or

$$\liminf_{R\to\infty} \frac{m_R}{R^\gamma} > 0, \tag{4.4}$$

for any γ such that

$$\gamma \leq \frac{(q-n)\frac{p^-}{p^+} + q(\frac{p^-}{\delta} - 1)}{q(\frac{p^+}{\delta} - \frac{q-1}{q}\alpha^+)}. \tag{4.5}$$

Before proving theorem we will discuss some of its consequences and show that Theorem 4.3 generalizes the case of constant exponent p both in the nonlinear case of $p \neq 2$ likewise for the Laplace operator.

Corollary 4.4. *Suppose that assumptions of Theorem 4.3 hold. If $p^+ = p^- = p = const$ and $\alpha^+ > \frac{n}{q-1}$, then in (4.5) one may take $\gamma = 1$. In such a case we obtain the nonhomogeneous counterpart of the result by Lindqvist ([21, Theorem 4.6]) for p-subsolutions.*

Proof. If $p^+ = p^- = p$, then the $p(\cdot)$-Laplacian becomes p-Laplace operator and condition (4.3) reads: $uf(x,u,\nabla u) \geq c|u|^{\alpha(x)}(1+|\nabla u|^{p\frac{p-\alpha^+}{\alpha^+}\frac{q}{n-1}})$ with $\alpha(x) \leq p$ for $x \in \Omega$. In such a case condition (4.5) takes the form $\gamma \leq 1 + \frac{(q-1)\alpha^+ - n}{q\frac{\delta}{p} - (q-1)\alpha^+}$. The latter expression equals at least one, provided that $\alpha^+ > \frac{n}{q-1}$. Hence, we can choose $\gamma = 1$ and retrieve the rate of growth of u as in [21]. □

Corollary 4.5. *Suppose that assumptions of Theorem 4.3 hold. If $p = n = 2$ and $\alpha^+ > \frac{2}{q-1}$, then in (4.5) one may take $\gamma = 1$. In such a case we retrieve the growth estimate of m_R as in the classical Phragmén-Lindelöf theorem for the planar Laplace operator.*

Proof. If $p = n = 2$, then condition (4.3) reads: $uf(x,u,\nabla u) \geq c|u|^{\alpha(x)}(1+|\nabla u|^{2q(2/\alpha^+-1)})$ for $q > 2$, $1 \leq \delta < \frac{2}{\alpha^+}$ and $\alpha(x) \leq 2$ for $x \in \Omega$. Moreover, (4.5) takes the form:

$$\gamma \leq \frac{2(\frac{q}{\delta} - 1)}{q(\frac{2}{\delta} - \frac{q-1}{q}\alpha^+)} = \frac{2(\frac{q}{\delta} - 1)}{2(\frac{q}{\delta} - 1) + 2 - (q-1)\alpha^+}$$

and hence γ can be chosen to be equal one, provided that $\alpha^+ > \frac{2}{q-1}$. □

Remark 4.6. According to our best knowledge Corollaries 4.4 and 4.5 were not known before in the literature.

Proof of Theorem 4.3. Suppose that $u \not\equiv 0$ in Ω. By the maximum principle for $p(\cdot)$-Laplacian (cf. Theorem 3.4 in [10] applied with $B \equiv -f$) and the assumption that $u_{|\partial\Omega} \equiv 0$, we know that there exists an unbounded component of Ω such that u has a constant sign inside. For the sake of simplicity of notation denote this component Ω.

Let $\psi \in C_0^\infty(\Omega \cap B_R)$, such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ on $\Omega \cap B_r$ for some $0 < r < R$. Set $s := \frac{q}{\delta}p^+$. Define $\phi := \psi^s u$. Then

$$\nabla\phi = \psi^s\nabla u + s\psi^{s-1}u\nabla\psi. \tag{4.6}$$

Using this as a test function in (4.2) we get

$$-\int_{\Omega\cap B_R} |\nabla u|^{p(x)}\psi^s - \int_{\Omega\cap B_R} s|\nabla u|^{p(x)-2}u\psi^{s-1}\langle\nabla u, \nabla\psi\rangle \geq \int_{\Omega\cap B_R} f(x,u,\nabla u)\psi^s u. \tag{4.7}$$

By using the Young inequality (cf. (2.3)) we estimate the second integral on the left-hand side:

$$\begin{aligned}
&-\int_{\Omega\cap B_R} s|\nabla u|^{p(x)-2}u\psi^{s-1}\langle\nabla u, \nabla\psi\rangle \\
&\leq \int_{\Omega\cap B_R} \left(|\nabla u|^{p(x)-\delta}\psi^{s\frac{p(x)-\delta}{p(x)}}\right)\left(s|\nabla u|^{\delta-1}\psi^{\frac{s\delta}{p(x)}-1}|u||\nabla\psi|\right) \\
&\leq \int_{\Omega\cap B_R} |\nabla u|^{p(x)}\psi^s + s^{\frac{p^+}{\delta}}\int_{\Omega\cap B_R} |\nabla u|^{p(x)(1-\frac{1}{\delta})}\psi^{s-\frac{p(x)}{\delta}}(|u||\nabla\psi|)^{\frac{p(x)}{\delta}}.
\end{aligned}$$

