Maximum principles and nonexistence results for radial solutions to equations involving p-Laplacian

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Abstract

We obtain the variant of maximum principle for radial solutions of, possibly singular, p-harmonic equations of the form $-a(|x|)\Delta_p(w) + h(|x|, w, \nabla w(x) \cdot \frac{x}{|x|}) = \phi(w)$, as well as for solutions of the related ODE. We show that for the considered class of equations local maximas of $|w|$ form a monotone sequence in $|x|$ and constant sign solutions are monotone. The results are applied to nonexistence and nonlinear eigenvalue problems and generalize our previous work.

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1 Introduction

The study of the so-called nonlinear eigenvalue problems is one of the main areas of p-harmonic theory, e.g. [18, 22, 23, 32, 40, 42]. The starting point for such considerations is the following equation

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u, \quad \text{where } \lambda \in \mathbf{R}.$$

Clearly, when $p = 2$ one retrieves the classical harmonic eigenvalue problem. In the course of development of this theory various perturbations of the right hand side have been considered, allowing such functions to depend on points in the domain, solution and its gradient. In this setting the natural problems include: maximum principles,

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radial solutions and their properties, (non)existence of constant sign solutions.
In this note we investigate the following PDE:

$$-a(|x|)\mathrm{div}\left(|\nabla w(x)|^{p-2}\nabla w(x)\right) + h(|x|, w(x), \langle \nabla w(x), \frac{x}{|x|}\rangle) = \phi(w(x)) \text{ a.e. in } B, \quad (1)$$

where B is assumed to be a ball in $\mathbf{R}^n$ centered at 0 with radius R and w is a radial function in the space $W^{1,1}_{loc}(B \setminus \{0\})$. We discuss Equation (1) also in the case $n = 1$, $B = (0, R)$. Such a weak regularity assumption on w admits solutions to be in the weighted Sobolev spaces with radial weights; moreover, the admitted weights may explode or vanish near the origin or the boundary of B.

Our equation is given in the nondivergent form. Function a may change sign, in particular our PDE can be singular.

In a consequence of radiality of the solutions, Equation (1) reduces to the related ODE:

$$a(\tau)(\Phi_p(u'(\tau)))' + (n-1)\frac{a(\tau)}{\tau}|u'(\tau)|^{p-2}u'(\tau) - h(\tau, u(\tau), u'(\tau)) + \phi(u(\tau)) = 0, \quad (2)$$

for a.e. $\tau \in (0, R)$, where $\Phi_p(\lambda) = |\lambda|^{p-2}\lambda$, for $\lambda \neq 0$ and $\Phi_p(0) = 0$.

Our main result, Theorem 2.1, is the maximum principle for radial solutions of Equation (1), stating that, under the appropriate additional assumptions, an absolute value of the solution $|w|$ attains supremum at 0 or at the boundary of B. This maximum principle is then applied to nonlinear eigenvalue problems and to nonexistence type results.

The result of Theorem 2.1 is developed further in Theorem 3.1. Roughly speaking, it says that under the assumptions of Theorem 2.1 a constant sign solution is monotone in $|x|$. Moreover, in general the local extrema of $|w|$, together with an upper limits of $|w|$ at 0 and R form the monotone sequence with respect to $|x|$.
In the Section 4 we discuss some methods of generating the admissible perturbations h and give examples.

Our results can be applied directly to the related ODE, Equation (2), see Section 5 for presentation of results and examples.

This work extends results obtained in [2], where we discussed the case of $h \equiv 0$, under stronger assumptions on u and a. Contrary to the previous approach here we *do not* assume that the solution w is of class C^1 close to the origin; moreover, it may even be discontinuous at 0. In particular, our goal is to propose the possibly general class of PDEs to which such maximum principles apply, as well as to consider the possibly wide class of their radial solutions.
Equations involving p-Laplacian appear in various areas of pure and applied mathematics including geological sciences [8], fluid dynamics [1, 14, 19, 20, 41], electrostatics [24], cosmology [30], analysis on Riemannian manifolds ([15, 35], Section 8 in [36]), theory of Carnot-Carathéodory groups and analysis on metric spaces ([7, 11, 33] and references therein), as well as in relation to inequalities of Poincaré, Writinger, Sobolev type and isoperimetric inequalities [9, 22, 23, 31]. Motivations for studying radial solutions can be found e.g. in [12, 13, 16, 17, 21, 25, 31].

2 Derivation of the maximum principle

Basic notation. We use the standard notation $W^{k,p}(\Omega)$ and $W^{k,p}_{loc}(\Omega)$ to denote Sobolev spaces, where Ω is a given domain in $\mathbf{R}^n$. By ∇f we denote the distributional gradient of f. The k–th distributional derivative of a one-variable function is denoted by $f^{(k)}$. By $B(0,R)$ we denote a ball in $\mathbf{R}^n$ centered at 0 with radius R. If $p \in (1,\infty)$ we define continuous function

$$\Phi_p(\lambda) = |\lambda|^{p-2}\lambda, \ \text{ for } \lambda \neq 0, \quad \Phi_p(0) = 0.$$

Here λ can be either scalar or vector. We follow convention and denote by ω_{n-1} the measure of the unit sphere in $\mathbf{R}^n$.

The equation on the ball in $\mathbf{R}^n$. The main subject of our considerations is the following equation:

$$-a(|x|)\mathrm{div}\left(|\nabla w(x)|^{p-2}\nabla w(x)\right) + h(|x|, w(x), \langle \nabla w(x), \frac{x}{|x|}\rangle) = \phi(w(x)) \text{ a.e. in } B. \quad (3)$$

We assume that $p > 1, B = B(0,R), R \in (0,\infty]$, $w(x) = u(|x|)$ is a radial function such that $w \in W^{1,1}_{loc}(B \setminus \{0\})$ and $\Phi_p(\nabla w) \in W^{1,1}_{loc}(B \setminus \{0\}, \mathbf{R}^n)$. The expression $\mathrm{div}\,(|\nabla w(x)|^{p-2}\nabla w(x))$ is understood in the sense of distributions. Next, let h be a Carathéodory function (i.e. measurable with respect to first variable and continuous with respect to remaining ones); also let ϕ be continuous.

The related equation on an interval. Our considerations will be based on the following observation.

Fact 2.1. *Let $n > 1$, $g(x) = v(|x|)$ for $g \in W^{1,1}_{loc}(B \setminus \{0\})$. Then we have*

1) $v \in W^{1,1}_{loc}((0,R))$

2) *If $\Phi_p(\nabla g) \in W^{1,1}_{loc}(B \setminus \{0\})$ then $\Phi_p(v') \in W^{1,1}_{loc}((0,R))$ and $|v'|^p \in W^{1,1}_{loc}((0,R))$. In particular $|v'|$ is continuous on $(0,R)$.*

