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McMullen's and geometric pressures and approximating the Hausdorff dimension of Julia sets from below

by

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Summary. We introduce new variants of the notion of geometric pressure for rational functions on the Riemann sphere, including non-hyperbolic functions, in the hope that some of them will turn out useful to achieve fast approximation from below of the hyperbolic Hausdorff dimension of Julia sets.

Contents

1. Introduction. Iterating a mapping f (roughly) increasing distances, for example a rational function on a neighbourhood of its "chaotic" invariant

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Julia set, transforms its small subsets into large ones, roughly preserving shapes (dynamical "escalator"). So long time behaviour under the action of f provides an insight into the local structure of $J(f)$—here, in complex dimension 1, into its Hausdorff dimension characteristic.

A tool is a *geometric pressure* with respect to the potential $\phi = \phi_t = -t\log|f'|$ on the Julia set, here for $t > 0$. The pressure (free energy) can be defined for any ϕ in a variational way:

$$P_{\mathrm{var}}(f,\phi) = \sup_{\mu}\Big(h_\mu(f) + \int \phi\, d\mu\Big), \tag{1.1}$$

with the supremum over all f-invariant probability measures μ on $J(f)$; $h_\mu(f)$ means the measure-theoretical (Kolmogorov) entropy, and the μ for which the supremum is attained (if it exists) is an equilibrium state. Below we provide different definitions. There is an analogy with equilibria in statistical physics, e.g. Ising model of ferromagnetics, where equilibria are distributed on the space of all configurations of $+$ or $-$ over $\mathbb{Z}^2$ with a potential depending on a Hamiltonian function expressed in terms of interactions between elements of the configuration.

The founders of applications of such models in dynamics are in particular Y. Sinai, D. Ruelle and R. Bowen (SRB measures). Here we consider forward trajectories, so the "configurations" are over $\mathbb{N}$. In particular a geometric application with the use of ϕ_t, hence $\exp S_n(\phi_t) = |(f^n)'|^{-t}$, is to relate (roughly) an equilibrium measure (a mass) of a disc of diameter $|(f^n)'|^{-1}$ to this diameter, for each t up to a normalizing coefficient $\exp nP(f,\phi_t)$. It is equal to 1 if $t = t_0$ is a zero of the pressure $P(f,\phi_t)$, denoted also $P(f,t)$. This t_0 is called the *hyperbolic Hausdorff dimension* of $J(f)$, $\mathrm{HD}_{\mathrm{hyp}}(f,t)$, defined as the supremum of the Hausdorff dimensions of invariant hyperbolic subsets of $J(f)$.

An introduction to this theory is provided in [PU]. Closer to the content of this paper are [P2] and [PRS2], where geometric pressure for general rational functions was first defined and studied. See also [P3].

The aim of this note is to introduce more variants of this notion, in particular, some notions close to McMullen's pressure defined for hyperbolic rational functions in [McM], useful to numerically calculate the Hausdorff dimension of the underlying Julia sets, or estimate it from below.

The pressure function $t \mapsto P(f,t)$ will occur to be the limit from below of a sequence of functions specific to each notion of pressure we introduce. Therefore their first zeros will converge to $\mathrm{HD}_{\mathrm{hyp}}(f,t)$ from below. There are two ideas in these notions of pressure:

(1) To replace a potential along each trajectory by a "fuzzy" one, mainly by replacing the value at a point by the infimum of the values in a small

neighbourhood, or the smaller of the values at two sampling points close to it.

(2) To restrict the pressure to trajectories not passing too close to the set of critical points.

The key issue we do not address here is how efficient is the calculation of these functions and how fast are the convergences. This will be dealt with in [DGT].

2. Definitions and the statement of the main theorem

2.1. Topological pressures. We start by recalling the basic definition of topological pressure for a continuous transformation of a metric compact space and a real continuous potential (see e.g. [Wal] or [PU]).

DEFINITION 2.1 (Topological pressure via separated sets). Let $f : X \to X$ be a continuous map on a compact metric space X, and $\phi : X \to \mathbb{R}$ a continuous real-valued function (a *potential*) on X. For S_n defined in (2.2) below, consider

$$P_{\mathrm{sep}}(f,\phi) := \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\Big(\sup_Y \sum_{y\in Y}\exp S_n\phi(y)\Big), \tag{2.1}$$

where the supremum is taken over all (n,ε)-separated sets $Y \subset X$, that is, such that $\rho_n(y_1,y_2)\ge\varepsilon$ for any distinct $y_1,y_2\in Y$, where ρ_n is the metric defined by $\rho_n(x,y)=\max\{\rho(f^j(x),f^j(y)) : j=0,\dots,n\}$.