This, together with (4.7) results in the following inequality:

$$\int_{\Omega\cap B_R} f(x,u,\nabla u)\psi^s u \le s^{\frac{p^+}{\delta}} \int_{\Omega\cap B_R} |\nabla u|^{p(x)(1-\frac{1}{\delta})}\psi^{s-\frac{p(x)}{\delta}}(|u||\nabla\psi|)^{\frac{p(x)}{\delta}}. \tag{4.8}$$

Observe also, that $\psi^{(s-p(x)/\delta)\frac{q}{q-1}} \le \psi^s$ and recall that $q > n$. By applying in (4.8) the constant exponent Hölder inequality we have

$$\begin{aligned}
&\int_{\Omega\cap B_R} f(x,u,\nabla u)\psi^s u \\
&\le cs^{\frac{p^+}{\delta}}\left(\int_{\Omega\cap B_R}|\nabla\psi|^{q\frac{p(x)}{\delta}}\right)^{\frac1q}\left(\int_{\Omega\cap B_R}|\nabla u|^{p(x)\frac{\delta-1}{\delta}\frac{q}{q-1}}|u|^{\frac{p(x)}{\delta}\frac{q}{q-1}}\psi^{(s-\frac{p(x)}{\delta})\frac{q}{q-1}}\right)^{\frac{q-1}{q}}\\
&\le c\left(\int_{\Omega\cap B_R}|\nabla\psi|^{q\frac{p(x)}{\delta}}\right)^{\frac1q}\left(\int_{\Omega\cap B_R}|\nabla u|^{p(x)\frac{\delta-1}{\delta}\frac{q}{q-1}}|u|^{\alpha(x)}|u|^{\left(\frac{p(x)}{\delta}-\alpha(x)\frac{q-1}{q}\right)\frac{q}{q-1}}\psi^{s}\right)^{\frac{q-1}{q}}\\
&\le c\left(\int_{\Omega\cap B_R}|\nabla\psi|^{q\frac{p(x)}{\delta}}\right)^{\frac1q}\left(\int_{\Omega\cap B_R}|u|^{\alpha(x)}|\nabla u|^{p(x)\frac{\delta-1}{\delta}\frac{q}{q-1}}\psi^{s}\right)^{\frac{q-1}{q}}\mathrm{m}_R^{\frac{p^+}{\delta}-\frac{q-1}{q}\alpha^+}.
\end{aligned}\tag{4.9}$$

The variable exponent Hölder inequality (2.2) allows us to estimate the first integral on the right-hand side as follows:

$$\begin{aligned}
\left(\int_{\Omega\cap B_R}|\nabla\psi|^{q\frac{p(x)}{\delta}}\right)^{\frac1q} &\le 2\|\,|\nabla\psi|^{q\frac{p(\cdot)}{\delta}}\,\|^{\frac1q}_{L^{\frac{p^+}{p(\cdot)}}(\Omega\cap B_R)}\|1\|^{\frac1q}_{L^{\frac{p^+}{p^+-p(\cdot)}}(\Omega\cap B_R)}\\
&\le 2\max\left\{\int_{\Omega\cap B_R}|\nabla\psi|^{q\frac{p^+}{\delta}},\Big(\int_{\Omega\cap B_R}|\nabla\psi|^{q\frac{p^+}{\delta}}\Big)^{p^-/p^+}\right\}^{\frac1q}\max\{1,|\Omega\cap B_R|^{1-\frac{p^-}{p^+}}\}.
\end{aligned}\tag{4.10}$$

Here, we have also used estimate (2.1). Since $|\nabla u|^{p(x)} \le 2^{p^+}(1+|\nabla u|^{p^+})$ for all $x\in\Omega$, we get

$$\int_{\Omega\cap B_R}|u|^{\alpha(x)}|\nabla u|^{p(x)\frac{\delta-1}{\delta}\frac{q}{q-1}}\psi^s \le \int_{\Omega\cap B_R}|u|^{\alpha(x)}(1+|\nabla u|)^{p^+\frac{\delta-1}{\delta}\frac{q}{q-1}}\psi^s. \tag{4.11}$$

Note further, that $\frac{\delta-1}{\delta} < \frac{p^- - \alpha^+}{\alpha^+}$. This, the assumption that $q > n$ and the growth condition (4.3) together with (4.11) imply that

$$\begin{aligned}
c\int_{\Omega\cap B_R}|u|^{\alpha(x)}|\nabla u|^{p(x)\frac{\delta-1}{\delta}\frac{q}{q-1}}\psi^s &\le c\int_{\Omega\cap B_R}|u|^{\alpha(x)}(1+|\nabla u|^{p^+\frac{p^--\alpha^+}{\alpha^+}\frac{q}{n-1}})\psi^s\\
&\le \int_{\Omega\cap B_R} f(x,u,\nabla u)\psi^s u,
\end{aligned}\tag{4.12}$$

where the first inequality holds for large enough R. We use (4.10) and (4.12) in (4.9) to obtain inequality