Proof of Fact 2.1.
1) Observe first, that $g \in W^{1,1}(P_{\epsilon,r})$ for every $0 < \epsilon < r < R$, where $P_{\epsilon,r} := B(0,r) \setminus B(0,\epsilon)$ is a ring. Using the variant of Nikodym ACL Characterization Theorem (see e.g. Theorem 1, Chapter 1.1.3 in [34]) we deduce that for every $\theta \in \mathbb{S}^{n-1}$ the mapping $\tau \mapsto g(\tau\,\theta) = v(\tau)$ is an absolutely continuous function on $[\epsilon, r]$. Since $\nabla g(x) = v'(|x|)\frac{x}{|x|}$, then for $s \in (\epsilon, r)$ and $\theta \in \mathbb{S}^{n-1}$ we have $|v'(s)| = |\nabla g(s\theta)|$. In particular

$$\begin{aligned} \omega_{n-1}\int_\epsilon^r |v'(s)|ds &\leq \frac{\omega_{n-1}}{\epsilon^{n-1}}\int_\epsilon^r |v'(s)|s^{n-1}ds \\ &= \frac{1}{\epsilon^{n-1}}\int_{\mathbb{S}^{n-1}}\int_\epsilon^r |\nabla g(s\theta)|s^{n-1}ds\,d\theta = \frac{1}{\epsilon^{n-1}}\int_{P_{\epsilon,r}} |\nabla g(x)|dx < \infty. \end{aligned}$$

2) Let

$$h(\lambda_1) := |\lambda_1|^{\frac{p}{p-1}}, \quad \lambda_1 \in \mathbf{R}. \tag{4}$$

As $p > 1$, the mapping is locally Lipshitz. Moreover, as $\Phi_p(\nabla w(x)) = \Phi_p(v'(|x|)) \cdot \frac{x}{|x|} \in W^{1,1}_{loc}(B(0,R) \setminus \{0\}, \mathbf{R}^n)$, then $\Phi_p(v'(|x|)) = \Phi_p(\nabla w(x)) \cdot \frac{x}{|x|} \in W^{1,1}_{loc}(B(0,R) \setminus \{0\})$. From the previous part we have that $\Phi_p(v') \in W^{1,1}_{loc}((0,R))$. Hence $|v'|^{p-1} = |\Phi_p(v^{'})| \in W^{1,1}_{loc}((0,R))$ and $|v'|^p = h(|v'|^{p-1}) \in W^{1,1}_{loc}((0,R))$ by the ACL Characterization Theorem. □

Equation (3) together with Fact 2.1 imply that u solves the ODE:

$$a(\tau)(\Phi_p(u^{'}(\tau)))^{'} + (n-1)\frac{a(\tau)}{\tau}|u^{'}(\tau)|^{p-2}u^{'}(\tau) - h(\tau, u(\tau), u^{'}(\tau)) + \phi(u(\tau)) = 0,$$
$$\text{a.e. for } \tau \in (0,R), \tag{5}$$

where $(\Phi_p(u^{'}(\tau)))^{'}$ is understood in the sense of distributions. Moreover, $u \in W^{1,1}_{loc}((0,R))$ and $\Phi_p(u')$, $|u'|^p \in W^{1,1}_{loc}((0,R))$. In particular $|u'|^p$ is continuous on $(0,R)$.

Let us introduce the following set of assumptions.

Assumptions $\mathcal{A}$

1. $p > 1$, $n > 1$, $R \in (0,\infty]$, $B = B(0,R) \subset \mathbf{R}^n$ (for $R = \infty$ the ball B is the whole $\mathbf{R}^n$).
2. $w \in W^{1,1}_{loc}(B \setminus \{0\})$ and $\Phi_p(\nabla w) \in W^{1,1}_{loc}(B \setminus \{0\})$, w is radial function and $w(x) = u(|x|)$.
3. ϕ is an integrable odd continuous function on $\mathbf{R}$ such that $\tau\phi(\tau)$ is either positive or negative for almost all τ's.
4. $a \in W^{1,1}_{loc}((0,R))$.
5. $h = h(\tau, \lambda_0, \lambda_1)$ is a Carathéodory function defined on $(0,R) \times \mathbf{R}^2$ i.e. measurable with respect to τ, continuous with respect to the remaining variables. Additionally, for every $K > 0$

$$\sup_{|\lambda_0|<K, |\lambda_1|<K} |h(\tau, \lambda_0, \lambda_1)| \in L^1_{loc}(0,R).$$

For $a \in W^{1,1}_{loc}((0,R))$ we define

$$\delta_a(\tau) := (n-1)\frac{a(\tau)}{\tau} - \left(1 - \frac{1}{p}\right) a^{'}(\tau) \tag{6}$$

and consider also the following assumptions:

$\mathbf{B_1}$) $\tau\phi(\tau) > 0$, $\liminf_{s\to 0} a(s)|u'(s)|^p \leq 0$, $\liminf_{s\to R} a(s)|u'(s)|^p \geq 0$ and $h(\tau, \lambda_0, \lambda_1)\lambda_1 \leq \delta_a(\tau)|\lambda_1|^p$ for a. e. $\tau \in (0,R)$ and every $\lambda_0, \lambda_1 \in \mathbf{R}$,

$\mathbf{B_2}$) $\tau\phi(\tau) < 0$, $\limsup_{s\to 0} a(s)|u'(s)|^p \geq 0$, $\limsup_{s\to R} a(s)|u'(s)|^p \leq 0$ and $h(\tau, \lambda_0, \lambda_1)\lambda_1 \geq \delta_a(\tau)|\lambda_1|^p$ for a. e. $\tau \in (0,R)$ and every $\lambda_0, \lambda_1 \in \mathbf{R}$,

C_1) $\tau\phi(\tau) > 0$, $\liminf_{s\to 0} a(s)|u'(s)|^p \geq 0$, $\liminf_{s\to R} a(s)|u'(s)|^p \leq 0$ and $h(\tau,\lambda_0,\lambda_1)\lambda_1 \geq \delta_a(\tau)|\lambda_1|^p$ for a. e. $\tau \in (0,R)$ and every $\lambda_0,\lambda_1 \in \mathbf{R}$,

C_2) $\tau\phi(\tau) < 0$, $\limsup_{s\to 0} a(s)|u'(s)|^p \leq 0$, $\limsup_{s\to R} a(s)|u'(s)|^p \geq 0$ and $h(\tau,\lambda_0,\lambda_1)\lambda_1 \leq \delta_a(\tau)|\lambda_1|^p$ for a. e. $\tau \in (0,R)$ and every $\lambda_0,\lambda_1 \in \mathbf{R}$.

Remark 2.1. 1) Conditions: $\liminf_{s\to 0} a(s)|u'(s)|^p \leq 0$, $\liminf_{s\to R} a(s)|u'(s)|^p \geq 0$ in $\mathbf{B_1}$) are satisfied if we assume for example that $a \geq 0$ (ellipticity condition) and $\liminf_{s\to 0} a(s)|u'(s)|^p = 0$. Similar comment can be made in relation to the remaining conditions: $\mathbf{B_2}$), $\mathbf{C_1}$), $\mathbf{C_2}$).
2) The condition $a(s)|u'(s)|^p \to 0$ for s converging to 0 or R is satisfied under additional assumptions or as a consequence of certain regularity results. For example, if we assume that $w \in C^1(B)$ then u' is right-continuous at 0 and $u'(0) = 0$. If additionally a is bounded for s close to 0 then we have that $\lim_{s\to 0} a(s)|u'(s)|^p = 0$. The same holds if we know that u' is bounded close to 0 and $\lim_{s\to 0} a(s) = 0$. For $C^1(\overline{B})$-solutions and for a bounded close to R we have that $a(s)|u'(s)|^p \to 0$ for $s \to R$ under Neumann condition on the solution w.
3) Conditions of type $\lim_{s\to t} a(s)|u'(s)|^p \to 0$ for the appropriate choice of t are often present in the literature (see e.g. [6]).

We are now in a position to present the main result of this note.