We can slightly modify this definition, defining the *fuzzy pressure* or *inf-pressure* $P^0_{\mathrm{sep}}(f,\phi)$ by first replacing

$$S_n\phi(y) := \sum_{j=0}^{n-1}\phi(f^j(y)) \tag{2.2}$$

in (2.1) by

$$S_n^\delta\phi(y) := \sum_{j=0}^{n-1}\inf\{\phi(z) : z\in B(f^j(y),\delta)\},$$

obtaining $P^\delta_{\mathrm{sep}}(f,\phi)$, and then defining

$$P^0_{\mathrm{sep}}(f,\phi) := \lim_{\delta\to 0}P^\delta_{\mathrm{sep}}(f,\phi).$$

By the uniform continuity of ϕ, an easy calculation gives $P_{\mathrm{sep}}(f,\phi) = P^0_{\mathrm{sep}}(f,\phi)$.

A related notion is *tree pressure* (or *Gurevitch pressure*), interesting for a non-invertible f, defined for $z \in X$ by

$$P_{\mathrm{tree}}(f,\phi,z) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{y\in f^{-n}(z)} \exp S_n\phi(y), \tag{2.3}$$

which can also be defined in a "fuzzy" way as *fuzzy tree pressure*, or *infimum tree pressure*,

$$P^0_{\mathrm{tree}}(f,\phi,z) := \lim_{\delta\to 0} P^\delta_{\mathrm{tree}}(f,\phi,z), \tag{2.4}$$

where

$$P^\delta_{\mathrm{tree}}(f,\phi,z) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{y\in f^{-n}(z)} \exp S^\delta_n\phi(y). \tag{2.5}$$

Later on we shall use the following easy result [PU, Chapter 4].

Remark 2.2. Suppose $f : X \to X$ is open, distance expanding, and *topologically exact*: for every open $U \subset X$ there exists $n \in \mathbb{N}$ such that $f^n(U) = X$. Also, let $\phi : X \to \mathbb{R}$ be continuous. Then all the above pressures coincide and are independent of z. So we can denote them just by $P(f,\phi)$.

2.2. Geometric pressures. From now on we shall consider a rational transformation of the Riemann sphere, $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, and its restriction to the Julia set $J(f)$. We shall consider geometric potentials $\phi = \phi_t = -t\log|f'|$ for $t > 0$. Note that in this case we can write $\exp S_n\phi_t$ in Definition 2.1 in the form $|(f^n)'|^{-t}$. The derivative f' will be considered only with respect to the spherical Riemann metric. We shall use only its absolute value $|f'|$ so there will be no ambiguity caused by its argument.

The points $x \in \overline{\mathbb{C}}$ where $f'(x) = 0$ are called *critical* and their set is denoted by $\mathrm{Crit}(f)$. For $c \in \mathrm{Crit}(f)$, if $f(z) = a(z-c)^\nu + b(z-c)^{\nu+1} + \cdots$ in the complex plane coordinates, with $a \neq 0$, then $\nu = \nu(c)$ is called the *multiplicity* of f at c. We shall also consider the *post-critical set*

$$\mathrm{PC}(f) := \bigcup_{n=1}^{\infty} f^n(\mathrm{Crit}(f)).$$

If the forward trajectory of no critical point accumulates at $J(f)$, that is, there are f-critical points neither in $J(f)$, nor attracted to parabolic periodic orbits, then $f|_{J(f)}$ is open expanding. (Another term for this is *hyperbolic*.) It means there exist $C > 0$ and $\lambda > 1$ such that $|(f^n)'(z)| > C\lambda^n$ for all $z \in J(f)$ and $n \in \mathbb{N}$. This is an easy case, covered by Remark 2.2.

From now on we shall consider the general case with critical points whose forward trajectories can accumulate at $J(f)$. The above definitions of $P_{\mathrm{tree}}(f,\phi,z)$ and $P^0_{\mathrm{tree}}(f,\phi,z)$ make sense even though ϕ_t is infinite at critical points (yielding ∞) and for z outside $J(f)$. The pressure $P_{\mathrm{tree}}(f,\phi_t,z)$

does not depend on $z \in \overline{\mathbb{C}}$ for non-exceptional z (not in or fast accumulated by $\mathrm{PC}(f)$, see Definition 2.10). See [P2] and [PRS2]. So we can denote it just by $P_{\mathrm{tree}}(f,t)$. The independence of $P^0_{\mathrm{tree}}(f,\phi,z)$ from non-exceptional z also holds, via $P_{\mathrm{hyp}}(f,t)$ and $P_{\mathrm{tree}}(f,t)$, see Main Theorem 2.8(1).

To fix notation for $\phi = \phi_t = -t \log|f'|$ let us rewrite:

Notation 2.3 (Geometric tree pressure).

$$P_{\mathrm{tree}}(f,t,z) = P_{\mathrm{tree}}(f,\phi_t,z) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{v\in f^{-n}(z)} \Pi_n(t,v), \tag{2.6}$$

where

$$\Pi_n(t,v) := \prod_{k=1}^{n} |f'(f^{n-k}(v))|^{-t} = |(f^n)'(v)|^{-t}. \tag{2.7}$$

Notation 2.4 (Geometric fuzzy tree pressure).