$$\begin{aligned}
&c\left(\int_{\Omega\cap B_R}|u|^{\alpha(x)}(1+|\nabla u|^{p^+\frac{p^--\alpha^+}{\alpha^+}\frac{q}{n-1}})\psi^s\right)^{\frac1q}\\
&\le 2\max\left\{\int_{\Omega\cap B_R}|\nabla\psi|^{q\frac{p^+}{\delta}},\Big(\int_{\Omega\cap B_R}|\nabla\psi|^{q\frac{p^+}{\delta}}\Big)^{\frac{p^-}{p^+}}\right\}^{\frac1q}|\Omega\cap B_R|^{1-\frac{p^-}{p^+}}\mathrm{m}_R^{\frac{p^+}{\delta}-\frac{q-1}{q}\alpha^+}.
\end{aligned}\tag{4.13}$$

Clearly,

$$I(r) := \int_{\Omega\cap B_r}|u|^{\alpha(x)}(1+|\nabla u|^{p^+\frac{p^--\alpha^+}{\alpha^+}\frac{q}{n-1}}) \le \int_{\Omega\cap B_R}|u|^{\alpha(x)}(1+|\nabla u|^{p^+\frac{p^--\alpha^+}{\alpha^+}\frac{q}{n-1}})\psi^s.$$

Apply power q to both sides of inequality (4.13) with $I(r)$ on the left-hand side. We are now in a position to take the infimum over all test functions ψ as defined above. In a consequence we get from (4.13) that

$$c\,I(r) \le \max\left\{\mathrm{cap}_{q\frac{p^+}{\delta}}(\overline{\Omega\cap B_r},\Omega\cap B_R),\mathrm{cap}^{p^-/p^+}_{q\frac{p^+}{\delta}}(\overline{\Omega\cap B_r},\Omega\cap B_R)\right\}|\Omega\cap B_R|^{q\left(1-\frac{p^-}{p^+}\right)}\mathrm{m}_R^{q\frac{p^+}{\delta}-(q-1)\alpha^+} \tag{4.14}$$

Since $q > n$ and $\delta \le \frac{p^-}{\alpha^+}$ it holds that $q\frac{p^+}{\delta} > q\alpha^+\frac{p^+}{p^-} > n$. Set $R = 2r$ in (4.14). Then, by discussion similar to that in [14, Chapter 2.11] we have that (cf. (2.5))

$$\mathrm{cap}_{q\frac{p^+}{\delta}}(\overline{\Omega\cap B_r},\Omega\cap B_{2r}) \le c(n,q,p^+,\delta)r^{n-q\frac{p^+}{\delta}}. \tag{4.15}$$

For the sake of simplicity, we decide not to step into details regarding how the structure of Ω for large R affects the capacity, cf. paragraph following estimate (3.19) in Theorem 3.3 and Section 3.3. Hence, estimate (4.15) is sufficient for the proof of this theorem.

By combining (4.14) and (4.15) we get for $r>1$ the following inequality:

$$c\,I(r) \leq c(n,q,p^-,p^+)\omega_n r^{(n-q\frac{p^+}{\delta})\frac{p^-}{p^+}} r^{q\left(1-\frac{p^-}{p^+}\right)} \mathrm{m}_{2r}^{q\frac{p^+}{\delta}-(q-1)\alpha^+}. \tag{4.16}$$

On the contrary to the assertion of theorem (4.4) let us assume that

$$\liminf_{r\to\infty} \frac{\mathrm{m}_r}{r^\gamma} = 0, \tag{4.17}$$

where

$$\gamma \leq \frac{(q-n)\frac{p^-}{p^+} + q(\frac{p^-}{\delta}-1)}{q(\frac{p^+}{\delta} - \frac{q-1}{q}\alpha^+)}.$$

Denote $\{r_i\}_{i=1}^\infty$ a sequence of radii along which liminf in (4.17) is attained. From (4.16) and (4.17) we have that

$$c\,I(r_i) \leq r_i^{(n-q\frac{p^+}{\delta})\frac{p^-}{p^+}+q(1-\frac{p^-}{p^+})+\gamma(q\frac{p^+}{\delta}-(q-1)\alpha^+)} \left(\frac{\mathrm{m}_{2r_i}}{r_i^\gamma}\right)^{q\frac{p^+}{\delta}-(q-1)\alpha^+} \longrightarrow 0, \quad \text{for } r_i\to\infty, \tag{4.18}$$

as under assumption on γ the power of r_i in the first factor in (4.18) is negative. However, by the definition of $I(r)$ and by (2.1) we have the following estimate:

$$I(r) \geq \int_{\Omega\cap B_r} |u|^{\alpha(x)} \geq \min\{\, \|u\|^{\alpha^+}_{L^{\alpha(\cdot)}(B_r\cap\Omega)}, \|u\|^{\alpha^-}_{L^{\alpha(\cdot)}(B_r\cap\Omega)}\,\}.$$

The right-hand side of this inequality is positive and increases for all $r\to\infty$ and so, in particular, for r_i with $i=1,\ldots$. This observation together with (4.18) result in contradiction. Hence assumption (4.17) is false which proves theorem. □

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