Theorem 2.1. *Let Assumptions $\mathcal{A}$ be satisfied and w be a radial solution to (3). If either condition $\mathbf{B_1}$) or $\mathbf{B_2}$) holds we have*

$$\sup_{x\in B}|w(x)| = \limsup_{x\to 0} |w(x)|, \tag{7}$$

while if either condition $\mathbf{C_1}$) or $\mathbf{C_2}$) holds we have

$$\sup_{x\in B}|w(x)| = \limsup_{|x|\to R} |w(x)|. \tag{8}$$

Proof of Theorem 2.1.
For the sake of simplicity we will assume that $R < \infty$. Observe that u solves the ODE:

$$\phi(u(\tau))u^{'}(\tau) = -a(\tau)(\Phi_p(u'(\tau)))'u'(\tau) - (n-1)\frac{a(\tau)}{\tau}|u^{'}|^p + h(\tau,u(\tau),u^{'}(\tau))u^{'}(\tau). \tag{9}$$

Define

$$\Phi(\tau) := \int_0^\tau \phi(s)ds$$

and

$$A(\tau_1,\tau_2) := \Phi(|u(\tau_1)|) - \Phi(|u(\tau_2)|) \quad \text{for } \tau_1 > \tau_2. \tag{10}$$

In order to prove (7) it suffices to show that $\liminf_{\epsilon\to 0} A(\tau,\epsilon) \leq 0$ for every $\tau \in (0,R)$ being the critical point of u and that $\limsup_{\tau\to r} \liminf_{\epsilon\to 0} A(\tau,\epsilon) \leq 0$. Similarly, for the proof of (8) it is enough to show that $\limsup_{r\to R} A(r,\tau) \geq 0$ for every $\tau \in (0,R)$ being the critical point of u and $\limsup_{\tau\to R} \liminf_{\epsilon\to 0} A(\tau,\epsilon) \geq 0$. Note that according to Fact 2.1 $|u^{'}|$ is continuous inside $(0,R)$. Observe that under the above assumptions $\Phi(|u(\tau)|) \in W^{1,1}_{loc}((0,R))$. Indeed, using Fact 2.1 with $w(x) = u(|x|)$ one gets $u \in W^{1,1}_{loc}((0,R))$. The Nikodym ACL Characterization Theorem implies that $|u| \in W^{1,1}_{loc}((0,R))$ and, since Φ is locally Lipshitz, $\Phi \circ |u| \in W^{1,1}_{loc}((0,R))$. Thus, for any r and ϵ such that $0 < \epsilon < r < R$ we have

$$A(r,\epsilon) \;=\; \int_\epsilon^r \frac{d}{d\tau}(\Phi(|u(\tau)|))\,d\tau \;=\; \int_\epsilon^r \Phi^{'}(|u(\tau)|)\mathrm{sgn}u(\tau)u^{'}(\tau)\,d\tau = \int_\epsilon^r \phi(u(\tau))u^{'}(\tau)\,d\tau. \tag{11}$$

We split the discussion into two cases.
Case 1: $\tau\phi(\tau) > 0$.
If $\mathbf{B_1}$) holds, then the right hand side of inequality (9) does not exceed value

$$-a(\tau)\left(\Phi_p(u'(\tau))\right)' u'(\tau) - \left(1 - \frac{1}{p}\right) a^{'}(\tau)|u^{'}(\tau)|^p. \tag{12}$$

To proceed further we consider expression

$$\Psi(\tau,\lambda_1) \;:=\; -\left(1 - \frac{1}{p}\right) a(\tau)|\lambda_1|^p. \tag{13}$$

Applying Fact 2.1 we get

$$|u^{'}|^p \in W^{1,1}_{loc}((0,R)).$$

This, together with the fact that $a(\cdot) \in W^{1,1}_{loc}((0,R))$ imply that $\Psi(\tau,u^{'}(\tau)) \in W^{1,1}_{loc}((0,R))$. By direct computation we obtain for $H(\lambda) = |\lambda|^{\frac{p}{p-1}}$:

$$\begin{aligned}\left(|u^{'}|^p\right)^{'} \;&=\; \left(H\left(\Phi_p(u^{'})\right)\right)^{'} = H^{'}(\Phi_p(u^{'}))\cdot\left(\Phi_p(u^{'})\right)^{'} = \\ &=\; \frac{p}{p-1}\left(|\Phi_p(u^{'})|^{\frac{1}{p-1}}\mathrm{sgn}(\Phi_p(u^{'}))\right)(\Phi_p(u^{'}))^{'} = \frac{p}{p-1}u^{'}\cdot(\Phi_p(u^{'}))^{'} \quad \text{a.e.}\end{aligned}$$

Therefore (12) equals $\frac{d}{d\tau}(\Psi(\tau,u^{'}(\tau)))$ where $\Psi(\tau,u^{'}(\tau)) \in W^{1,1}_{loc}((0,R))$ and $\Psi(\cdot,\cdot)$ is given by (13). Hence (11) implies

$$A(r,\epsilon) \leq \Psi(\tau,u^{'}(\tau))|_\epsilon^r = -\left(1 - \frac{1}{p}\right)a(r)|u^{'}(r)|^p + \left(1 - \frac{1}{p}\right)a(\epsilon)|u^{'}(\epsilon)|^p. \tag{14}$$

Thus by $\mathbf{B_1}$), if r is a critical point of u, we get

$$\Phi(|u(r)|) - \limsup_{\epsilon\to 0}\Phi(|u(\epsilon)|) \;=\; \liminf_{\epsilon\to 0} A(r,\epsilon) \leq\; \liminf_{\epsilon\to 0}(1 - \frac{1}{p})a(\epsilon)|u^{'}(\epsilon)|^p \leq 0.$$

Moreover, (14) shows that

$$\limsup_{r\to R}\Phi(|u(r)|)-\limsup_{\epsilon\to 0}\Phi(|u(\epsilon)|)=\limsup_{r\to R}\liminf_{\epsilon\to 0}\left(\Phi(|u(r)|)-\Phi(|u(\epsilon)|)\right)\leq$$
$$\leq \limsup_{r\to R}\liminf_{\epsilon\to 0}\left(\left(1-\frac{1}{p}\right)\left(-a(r)|u^{'}(r)|^p+a(\epsilon)|u^{'}(\epsilon)|^p\right)\right)=$$
$$=\left(1-\frac{1}{p}\right)\limsup_{r\to R}\left(-a(r)|u^{'}(r)|^p+\liminf_{\epsilon\to 0}a(\epsilon)|u^{'}(\epsilon)|^p\right)=$$
$$=\left(1-\frac{1}{p}\right)\left(-\liminf_{r\to R}a(r)|u^{'}(r)|^p+\liminf_{\epsilon\to 0}a(\epsilon)|u^{'}(\epsilon)|^p\right)\leq 0.$$

If $\mathbf{C_1}$) holds then the inequalities at (12) become reversed, and therefore for $0<r<s<R$

$$A(s,r)\geq \Psi(\tau,u^{'}(\tau))|_r^s=-\left(1-\frac{1}{p}\right)a(s)|u^{'}(s)|^p+\left(1-\frac{1}{p}\right)a(r)|u^{'}(r)|^p.$$

Hence, if r is a critical point of u

$$\limsup_{s\to R}\Phi(|u(s)|)-\Phi(|u(r)|)\;=\;\limsup_{s\to R}A(s,r)\geq\;\limsup_{s\to R}\left(-\left(1-\frac{1}{p}\right)a(s)|u^{'}(s)|^p\right)\geq 0.$$

Furthermore

$$\limsup_{s\to R}\Phi(|u(s)|)-\limsup_{r\to 0}\Phi(|u(r)|)\geq$$
$$\geq\left(1-\frac{1}{p}\right)\left(-\liminf_{s\to R}a(s)|u^{'}(s)|^p+\liminf_{r\to 0}a(r)|u^{'}(r)|^p\right)\geq 0$$

and Case 1 follows.