$$\begin{aligned} &P^0_{\mathrm{tree}}(f,t,z) \\ &\quad := \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sum_{v\in f^{-n}(z)} \prod_{k=0}^{n} \inf\{|f'(y)|^{-1} : y \in B(f^{n-k}(v),\delta)\}. \end{aligned} \tag{2.8}$$

We shall also introduce a new notion:

Definition 2.5 (Pullback infimum tree pressure).

$$P^{\mathrm{pullinf}}_{\mathrm{tree}}(f,t,z) := \lim_{r\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sum_{v\in f^{-n}(z)} \Pi^{\mathrm{pullinf}}_n(t,v), \tag{2.9}$$

where

$$\begin{aligned} &\Pi^{\mathrm{pullinf}}_n(t,v) \\ &\quad := \prod_{k=1}^{n} \inf\{|f'(y)|^{-t} : y \in \mathrm{Comp}_{f^{n-k}(v)}\, f^{-k}(B(f^n(v),r))\}, \end{aligned} \tag{2.10}$$

where Comp_x means the component containing x.

The limit as $r \to 0$ exists since the relevant family is increasing as $r \to 0$ because the infima in (2.10) are taken over shrinking sets.

Let the following definition of $P(f, -t\log|f'|)$, called the *hyperbolic pressure*, be considered as a default one, to be denoted $P(f,t)$ (see e.g. [P3]):

Definition 2.6 (Hyperbolic pressure).

$$P(f,t) = P_{\mathrm{hyp}}(f,t) := \sup_{X\in\mathscr{H}(f,J(f))} P(f|_X, -t\log|f'|),$$

where $\mathscr{H}(f,J(f))$ is defined as the family of all compact forward f-invariant ($f(X) \subset X$) hyperbolic topologically exact subsets of $J(f)$ and repelling for f, that is, if a forward trajectory of a point is in a sufficiently small neighbourhood of X then it is entirely in X.

The repelling assumption can be omitted without affecting $P_{\mathrm{hyp}}(f,t)$. A direct proof is, given X maybe not repelling, to find a Cantor hyperbolic repeller X' in a neighbourhood of X with

$$P(f|_{X'}, -t\log|f'|) \geq P(f|_X, -t\log|f'|)$$

by shadowing, as in [PU, Theorem 11.6.1] (but simpler due to uniform hyperbolicity).

2.3. Hausdorff dimension. The definition and Bowen's formula saying that, for every hyperbolic repeller X, the Hausdorff dimension $\mathrm{HD}(X)$ is the only zero of the function $t \mapsto P(f|_X, t)$ [Bow] immediately yield

PROPOSITION 2.7 (Generalized Bowen formula). *The first zero t_0 of the function $t \mapsto P_{\mathrm{hyp}}(t)$ is equal to the hyperbolic dimension $\mathrm{HD}_{\mathrm{hyp}}(J(f))$, defined by*

$$\mathrm{HD}_{\mathrm{hyp}}(J(f)) := \sup_{X\in\mathscr{H}(f,J(f))} \mathrm{HD}(X).$$

Note that this zero exists, since all $P(f|_X, -t\log|f'|)$ are decreasing, hence their limit $P_{\mathrm{hyp}}(f,t)$ is decreasing and $\mathrm{HD}(X) \leq 2$ since $X \subset \overline{\mathbb{C}}$, which is of dimension 2.

2.4. Main Theorem. We shall prove the following (some definitions will be provided later on):

THEOREM 2.8 (Main Theorem). *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function of degree at least 2. Then:*

(1) $P(f,t) = P_{\mathrm{tree}}(f,t,z) = P^0_{\mathrm{tree}}(f,t,z)$ *for all non-exceptional $z \in \overline{\mathbb{C}}$.*
(2) $P(f,t) = \widehat{P}_{\mathrm{McM}}(f,t) = P^0_{\mathrm{McM}}(f,t) \leq P_{\mathrm{McM}}(f,t)$ *(restricted and fuzzy McMullen pressures) provided for the puzzle structure $\mathscr{P}$ in the definition of McMullen's pressures the diameters of the puzzle pieces of the renormalizations $\mathscr{R}^N(\mathscr{P})$ tend uniformly to 0 as $N \to \infty$.*
(3) $P(f,t) = P^{\mathrm{pullinf}}_{\mathrm{tree}}(f,t)$ *for all r small enough, provided f is* backward uniformly asymptotically stable. *Moreover for all r small enough, all non-exceptional z and every backward trajectory (z_n) of z (that is, $f(z_n) = z_{n-1}$ for all $n \in \mathbb{N}$, and $z_0 = z$)*

$$\frac{1}{n}\log\left(\frac{\Pi^{\mathrm{pullinf}}_n(t,z_n)}{|(f^n)'(z_n)|^{-t}}\right) \to 0 \tag{2.11}$$

as $n \to \infty$, uniformly with respect to the backward trajectory (z_n), where $z_0 = z$.

The equalities in (1) say in particular that the pressures there do not depend on z for all non-exceptional z.

DEFINITION 2.9. f is said to be *backward uniformly asymptotically stable* (buas) if there exists $r_0 > 0$ such that for each $z \in J(f)$, the diameters of all *pullbacks* of $B(z, r_0)$ (components of preimages) under f^n tend to 0 uniformly fast, with respect to z and to the pullback, as $n \to \infty$.

Notice that this property is hereditary, that is, if it holds for r, then it holds for every $\tilde{r} \leq r$.

DEFINITION 2.10. We call a point $z \in \overline{\mathbb{C}}$ *non-exceptional* if for each $\epsilon > 0$ and n large enough $B(z, \exp(-n\epsilon))$ is disjoint from $f^j(\mathrm{Crit}(f))$ for all $j = 1, \ldots, n$. In particular, the point z is not *post-critical*, that is, $z \notin \mathrm{PC}(f) = \bigcup_{n=1}^{\infty} f^n(\mathrm{Crit}(f))$. The other points are called *exceptional* and the set of all exceptional points is denoted by E.

It is clear from the definition that the exceptional set $E \subset \widehat{\mathbb{C}}$ has Hausdorff dimension 0. Sometimes it is convenient to consider $z \in \overline{\mathbb{C}} \setminus J(f)$ so close to $J(f)$ that it cannot be post-critical itself or accumulated by the post-critical set. Then of course it must be non-exceptional.

REMARK 2.11. The $\limsup_{n\to\infty}$ in the definition (2.3) can be replaced by lim, which always exists [PU, Remark 12.5.18]. So the limits exist also in the definition of $P_{\mathrm{tree}}^{\mathrm{pullinf}}(f, t, z)$ for $z \in J(f)$, provided f is buas. This will follow easily from the proof of Theorem 2.8 (see (2.11) and (3.4)). The same holds for $P_{\mathrm{tree}}^{\mathrm{infW}}(f, t, z)$, to be defined in (3.5).

3. Fuzzy (infimum) and pullback infimum tree pressures

Proof of Theorem 2.8(1). The result follows immediately from known theory and easy observations.

Indeed, the inequality $P_{\mathrm{hyp}}(f, t) \leq P_{\mathrm{tree}}(f, t, z)$ follows from the trivial observation that for every $X \in \mathscr{H}(f, J(f))$ in Definition 2.6 and $z \in X$,

$$P_{\mathrm{tree}}(f|_X, -t \log |f'|) \leq P_{\mathrm{tree}}(f, -t \log |f'|, z)$$

since in the former pressure we count only backward trajectories in X of a non-exceptional $z \in X$, whereas in the latter we count all trajectories in $J(f)$. A non-exceptional $z \in X$ exists since $\mathrm{HD}_{\mathrm{hyp}}(J(f)) > 0$, hence in its definition it is sufficient to consider only X satisfying $\mathrm{HD}(X) > 0$, and we have $\mathrm{HD}(E) = 0$.

The same reasoning holds for $P_{\mathrm{hyp}}(f, t, z) \leq P_{\mathrm{tree}}^{0}(f, t, z)$, defined in (2.8). Take into account that for every backward trajectory (z_n) of z in X we have $\log |f'(z_k)| \leq \log |f'(v)| + \epsilon$ for every $v \in \overline{\mathbb{C}}$ with $\rho(x_k, v) \leq \delta$, for every ϵ and δ small enough. This is so because X is disjoint from $\mathrm{Crit}(f)$, hence, being compact, bounded away from it.

The inequality $P^0_{\mathrm{tree}}(f,t,z) \le P_{\mathrm{tree}}(f,t,z)$ holds trivially for every z since $\inf\{|f'(y)|^{-1} : v \in B(z_k,\delta)\} \le |f'(z_k)|^{-1}$ for all k. The latter also implies the monotone increasing of $P^\delta_{\mathrm{tree}}(f,t,z)$ as $\delta \to 0$.

The proof that $P_{\mathrm{tree}}(f,t,z) \le P_{\mathrm{hyp}}(f,t)$ for non-exceptional $z \in J(f)$ is harder, but fortunately it is known. First one assumes that z is non-exceptional and additionally *hyperbolic*, which means by definition that there exist $r>0$ and $\lambda>1$ such that for every disc $B(z,\tau)$ there exists $n \in \mathbb{N}$ such that $|(f^n)'(y)| \ge \lambda^n$ for every $y \in B(z,\tau)$, $f^n|_{B(z,\tau)}$ is univalent and $f^n(B(z,\tau)) \supset B(f^n(z),r)$ [PRS2, Proposition 2.1]. An idea of the proof is to capture a large hyperbolic set using "shadowing".

Next one proves that $P_{\mathrm{tree}}(f,t,z^1) = P_{\mathrm{tree}}(f,t,z^2)$ for any two non-exceptional z^1 and z^2 ([P2, Theorem 3.3] and [PRS1, Geometric Lemma]). The idea is to find a curve (or a curve for each n) joining z_1 to z_2 in $\overline{\mathbb{C}}$ not fast accumulated by $f^n(\mathrm{Crit}(f))$, therefore with a controllable distortion for all branches of f^{-n} on appropriate neighbourhoods.