Case 2: $\tau\phi(\tau)<0$. Consider $\tilde{\phi}(\tau):=-\phi(\tau)$, $\tilde{\Psi}=-\Psi$, $\tilde{\Phi}(\tau):=\int_0^\tau\tilde{\phi}(\tau)d\tau$ and $\tilde{A}(x_1,x_2):=\tilde{\Phi}(|u(x_1)|)-\tilde{\Phi}(|u(x_2)|)$. Under assumption $\mathbf{C_2}$) we arrive at inequality

$$\tilde{\phi}(u(\tau))u^{'}(\tau)\geq\frac{d}{d\tau}\tilde{\Psi}(\tau,u^{'}(\tau)),\qquad\text{where }\tau\in(0,R). \tag{15}$$

Upon integration over (s,r) with $0<s<r<R$ the inequality gives that if r is a critical point of u or r is close to R, we have

$$\tilde{A}(r,s)\geq\tilde{\Psi}(\tau,u^{'}(\tau))|_s^r=\left(1-\frac{1}{p}\right)a(r)|u^{'}(r)|^p-\left(1-\frac{1}{p}\right)a(s)|u^{'}(s)|^p\geq$$
$$\geq-\left(1-\frac{1}{p}\right)a(s)|u^{'}(s)|^p. \tag{16}$$

Therefore, $\limsup_{r\to R}\tilde{A}(r,s)\geq 0$ provided $u^{'}(s)=0$. To complete the proof it suffices to note that

$$\limsup_{r\to R}\tilde{\Phi}(|u(r)|)-\limsup_{s\to 0}\tilde{\Phi}(|u(s)|)\geq$$
$$\geq\left(1-\frac{1}{p}\right)\left(\limsup_{r\to R}a(r)|u^{'}(r)|^p-\limsup_{s\to 0}a(s)|u^{'}(s)|^p\right)\geq 0.$$

In the case $\mathbf{B_2}$) instead of (15) the converse inequality holds. Again, we integrate it over (s,r) to obtain:

$$\tilde{A}(r,s) \leq \tilde{\Psi}(\tau, u^{'}(\tau))|_s^r = \left(1-\frac{1}{p}\right) a(r)|u^{'}(r)|^p - \left(1-\frac{1}{p}\right) a(s)|u^{'}(s)|^p \tag{17}$$

and the computations follow than the same lines as in the case $\mathbf{C_2}$). This completes the proof. $\square$

We will apply the above result to the nonlinear eigenvalue problems. For this purpose we consider the following set of assumptions.

$\mathbf{B_1'}$) $\lambda > 0$, $\liminf_{s\to 0} a(s)|u'(s)|^p \leq 0$, $\liminf_{s\to R} a(s)|u'(s)|^p \geq 0$ and $h(\tau,\lambda_0,\lambda_1)\lambda_1 \leq \delta_a(\tau)|\lambda_1|^p$ for a. e. $\tau \in (0,R)$ and every $\lambda_0, \lambda_1 \in \mathbf{R}$,

$\mathbf{B_2'}$) $\lambda < 0$, $\limsup_{s\to 0} a(s)|u'(s)|^p \geq 0$, $\limsup_{s\to R} a(s)|u'(s)|^p \leq 0$ and $h(\tau,\lambda_0,\lambda_1)\lambda_1 \geq \delta_a(\tau)|\lambda_1|^p$ for a. e. $\tau \in (0,R)$ and every $\lambda_0, \lambda_1 \in \mathbf{R}$,

$\mathbf{C_1'}$) $\lambda > 0$, $\liminf_{s\to 0} a(s)|u'(s)|^p \geq 0$, $\liminf_{s\to R} a(s)|u'(s)|^p \leq 0$ and $h(\tau,\lambda_0,\lambda_1)\lambda_1 \geq \delta_a(\tau)|\lambda_1|^p$ for a. e. $\tau \in (0,R)$ and every $\lambda_0, \lambda_1 \in \mathbf{R}$,

$\mathbf{C_2'}$) $\lambda < 0$, $\limsup_{s\to 0} a(s)|u'(s)|^p \leq 0$, $\limsup_{s\to R} a(s)|u'(s)|^p \geq 0$ and $h(\tau,\lambda_0,\lambda_1)\lambda_1 \leq \delta_a(\tau)|\lambda_1|^p$ for a. e. $\tau \in (0,R)$ and every $\lambda_0, \lambda_1 \in \mathbf{R}$.

We have the following result.

Corollary 2.1. *Let $q > 1$, Assumptions $\mathcal{A}$ be satisfied and w be a radial solution of*

$$-a(|x|)\mathrm{div}\left(|\nabla w(x)|^{p-2}\nabla w(x)\right) + h(|x|, w(x), \langle \nabla w(x), \frac{x}{|x|}\rangle) = \lambda |w|^{q-2} w, \text{ a.e. in } B. \tag{18}$$

If $\mathbf{B_1'}$) holds or if $\mathbf{B_2'}$) holds then

$$\sup_{x\in B} |w(x)| = \limsup_{x\to 0} |w(x)|.$$

Whereas, if $\mathbf{C_1'}$) holds or $\mathbf{C_2'}$) holds then

$$\sup_{x\in B} |w(x)| = \limsup_{|x|\to R} |w(x)|.$$

Direct application of Theorem 2.1 gives us also the following nonexistence result.

Proposition 2.1 (nonexistence of nontrivial solutions)**.** *Under the assumptions of Theorem 2.1 the following problems admit only the trivial solutions:*

1.

$$\begin{cases} -a(|x|)\mathrm{div}\left(|\nabla w(x)|^{p-2}\nabla w(x)\right) + h(|x|, w(x), \langle \nabla w(x), \frac{x}{|x|}\rangle) = \phi(w(x)) \text{ a.e. in } B, \\ w(0) = 0 \end{cases}$$

in the case $\mathbf{B_1}$) or $\mathbf{B_2}$) and for $w \in C(B)$.

2.

$$\begin{cases} -a(|x|)\mathrm{div}\,(|\nabla w(x)|^{p-2}\nabla w(x)) + h(|x|, w(x), \langle \nabla w(x), \frac{x}{|x|}\rangle) = \phi(w(x)) \text{ a.e. in } B, \\ w \equiv 0 \text{ on } \partial B(0,R) \text{ for } R < \infty \quad \text{or} \quad \lim_{|x|\to\infty} w(x) = 0 \text{ for } R = \infty. \end{cases}$$

in the case $\mathbf{C_1}$) *or* $\mathbf{C_2}$) *and* $w \in C(\overline{B})$ *for* $R < \infty$.

If $h \equiv 0$ one can find some other nonexistence results in [2] derived from the radial variant of Derrick-Pokhozhaev identity.

3 Qualitative properties of solutions

Our techniques allow us to study the oscillations of solutions to the considered class of equations. For this purpose let us introduce the following.

$$\begin{aligned} \Gamma &= \{r \in (0,R) : \text{ every } x \in \partial B(r) \text{ is a critical point for } w\} \cup \{0\} \cup \{R\} \\ M &: \Gamma \to [0,\infty],\ M(r) = \begin{cases} |w|_{\{|x|=r\}}, & 0 < r < R \\ \limsup_{x\to 0} |w(x)|, & r = 0 \\ \limsup_{|x|\to R} |w(x)|, & r = R \end{cases} \end{aligned} \tag{19}$$

With the above notation we have the following variant of Theorem 2.1.