So one gets equality of the tree pressures for all non-exceptional z, which justifies the definition of tree pressure:

$$P_{\mathrm{tree}}(f,t) := P_{\mathrm{tree}}(f,t,z) \tag{3.1}$$

for every non-exceptional z. ∎

Proof of Theorem 2.8(3). The inequality $P^{\mathrm{pullinf}}_{\mathrm{tree}}(f,t,z) \le P_{\mathrm{tree}}(t,f,z)$ holds trivially for every $z \in \overline{\mathbb{C}}$ as before since obviously

$$\inf\{v \in W_{k,z_k,r} : |f'(y)|^{-1}\} \le |f'(z_k)|^{-1}$$

for all k. Here $W_{k,x,\tau}$ is the pullback $\mathrm{Comp}_x\, f^{-k}(B(f^k(x),\tau))$.

So let us prove the opposite inequality for every non-exceptional z in $J(f)$, namely

$$P^{\mathrm{pullinf}}_{\mathrm{tree}}(f,t,z) \ge P_{\mathrm{tree}}(f,t,z).$$

Write $r_n := \sup_{z\in J(f),\, v\in f^n(z)} \operatorname{diam} W_{n,v,r}$ and $r_{\max} := \max\{r_n : n = 1,2,\ldots\}$ for all $n=1,2,\ldots$. Notice that

$$r_{\max} \to 0 \quad \text{as } r \to 0. \tag{$*$}$$

Indeed, by backward uniform asymptotic stability, for all $r \le r_0$ as in Definition 2.9,

$$\forall \epsilon>0\ \exists n(\epsilon)\ \forall n \ge n(\epsilon)\ \forall W_{n,v,r}, \quad \operatorname{diam} W_{n,v,r} \le \epsilon.$$

Notice also that for every $\delta>0$ there exists $\delta'>0$ such that for every $z \in \overline{\mathbb{C}}$ and every component we have $\operatorname{diam}\mathrm{Comp}\, f^{-1}(B(z,\delta')) \le \delta$. By iterating a number of times smaller than $n(\epsilon)$ we conclude that there exists $0<r' \le r$ such that $\operatorname{diam} W_{n,v,r'} \le \epsilon$ for every $n<n(\epsilon)$. Thus, $(r')_{\max} \le \epsilon$. This proves $(*)$.

Having chosen $z \in J(f)$ consider an arbitrary backward trajectory (z_n). To simplify notation assume that z is non-periodic, hence z_n determines n.

Consider r such that $2r_{\max} < r_0$. We define inductively a sequence of integers k_j. Let k_1 be the least $k \geq 1$ such that

$$\widetilde{W}^{k,1} := W_{k,z_k,2r}$$

contains a critical point. Note the coefficient 2 of r; it will guarantee bounded distortion on discs of radius r of the branches f^{-k}, $k = 1, \dots, k_1 - 1$, mapping z to z_k.

Having defined k_j we consider the least $k = k_{j+1} > k_j$ such that

$$\widetilde{W}^{k_{j+1},j+1} := W_{k_{j+1}-k_j, z_{k_{j+1}}, 2 \operatorname{diam} W_{k_j, z_{k_j}, r}}$$

contains a critical point.

Let C be an upper bound of the distortion of $f^{k_{j+1}-k_j-1}$ on $f(W_{k_{j+1},z_{k_{j+1}},r})$; by the Koebe distortion lemma it is universal for all $r \leq r_0$ [PU, Lemma 6.2.3]. Notice also that, for another C,

$$\frac{\operatorname{diam} W_{k_{j+1},z_{k_{j+1}},r}}{\operatorname{diam} f(W_{k_{j+1},z_{k_{j+1}},r})} \leq C|f'(y)|^{-1}$$

for all $y \in W_{k_{j+1},z_{k_{j+1}},r}$ and j, and C depending on the maximal degree of criticality at critical points.

Finally, notice that since $\operatorname{diam} W_{k_j,z_{k_j},r} \to 0$, by $(*)$,

$$\operatorname{diam} \widetilde{W}^{k_{j+1},j+1} \to 0$$

as $j \to \infty$, uniformly with respect to the backward trajectories (x_n) for $x \in J(f)$. So $k_{j+1} - k_j \to \infty$ uniformly if the same critical point above appears, since otherwise the trajectory (x_n) would not be in $J(f)$ (see e.g. [P1, Lemma 1]). Taking into account that f has only a finite number of critical points, we conclude that $\#\{j : k_j \leq n\}/n \to 0$.

Thus, for every $y \in W_{n,z_n,r}$,

$$\Pi_n(1,y) = |(f^n)'(y)|^{-1} \geq \Pi_n^{\text{pullinf}}(1,z_n) \geq \exp(-n\epsilon)\frac{\operatorname{diam} W_{n,z_n,r}}{r} \tag{3.2}$$

for all $n \geq n(\epsilon)$ uniformly with respect to the backward trajectory (z_n), for all $\epsilon > 0$.

Finally, we shall use the assumption that y is non-exceptional, set $v = z_n$, and prove an inequality roughly opposite to (3.2):

$$\begin{aligned} \frac{\operatorname{diam} W_{n,z_n,r}}{r} &\geq C\exp(-n\epsilon)\frac{\operatorname{diam} \operatorname{Comp}_{z_n} f^{-n}(B(z, C\exp(-n\epsilon)))}{C\exp(-n\epsilon)} \\ &\geq \text{const} \cdot \exp(-n\epsilon)|(f^n)'(z_n)|^{-1}. \end{aligned} \tag{3.3}$$

The latter inequality follows from the bounded distortion of the conformal mapping $f^{-n} : B(z, C\exp(-n\epsilon)) \to \mathrm{Comp}_{z_n} f^{-n}(B(z, C\exp(-n\epsilon)))$ for a constant $C > 0$ and every n since z is non-exceptional.

In other words, for all $r > 0$ small enough,

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{\Pi_n^{\mathrm{pullinf}}(1, z_n)}{|(f^n)'(z_n)|^{-1}}\right) = 1, \tag{3.4}$$

the convergence being uniform over all backward trajectories of z.

Raising suitable expressions in (3.2) and (3.3) to the power t, summing over z_n and letting $n \to \infty$ and $\epsilon \to 0$ yields equality of the pressures in Theorem 2.8. ∎

Notice that in $P_{\mathrm{tree}}^{\mathrm{pullinf}}$ some components $W_{n,z_n,r}$ can contain many elements of $f^{-n}(z)$, thus being counted many times, but these multiplicities are upper bounded by $\exp \tau n$ for τ arbitrarily small and n large, again due to scarcity of "critical" times: $k_{j+1} - k_j \to \infty$ for each critical point. This justifies

Definition 3.1 (Pullback infimum W-tree pressure).

$$\begin{aligned} & P_{\mathrm{tree}}^{\mathrm{pullW}}(f, t, z) \\ & \quad := \lim_{r\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sum_{W\in\mathcal{W}_n} \prod_{k=1}^{n} \inf\{|f'(y)|^{-t} : y \in f^{n-k}(W)\}, \end{aligned} \tag{3.5}$$

where $\mathcal{W}_n$ is the family of all pullbacks of $B(z, r)$ for f^n. The limit in (3.5) as $r \to 0$ exists due to obvious monotonicity (cf. Definition 2.5).

Thus we obtain

Corollary 3.2. $P_{\mathrm{tree}}^{\mathrm{pullW}}(f, t, z) = P_{\mathrm{tree}}(f, t)$ *for every non-exceptional* z.

Remark 3.3. The inequality (3.2) was proved in [P4] and named a "telescope lemma". However, there the exponential convergence of $|(f^n)'(v)|^{-1}$ to 0 was assumed and nothing was assumed on $\mathrm{diam}\, W_{n,z_n,r}$. Here the uniform convergence $\mathrm{diam}\, W_{n,z_n,r} \to 0$ is a priori assumed, but we assume nothing about the derivatives.

Remark 3.4. In $P_{\mathrm{tree}}^{0}(f, t)$ the fraction

$$\frac{\prod_{k=0}^{n} \inf\{|f'(y)|^{-t} : y \in B(f^{n-k}(z_n), \delta)\}}{|(f^n)'(z_n)|^{-t}}$$

can be very close to 0, unlike in (2.11). So an analogue of (2.11) need not hold. See Remark 4.6 and Section 5.

4. McMullen's pressures. We now define McMullen's pressure. Assume there is a puzzle structure for f (a Markov partition with singularities). Namely there exists a covering of a neighbourhood of $J(f)$ by a family $\mathscr{P}$ of closed topological Jordan discs P_i, small enough that none of them contains more than one critical point, with interiors $\operatorname{Int} P_i$ intersecting $J(f)$ and mutually disjoint, and such that if $f(\operatorname{Int} P_i)$ intersects $\operatorname{Int} P_j$ then $f(\operatorname{Int} P_i) \supset \operatorname{Int} P_j$. We also assume that all the maps $f|_{\operatorname{Int} P_i}$ are proper. We allow critical points to belong to the boundaries of P_i.

Note that this definition differs somewhat from McMullen's. Firstly, we do not use any measure in it. Secondly, McMullen assumed an expanding property, while we do not. On the other hand, he assumed f to be continuous (conformal) only piecewise, on each P_i.

Following McMullen [McM] define a refinement $\mathscr{R}(\mathscr{P})$ to be

$$\operatorname{cl}(\operatorname{Comp} \operatorname{Int} f^{-1}(\mathscr{P}) \vee \mathscr{P}),$$

that is, the family of the closures of all components of the sets $f^{-1}(\operatorname{Int} P_j) \vee \operatorname{Int} P_i$. This is also a covering of a neighbourhood of $J(f)$ (maybe smaller than the one for $\mathscr{P}$) by the complete f-invariance of $J(f)$, and has a puzzle structure. We consider the consecutive refinements $\mathscr{R}^N(\mathscr{P})$ and assume that the diameters of their elements shrink to 0 uniformly.