Theorem 3.1. *Let Assumptions $\mathcal{A}$ hold and w be a radial solution to Equation (3). If either condition $\mathbf{B_1}$) or $\mathbf{B_2}$) holds then the mapping $M|_\Gamma$ is nonincreasing with respect to r, while if either condition $\mathbf{C_1}$) or $\mathbf{C_2}$) holds then the mapping $M|_\Gamma$ is nondecreasing. In particular if w is nonpositive or nonnegative then it is monotone.*

Proof. We prove the case $\mathbf{B_1}$) only. Take an arbitrary $\tau_1, \tau_2 \in \Gamma$ such that $\tau_1 > \tau_2$. Then Inequality (14) implies $M(\tau_1) - M(\tau_2) \le 0$. □

We also have the following observation. The easy proof follows the same lines as the one of Corollary 2.2 in [2] and therefore is omitted.

Corollary 3.1. *Let Assumptions $\mathcal{A}$ hold and $w(x) = u(|x|)$ be a radial solution to Equation (3).*

i) *Let either condition $\mathbf{B_1}$) or $\mathbf{B_2}$) hold and $w(x) = 0$ for $|x| = \tau_0 \in (0,R)$. Then either $w \equiv 0$ for every x with $\tau_0 \le |x| \le R$ or the function $u(r)$ has an isolated zero at τ_0, in particular u must change sign at τ_0.*

ii) *If either condition $\mathbf{B_1}$) or $\mathbf{B_2}$) holds, w is either nonpositive or nonnegative and there exists $\tau_0 \in (0,R)$ such that $w(x) = 0$ for $|x| = \tau_0$ then $w \equiv 0$ for $|x| \in [\tau_0, R)$.*

iii) *If either condition $\mathbf{C_1}$) or $\mathbf{C_2}$) holds and there exists $\tau_0 \in (0,R]$ such that $w(x) = 0$ for $|x| = \tau_0$ ($\lim_{|x|\to\infty} w(x) = 0$ if $\tau_0 = \infty$) then $w \equiv 0$ for $|x| \in [0,\tau_0]$.*

4 The admissible perturbations

The purpose of this section is to extract and analyze features of functions h considered in Assumptions $\mathcal{A}$ and Theorem 2.1.

Let $\epsilon \in \{+,-\}$ and $a \in W^{1,1}_{loc}((0,R))$. Also let h satisfy Assumption $\mathcal{A}.5$. Define

$$\mathcal{H}_\epsilon\,(=\mathcal{H}_\epsilon(a)) := \{h(\tau,\lambda_0,\lambda_1) : \epsilon\,(h(\tau,\lambda_0,\lambda_1)\lambda_1 - \delta_a|\lambda_1|^p) \le 0\},$$

where δ_a is as in (6).

The following observation summarizes several properties of members of $\mathcal{H}_\epsilon$.

Proposition 4.1. *Let $\epsilon \in \{+,-\}$ and $a \in W^{1,1}_{loc}((0,R))$.*
i) *If $\{\phi_i\}_{i\in\mathbf{N}}$ is a partition of unity defined on $[0,R]\times\mathbf{R}^2$ and $h_i \in \mathcal{H}_\epsilon$ then $\sum_i \phi_i h_i \in \mathcal{H}_\epsilon$. In particular $\mathcal{H}_\epsilon$ is convex.*
ii) *$\mathcal{H}_\epsilon$ is invariant under addition of elements from the class*

$$\mathcal{R}_\epsilon := \{r = r(\tau,\lambda_0,\lambda_1)\;:\; r \text{ satisfies Assumption } \mathcal{A}.5,\;\; \epsilon r(\tau,\lambda_0,\lambda_1)\lambda_1 \le 0\},$$

i. e. if $h \in \mathcal{H}_\epsilon$ and $r \in \mathcal{R}_\epsilon$ then $h + r \in \mathcal{H}_\epsilon$.
iii) *Let b satisfy Assumption $\mathcal{A}.4$. If $b \ge 0$ and $h \in \mathcal{H}_+$, then $bh \in \mathcal{H}_+$. Whereas, if $b \le 0$ and $h \in \mathcal{H}_-$, then $bh \in \mathcal{H}_-$.*
iv) *Let*

$$h(\tau,\lambda_0,\lambda_1) = s(\tau,\lambda_0,\lambda_1)|\lambda_1|^{p-2}\lambda_1, \tag{20}$$

for s satisfying Assumption $\mathcal{A}.5$. If $\sup_{(\lambda_0,\lambda_1)} s(\tau,\lambda_0,\lambda_1) \le \delta_a(\tau)$ we have $h \in \mathcal{H}_+$, while if $\inf_{(\lambda_0,\lambda_1)} s(\tau,\lambda_0,\lambda_1) \ge \delta_a(\tau)$ we have $h \in \mathcal{H}_-$.

Let us illustrate the above proposition with a couple of examples.

Example 4.1.
1. Let $s(\tau) = s_a(\tau) := \alpha b(\tau) - (1-1/p)\tau b'(\tau)$, $b(\tau) = a(\tau)/\tau$, $\alpha \in \mathbf{R}$, $a \in W^{1,1}_{loc}((0,R))$ and h be as in (20). Then for $\alpha \le n - 2 + \frac{1}{p}$ function h belongs to $\mathcal{H}_+$, while for $\alpha \ge n-2+\frac{1}{p}$ it holds that $h \in \mathcal{H}_-$.
2. Consider $\alpha_1,\alpha_2,\alpha_3 \ge 0$ and s as in part 1 of this example. Then

$$\begin{aligned} h(\tau,\lambda_0,\lambda_1) &= s(\tau)\frac{|\lambda_1|^{p-2}\lambda_1}{1+\tau^{\alpha_1}+|\lambda_0|^{\alpha_2}+|\lambda_1|^{\alpha_3}} \in \mathcal{H}_+ \\ h(\tau,\lambda_0,\lambda_1) &= s(\tau)\left(1+\tau^{\alpha_1}+|\lambda_0|^{\alpha_2}+|\lambda_1|^{\alpha_3}\right)|\lambda_1|^{p-2}\lambda_1 \in \mathcal{H}_-. \end{aligned}$$

3. Consider $\alpha_1,\alpha_2,\alpha_3 \ge 0$, s as in part 1 of this example and r satisfying Proposition 4.1 ii). Then

$$\begin{aligned} h(\tau,\lambda_0,\lambda_1) &= \left(\frac{s(\tau)}{1+\tau^{\alpha_1}+|\lambda_0|^{\alpha_2}+|\lambda_1|^{\alpha_3}} + r(\tau,\lambda_0,\lambda_1)\right)|\lambda_1|^{p-2}\lambda_1 \in \mathcal{H}_+ \\ h(\tau,\lambda_0,\lambda_1) &= \left(s(\tau)\left(1+\tau^{\alpha_1}+|\lambda_0|^{\alpha_2}+|\lambda_1|^{\alpha_3}\right) - r(\tau,\lambda_0,\lambda_1)\right)|\lambda_1|^{p-2}\lambda_1 \in \mathcal{H}_-. \end{aligned}$$