For $N = 0, 1, \ldots$ and $P_{N,i}, P_{N,j} \in \mathscr{R}^N(\mathscr{P})$ denote by $s_N(i,j)$ the number of components of $\operatorname{Int} P_{N,i} \cap \operatorname{Int} f^{-1}(P_{N,j})$. If N is fixed we write simply $s(i,j)$. Of course the number $s(i,j)$ of components is larger than 1 if and only if there is an f-critical point $c \in \operatorname{Int} P_{N,i} \setminus \operatorname{Int} f^{-1}(P_{N,j})$ (remember that we assume the puzzle pieces are so small that each contains at most one f-critical point in $\operatorname{Int} P_{N,i}$). In that case $s(i,j) = \nu(c)$ (the multiplicity of f at c). If the above intersection is empty, we set $s(i,j) = 0$. In this notation the number of elements of $\mathscr{R}^{N+1}(\mathscr{P})$ is $\sum_{i,j} s(i,j)$.

In McMullen's hyperbolic setting all $s(i,j)$ are 0 or 1 since there are no critical points present. The same holds in Example 6.11 below.

For each $\mathscr{P}$ and $N = 1, 2, \ldots$ as above we distinguish a point $y_{N,i}$ in each $\operatorname{Int} P_{N,i}$ [1]. With $\mathscr{R}^N(\mathscr{P})$ we associate the matrix $\mathscr{R}^N(T)$ with entries

$$a_{ij} = \begin{cases} |f'(y_{N,i})|^{-1} & \text{if } s(i,j) > 0, \\ 0 & \text{if } s(i,j) = 0. \end{cases} \tag{4.1}$$

Similarly for $N = 0$ a matrix T is associated to $\mathscr{P}$, with distinguished points $y_{0,i}$. Observe that all the entries of these matrices are non-negative. (Note that writing $\mathscr{R}(T)$ is an abuse of notation (harmless), since it is not derived only from T, but from $\mathscr{R}(\mathscr{P})$ which carries more information.)

If T is an $M \times M$ matrix (a dimension M square matrix), then $\mathscr{R}(T)$ is a dimension $\sum_{i,j=1}^{M} s(i,j)$ matrix. Similarly we define $\mathscr{R}^{N+1}(T)$ for $\mathscr{R}^N(T)$ in place of T.

---

[1] Another variant is to distinguish $y_{N,i,j} \in \operatorname{Int} P_{N,i} \cap \operatorname{Int} f^{-1}(P_{N,j})$ whenever this set is non-empty. The same in Definition 4.2 later on (cf. [DGT]).

We can consider a simplified integer-valued square matrix $\widehat{T}$ of dimension M, with each ij entry equal to $s(i,j)$. It can be interpreted as the directed graph $\Gamma(\widehat{T})$, with vertices corresponding to the numbers $i = 1,\ldots,M$ and every pair of vertices i,j joined with multiplicity $s(i,j)$, i.e. by $s(i,j)$ edges starting at i and ending at j. Then we can consider the derived graph $\mathscr{R}(\Gamma(\widehat{T})) := \Gamma(\mathscr{R}(\widehat{T}))$, that is, the directed graph where vertices are the edges of $\Gamma(\widehat{T})$ and edges are ordered pairs of edges of $\Gamma(\widehat{T})$ such that the end of the former coincides with the beginning of the latter, with newly gained multiplicities. See [Ore]. These simplified matrices play here an explanatory role. We shall not use them.

Note that there is no need here to consider for $N = 1,2,\ldots$ the partitions just of the form $\mathscr{R}^N(\mathscr{P})$ with the associated matrices $\mathscr{R}^N(T)$. See Remark 6.10. But for computational simplicity it is better to produce everything from one matrix T (the partition $\mathscr{P}$).

Let $\lambda(\mathscr{R}^N(T))$ denote the spectral radius of $\mathscr{R}^N(T)$. For each $t > 0$ we use the notation T^t for the matrix with each entry being the corresponding entry of T raised to the power t. Similarly we define $\mathscr{R}^N(T)^t$. In particular, we denote its spectral radius by $\lambda(\mathscr{R}^N(T)^t)$. Due to the topological exactness of f on $J(f)$,

$$\lambda(\mathscr{R}^N(T)^t) := \lim_{n\to\infty} \sqrt[n]{\|(\mathscr{R}^N(T)^t)^n\|} = \lim_{n\to\infty} \sqrt[n]{((\mathscr{R}^N(T)^t)^n)_{ij}} \tag{4.2}$$

independently of $1 \le i,j \le \dim(\mathscr{R}^N(T)^t)$. Indeed, in our situation the topological exactness says that all the matrices $\mathscr{R}^N(T)$ are *primitive*, that is, $((\mathscr{R}^N(T))^n)_{ij} > 0$ for n large enough and all i,j. Let us call the sequence $(i_0,\ldots,i_n)$ *admissible* if $\mathscr{R}^N(T)_{i_k i_{k-1}} > 0$ for all $k = 1,\ldots,n$. We can then say that $\mathscr{R}^N(T)$ is primitive if for all n large enough and all i,j there exists an admissible sequence (path) $(i_0,\ldots,i_n)$ such that $i_0 = j$ and $i_n = i$.