5 The ODE and special functions

All the results presented in this paper can be applied to solutions of Equation (5) considered for $n = 1$. Namely, instead of n-dimensional ball we consider assumptions $\mathbf{B_1}), \mathbf{B_2}), \mathbf{C_1})$ and $\mathbf{C_2})$ on the interval $(0, R)$. Furthermore, we substitute all the conditions for $w, \nabla w$ with the corresponding conditions for u, u'. In particular one can link our results with several results which deal with ODEs, e.g. [3, 4, 10, 22, 28, 29, 39].
For this purpose we consider the following set of assumptions (compare with **Assumptions** $\mathcal{A}$).
Assumptions $\mathcal{A}1$
1. $p > 1$, $R \in (0, \infty]$, $B = (0, R)$ (for $R = \infty$ the ball $B = (0, \infty)$).
2. $u, \Phi_p(u^{'}) \in W^{1,1}_{loc}((0, R))$.
3. ϕ is an integrable odd continuous function on $\mathbf{R}$ such that $\tau\phi(\tau)$ is either positive or negative for almost all τ's.
4. $a \in W^{1,1}_{loc}((0, R))$.
5. $h = h(\tau, \lambda_0, \lambda_1)$ is a Carathéodory function defined on $(0, R) \times \mathbf{R}^2$ i.e. measurable with respect to τ, continuous with respect to the remaining variables. Additionally, for every $K > 0$

$$\sup_{|\lambda_0|<K, |\lambda_1|<K} |h(\tau, \lambda_0, \lambda_1)| \in L^1_{loc}(0, R).$$

Note that for $n = 1$ Equation (5) reduces to the following:

$$-a(\tau)(\Phi_p(u^{'}(\tau)))^{'} + h(\tau, u(\tau), u^{'}(\tau)) = \phi(u(\tau)) \quad \text{a.e. for } \tau \in (0, R), \tag{21}$$

where $(\Phi_p(u^{'}(\tau)))^{'}$ is understood in the sense of distributions, $u, \Phi_p(u')$, $|u'|^p \in W^{1,1}_{loc}((0, R))$. Therefore, by using exactly the same arguments as for $n > 1$ we obtain the following results.

Theorem 5.1. *Let Assumptions $\mathcal{A}1$ be satisfied and u be a solution of (21). If either condition $\mathbf{B_1})$ or $\mathbf{B_2})$ holds for $n = 1$ then*

$$\sup_{\tau\in(0,R)} |u(\tau)| = \limsup_{\tau\to 0} |u(\tau)|, \tag{22}$$

while if either condition $\mathbf{C_1})$ or $\mathbf{C_2})$ holds for $n = 1$ then

$$\sup_{\tau\in(0,R)} |u(\tau)| = \limsup_{\tau\to R} |u(\tau)|. \tag{23}$$

Corollary 5.1. *Let Assumptions $\mathcal{A}1$ be satisfied, $n = 1$, $p > 1$ and u be a solution of (21).
If $\mathbf{B_1'})$ holds or if $\mathbf{B_2'})$ holds then*

$$\sup_{\tau\in(0,R)} |u(\tau)| = \limsup_{\tau\to 0} |u(\tau)|.$$

Whereas, if $\mathbf{C_1'}$) *holds or* $\mathbf{C_2'}$) *holds then*

$$\sup_{\tau\in(0,R)}|u(\tau)| = \limsup_{\tau\to R}|u(\tau)|.$$

Proposition 5.1. *Under the assumptions of Theorem 5.1 the following problems admit only the trivial solutions:*

1.

$$\begin{cases} -a(\tau)(\Phi_p(u^{'}(\tau)))^{'} + h(\tau, u(\tau), u^{'}(\tau)) = \phi(u(\tau)) \text{ a.e. in } (0,R), \\ u(0) = 0 \end{cases}$$

in the case $\mathbf{B_1}$) *or* $\mathbf{B_2}$) *and for* $u \in C([0,R))$.

2.

$$\begin{cases} -a(\tau)(\Phi_p(u^{'}(\tau)))^{'} + h(\tau, u(\tau), u^{'}(\tau)) = \phi(u(\tau)) \text{ a.e. in } (0,R), \\ u(R) = 0 \text{ for } R < \infty \quad \text{or} \quad \lim_{\tau\to\infty} u(\tau) = 0 \text{ for } R = \infty. \end{cases}$$

in the case $\mathbf{C_1}$) *or* $\mathbf{C_2}$) *and* $u \in C((0,R])$ *for* $R < \infty$.

For our next purpose we set

$$\begin{aligned} \Gamma &= \{r \in (0,R): \ r \text{ is a critical point for, } u\} \cup \{0\} \cup \{R\} \\ M &: \Gamma \to [0,\infty], \ M(r) = \begin{cases} |u(r)|, & 0 < r < R \\ \limsup_{\tau\to 0}|u(\tau)|, & r = 0 \\ \limsup_{\tau\to R}|u(\tau)|, & r = R. \end{cases} \end{aligned} \tag{24}$$

Theorem 5.2. *Let Assumptions* $\mathcal{A}1$ *hold and* u *be a solution to Equation (21). Also, assume that assumptions* $\mathbf{B_1}$), $\mathbf{B_2}$), $\mathbf{C_1}$), $\mathbf{C_2}$) *are satisfied for* $n = 1$.
If either condition $\mathbf{B_1}$) *or* $\mathbf{B_2}$) *holds then the mapping* $M|_\Gamma$ *given by (24) is nonincreasing with respect to* r*, while if either condition* $\mathbf{C_1}$) *or* $\mathbf{C_2}$) *holds then the mapping* $M|_\Gamma$ *is nondecreasing. In particular if* w *is nonpositive or nonnegative then it is monotone.*

Corollary 5.2. *Let Assumptions* $\mathcal{A}1$ *hold and* u *be a solution to Equation (21). Furthermore, assume that assumptions* $\mathbf{B_1}$), $\mathbf{B_2}$), $\mathbf{C_1}$), $\mathbf{C_2}$) *are satisfied for* $n = 1$.

i) *Let either condition* $\mathbf{B_1}$) *or* $\mathbf{B_2}$) *hold and* $u(\tau_0) = 0$ *for* $\tau_0 \in (0,R)$*. Then either* $u \equiv 0$ *for every* τ *with* $\tau_0 \le \tau \le R$ *or the function* $u(\tau)$ *has an isolated zero at* τ_0*, in particular must change sign at* τ_0.

ii) *If either condition* $\mathbf{B_1}$) *or* $\mathbf{B_2}$) *holds,* u *is either nonpositive or nonnegative and there exists* $\tau_0 \in (0,R)$ *such that* $u(\tau_0) = 0$ *then* $u \equiv 0$ *for* $\tau \in [\tau_0, R)$.

iii) *If either condition* $\mathbf{C_1}$) *or* $\mathbf{C_2}$) *holds and there exists* $\tau_0 \in (0,R]$ *such that* $u(\tau_0) = 0$ $(\lim_{\tau\to\infty} u(\tau) = 0$ *if* $\tau_0 = \infty)$ *then* $u \equiv 0$ *for* $\tau \in [0, \tau_0]$.

Let us illustrate our discussion with the following two examples.

Example 5.1 (nonlinear eigenvalue problems and Bessel functions)**.** Following Walter [40] we define the operator

$$L_p^\alpha u = \tau^{-\alpha}(\tau^\alpha \Phi_p(u^{'}))^{'} = (\Phi_p(u^{'}))^{'} + \frac{\alpha}{\tau}\Phi_p(u^{'}),$$

where $\tau \in \mathbf{R}$ is an independent variable, $\alpha \geq 0$, $p > 1$. Consider the following eigenvalue problem [40]:

$$L_p^\alpha u + (q(\tau) + \lambda s(\tau))\Phi_p(u) = 0 \ \text{ in } [0, R],\ u^{'}(0) = 0,\ u(R) = 0, \tag{25}$$

where the functions $q(\tau)$ and $s(\tau)$ are continuous and $s(\tau)$ is positive on $[0, R]$. In our setting this equation reads

$$-a(\tau)(\Phi_p(u^{'}(\tau)))^{'} \ + \ h(\tau, u(\tau), u^{'}(\tau)) = \phi(u(\tau)), \quad \text{a.e. for } \tau \in (0, R), \tag{26}$$

where $a(\tau) = \frac{1}{q(\tau)+\lambda s(\tau)}$, $h(\tau, \lambda_0, \lambda_1) = -\alpha\frac{a(\tau)}{\tau}\Phi_p(\lambda_1)$ and $\phi(\lambda_0) = \Phi_p(\lambda_0)$.

On page 183 in [40] Walter shows that the above eigenvalue problem has a countable number of simple eigenvalues $\beta_1 < \beta_2 < \ldots$, such that $\lim_{m\to\infty}\beta_m = \infty$, and no other eigenvalues. Each eigenfunction u_m has $m-1$ simple zeros in $(0, R)$. Between 0 and the first zero of u_m, between two consecutive zeros of u_m and between the last zero of u_m and R there is one and only one zero of u_{m+1}. Similar result with $s \equiv 1$ and $q \equiv 0$ was obtained by del Pino and Manásevich in [18], by Anane in [5] and by Binding and Volkmer in [10] for $p = 2$ (in the latter case the solution is the generalization of Bessel's function). For a given m let us consider the function $b_m(\tau) := \frac{1}{a_m(\tau)} := q(\tau) + \beta_m s(\tau)$ and assume additionally that $a_m, q, s \in W^{1,1}_{loc}((0, R))$. We find δ_{a_m} defined as in (6)

$$\delta_{a_m}(\tau) := (1 - \frac{1}{p})(q^{'} + \beta_m s^{'})a_m^2.$$

Let us verify that assumption $\mathbf{B_1}$) holds in our case. First, observe that $\tau\Phi_p(\tau) > 0$ for $\tau \neq 0$ and $\liminf_{s\to 0} a_m(s)|u^{'}(s)|^p = 0$ provided that a_m is bounded in the neighborhood of 0. Next, if we assume that $a_m \geq 0$ close to R then the condition $\liminf_{s\to R} a_m(s)|u^{'}(s)|^p \geq 0$ is satisfied trivially. Moreover, we verify the condition $h(\tau, \lambda_0, \lambda_1)\lambda_1 \leq \delta_{a_m}(\tau)|\lambda_1|^p$ which in our case reads

$$\frac{-\gamma}{\tau} \leq (q^{'} + \beta_m s^{'})a_m, \tag{27}$$

where $\gamma = \frac{\alpha}{1-\frac{1}{p}}$. Equivalently, $(\ln(b_m))^{'} \geq \frac{-\gamma}{\tau}$.

Corollary 5.1 reveals that if (27) holds for every $\tau \in (0, R)$ then each $|u_m|$ attains its maximum at 0. Moreover, the sequence of maximas of $|u_m(\tau)|$ is nonincreasing in τ, as a consequence of Theorem 5.2.
Another conclusion can be deduced if $q \equiv 0$, $b_m = \beta_m s$ and $\beta_m < 0$. In such a case Proposition 5.1 infers that Equation (25) admits no nontrivial solutions.

In the next example we deal with hipergeometric functions. Our presentation is based on [29] where this problem is considered for linear equations.

Example 5.2. **(1)** Let us assume that $p = 2$ and $a, b, c \in \mathbf{R}$ are given numbers and $u \in C^1([0,1]) \cap C^2((0,1))$ satisfies the hypergeometric equation of Gauss

$$\tau(1-\tau)u''(\tau) + (c - (1+a+b)\tau)u'(\tau) - abu(\tau) = 0, \quad \tau \in (0,1).$$

With our notation $a(\tau) = \tau(\tau-1)$, $\phi(\lambda_0) = ab\lambda_0$ and $h(\tau, \lambda_0, \lambda_1) = (c-(1+a+b)\tau)\lambda_1$. Henceforth, $\delta_a(\tau) = -(1-1/p)(2\tau-1) = -\tau + \frac{1}{2}$. If $ab < 0$, then $\tau\phi(\tau) < 0$ for $\tau \neq 0$. Since the weight a is negative continuous on $(0,1)$ and equal to 0 at the endpoints of this interval, the limsup conditions in $\mathbf{C_2}$) hold. Moreover, $h(\tau, \lambda_0, \lambda_1)\lambda_1 \le \delta_a(\tau)|\lambda_1|^2$ in this case reads

$$c - \frac{1}{2} \le (a+b)\tau, \quad \tau \in [0,1].$$

The above inequality is satisfied for all $\tau \in [0,1]$ provided that $c \le \frac{1}{2}$ and $a + b > 0$. From this $\mathbf{C_2}$) follows and we retrieve the maximum principles obtained in Corollary 3.1 1) in [29].

(2) The Legendre polynomials satisfy the equation:

$$(1-\tau^2)u''(\tau) - 2\tau u'(\tau) + n(n+1)u(\tau) = 0, \quad \tau \in (-1,1).$$

With our notation $a(\tau) = \tau^2 - 1$, $\phi(\lambda_0) = -n(n+1)\lambda_0$ and $h(\tau, \lambda_0, \lambda_1) = -2\tau\lambda_1$. Similarly to the previous example we verify that $\tau\phi(\tau) < 0$ for all $\tau \neq 0$. Also, $a \le 0$ on $[-1,1]$ and the continuity of a gives us the limsup conditions in $\mathbf{C_2}$). The inequality $h(\tau, \lambda_0, \lambda_1)\lambda_1 \le \delta_a(\tau)|\lambda_1|^2$ reduces to $-2 \le -1$. Hence our techniques allow us to prove that the maximum of Legendre polynomials is attained at $x = 1$. (The same reasoning gives us that the maximum is attained also at $x = -1$.)

(3) We will now consider Laguerre polynomials, an important example of so-called Jacobi polynomials (see [29]). Recall, that the class of Laguerre functions contain for instance Hermite polynomials. The governing equation for Laguerre polynomials is

$$\tau u''(\tau) + (1 + a - \tau)u'(\tau) + nu(\tau) = 0, \quad \tau \in \mathbf{R}.$$

Assume wlog that $\tau > 0$. Then, the discussion similar to the above examples gives us that $\tau\phi(\tau) = -n\tau^2 < 0$ and $h(\tau, \lambda_0, \lambda_1)\lambda_1 \ge (\le)\, \delta_a(\tau)|\lambda_1|^2$ holds if $a + \frac{1}{2} \ge (\le)\, \tau$ respectively. Therefore assumptions $\mathbf{B_2}$) ($\mathbf{C_2}$) respectively) hold and hence the sequence of maximas decreases for $a + \frac{1}{2} \ge \tau$, whereas for $a + \frac{1}{2} \le \tau$ this sequence increases (compare to Corollary 3.4 in [29]).

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References

[1] E. Acerbi and G. Mingione *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal. *164* (2002), no. 3, 213–259.

[2] T. Adamowicz, A. Kałamajska *On a variant of the maximum principle involving radial p-Laplacian with applications to nonlinear eigenvalue problems and nonexistence results*, Topol. Methods Nonlinear Anal. *34* (2009), 1–20.

[3] R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for second order dynamic equations*, Ser. Math. Anal. Appl., vol. 5, Taylor & Francis, 2003.

[4] R. P. Agarwal, S. R. Grace, *On the oscillation of certain second order differential equations*, Differential equations and applications. Vol. 4, 1–8, Nova Sci. Publ., New York, 2007.

[5] A. Anane, PhD Thesis, Université libre Bruxelles, 1988.

[6] R. P. Agarwal, D. O'Regan, B. Yan, *Nonlinear boundary value problems on semi-infinite intervals using weighted spaces: an upper and lower solution approach*, Positivity *11* (2007), no. 1, 171–189.

[7] H. Aikawa, N. Shanmugalingam, *Hölder estimates of p-harmonic extension operators*, J. Diff. Equations *220* (2006), no. 1, 18–45.

[8] D. Arcoya, J. Diaz, L. Tello, *S-Shaped bifurcation branch in a quasilinear multivalued model arising in climatology,* J. Diff. Equations *150* (1998), 215-225.

[9] M. Belloni, V. Ferrone, B. Kawohl, *Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic operators*, Z. Angew. Math. Phys. *54*, no. 5 (2003), 771-783.

[10] P. A. Binding, H. Volkmer, *Oscillation theory for Sturm-Liouville problems with indefinite coefficients,* Proc. Roy. Soc. Edinburgh Sect. A *131*, no. 5 (2001), 989-1002.

[11] A. Björn, J. Björn, N. Shanmugalingam, *The Dirichlet problem for p-harmonic functions on metric spaces*, J. Reine Angew. Math. *556*, (2003), 173–203.

[12] F. Brock, *Radial symmetry for nonnegative solutions of semilinear elliptic equations involving the p-Laplacian*, Proceedings of the conference Calculus of variations, applications and computations, Pont-a-Mousson (1997).

[13] J.F. Brothers, W.P. Ziemer, *Minimal rearrangements of Sobolev functions,* J. Reine Angew. Math. *384* (1988), 153-179.

[14] A. Callegari, A. Nachman *A nonlinear singular boundary value problem in the theory of pseudoplastic fluids,* SIAM J. Appl. Math. *38* (1980) 275-281.

[15] L. F. Cheung, C. K. Law, M. C. Leung, J. B. McLeodd, *Entire solutions of quasilinear differential equations corresponding to p-harmonic maps*, Nonlinear Anal. *31*, no. 5-6 (1998), 701–715.

[16] A. Cianchi, A. Ferone, *On symmetric functionals of the gradient having symmetric equidistributed minimizers,* SIAM J. Math. Anal. *38*, no. 1 (2006), 279-308.

[17] L. Damascelli, F. Pacella, M. Ramaswamy, *Symmetry of ground states of p-Laplace equations via the moving plane method,* Arch. Ration. Mech. Anal. *148*, no. 4 (1999), 291-308.

[18] M. Del Pino, R. Manásevich, *Global bifurcation from the eigenvalues of the p–Laplacian*, J. Differ. Equations *92* (1991), 226–251.

[19] J. I. Diaz, *Nonlinear partial differential equations and free boundaries, Vol. I, Elliptic equations* (Research Notes in Mat. Vol. 106) Boston, London, Melbourne: Pitman Publ. Ltd. 1985.

[20] L. Diening, A. Prohl, M. Ružička, *Semi-implicit Euler scheme for generalized Newtonian Fluids*, SIAM J. Numer. Anal. *44*, no. 3 (2006), 1172-1190.

[21] J. Dolbeault, P. Felmer, R. Monneau, *Symmetry and nonuniformly elliptic operators,* Differential Integral Equations *18* (2005), no. 2, 141-154.

[22] P. Drábek, R. Manásevich, *On the closed solution to some nonhomogeneous eigenvalue problem with p-Laplacian,* Differential Integral Equations *12*, no. 6 (1999), 773-788.

[23] P. Drábek, P. Takáč, *Poincaré inequality and Palais-Smale condition for the p-Laplacian,* Calc. Var. Partial Differential Equations, *29*, no. 1 (2007), 31-58.

[24] D. Fortunato, L. Orsina, L. Pisani, *Born-Infeld type equations for electrostatic fields*, J. Math. Phys. *43*, no. 11 (2002), 5698-5706.

[25] B. Franchi, E. Lanconelli, J. Serrin, *Existence and uniqueness of nonnegative solutions of quasilinear equations in* $\mathbf{R}^n$. Adv. Math. *118*, no. 2 (1996), 177-243.

[26] B. Gidas, W. M. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle,* Commun. Math. Phys. *68* (1979), 209-243.

[27] J-F. Grosjean, *p-Laplace operator and diameter of manifolds*, Ann. Global Anal. Geom. *28*, no. 3 (2005), 257-270.

[28] A. Kałamajska, K. Lira, *Maximum modulus principles for radial solutions of quasilinear and fully nonlinear singular PDE's*, Bull. Belg. Math. Soc., *14* (2007), 157-176.

[29] A. Kałamajska, A. Stryjek, *On maximum principles in the class of oscillating functions,* Aequationes Math. *69* (2005), 201–211.

[30] N. Kawano, E. Yanagida, S. Yotsutani *Structure theorems for positive radial solutions to* $\mathrm{div}(|Du|^{m-2}Du) + K(|x|)u^q = 0$ *in* R^n J. Math. Soc. Japan *45*, no. 4 (1993), 719–742.

[31] B. Kawohl, *Rearrangements and convexity of level sets in PDE*, *Lecture Notes in Mathematics*, 1150, Springer-Verlag, Berlin-Heidelberg, 1985.

[32] P. Lindqvist, *On the equation* $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0$, Proc. Amer. Math. Soc. *109*, No.1 (1990), 157-164. Addendum, ibid. 116 (1992), 583-584.

[33] J. Manfredi, G. Mingione, *Regularity Results for Quasilinear Elliptic Equations in the Heisenberg Group*, Mathematische Annalen *339* (2007), 485-544.

[34] V. G. Maz'ya, *Sobolev Spaces*, Springer–Verlag 1985.

[35] J. Mossino, *Inégalités isopérimétriques et applications en physique* (French), Travaux en Cours. Paris: Hermann (1984).

[36] P. Pucci, J. Serrin, *The strong maximum principle revisited*, J. Diff. Equations *196* (2004), 1–66.

[37] M. Ružička, *Electrorheological Fluids: Modeling and Mathematical Theory,* vol. 1748 of Lecture Notes in Mathematics, Springer, Berlin, Germany, 2000.

[38] J. Serrin, H. Zou, *Symmetry of Ground States of Quasilinear Elliptic Equations*, Arch. Rational Mech. Anal. 148 (1999), 265–290.

[39] J. Sugie and N. Yamaoka, *Growth conditions for oscillation of nonlinear differential equations with p-Laplacian,* J. Math. Anal. Appl. *306*, no. 1 (2005), 18–34.

[40] W. Walter, *Sturm-Liouville theory for the radial* Δ_p*-operator*, Math. Z. *227* (1998), no. 1, 175–185.

[41] Z. Wu, J. Zhao, J. Yin, H. Li, *Nonlinear diffusion equations,* World Scientific, Singapore, 2001.

[42] X. Zhu, *A Sturm Type Theorem for Nonlinear Problems*, J. Differ. Equations *129* (1996), 166–192.