Convergence to the spectral radius in (4.2) (the second equality) follows from the fact that for matrices with non-negative entries, one can replace the norms by suprema of the entries. Next consider paths related to the iterates of the matrices. It is visible that considering e.g. all of them, or only those starting at i and ending at j, gives the same limit due to primitivity.

Finally, we define *McMullen's pressure* (see [McM] in the hyperbolic case) by

$$P_{\mathrm{McM}}(f,t) := \limsup_{N\to\infty} \log \lambda(\mathscr{R}^N(T)^t). \tag{4.3}$$

Warning. Unfortunately, in the presence of critical points in $J(f)$ this notion has deficiencies if the distinguished points are critical or close to critical, making $P_{\mathrm{McM}}(f,t)$ too big, bigger than $P(f,t)$. A remedy is to consider

Definition 4.1 (Restricted McMullen pressure). Define the *restricted McMullen pressure* as

$$\widehat{P}_{\mathrm{McM}}(f,t) := \lim_{N\to\infty} \log \lambda(\widehat{\mathscr{R}^N(T)}^t), \tag{4.4}$$

where in each $\widehat{\mathscr{R}^N(T)}$ all the entries at positions ij such that

$$\frac{\mathrm{dist}(P_{N,i}, \mathrm{Crit}(f))}{\mathrm{diam}\, P_{N,i}} \geq A(N) \tag{4.5}$$

are the same as in $\mathscr{R}^N(T)$, and all others are 0. Here $A(N)$ is an arbitrary sequence of numbers tending to ∞ as $N \to \infty$ such that $A(N)\,\mathrm{diam}\,\mathscr{R}^N(\mathscr{P}) \to 0$; here diam denotes the supremum of the diameters of the sets of a partition.

The limit in (4.4) exists since the dimensions of the matrices are growing, acquiring a growing number of non-zero rows (puzzle pieces), where the puzzle pieces already considered become split with growing N; the distinguished points may move but they move a distance decreasing to 0. A detailed proof relies on (4.8) and (4.9). In fact, we do not need the existence of a limit a priori to prove that any limit (for a convergent subsequence) is equal to $P(f,t)$ as in the proof of Theorem 4.4. Still a deficiency of this notion is that with $y_{N,i}$ arbitrary, even far from $\mathrm{Crit}(f)$, there is no reason for the sequence to be increasing, or for its elements not to exceed $P(f,t)$ slightly.

So, we also consider versions suggested in [DGT] (see Example 6.11 here).

Definition 4.2 (Fuzzy McMullen pressures). To define $P^0_{\mathrm{McM}}(f,t)$, the *fuzzy McMullen pressure* (or infimum McMullen pressure), just replace $|f'(y_{N,i})|^{-t}$ by $\inf_{y\in P_{N,i}} |f'(y)|^{-t}$, for each N and $P_{N,i}$, in a_{ij} in the definition of McMullen's pressure (see (4.1)). Then consider the corresponding matrices $(\mathscr{R}^N(T)^t)^{\inf}$ and their spectral radii $\lambda^{\inf}_{N,t}$ and set

$$P^0_{\mathrm{McM}}(f,t) := \lim_{N\to\infty} \log \lambda^{\inf}_{N,t}. \tag{4.6}$$

The limit exists since the sequence is increasing, because when the puzzle pieces split for N growing, the infima are taken over smaller sets.

Definition 4.3 (Fuzzy restricted McMullen pressure). To define this pressure, denoted $\widehat{P}^0_{\mathrm{McM}}(f,t)$, keep unchanged the entries a_{ij} in the matrices accompanying $P^0_{\mathrm{McM}}(f,t)$ for i satisfying (4.5), putting 0 elsewhere. The monotonicity holds as before.

Now we can complete the proof of Theorem 2.8.

Theorem 4.4. *In the setting above, for each $t > 0$,*

$$P(f,t) = \widehat{P}_{\mathrm{McM}}(f,t) = \widehat{P}^0_{\mathrm{McM}}(f,t) = P^0_{\mathrm{McM}}(f,t) \leq P_{\mathrm{McM}}(f,t).$$

Proof. We prove that for any non-exceptional $z \in J(f)$,

$$
\begin{aligned}
(4.7)\qquad P_{\mathrm{hyp}}(f,t) \le \widehat{P}_{\mathrm{McM}}(f,t) = \widehat{P}^0_{\mathrm{McM}}(f,t)\\
\le P^0_{\mathrm{McM}}(f,t) \le P_{\mathrm{tree}}(f,t,z) \le P_{\mathrm{hyp}}(f,t).
\end{aligned}
$$

Consider an arbitrary set $X \in \mathscr{H}(f,J(f))$. and an arbitrary $\delta \ll \mathrm{dist}(X,\mathrm{Crit}(f))$, so that $\frac{|f'(x)|}{|f'(v)|}$ is close to 1 for $v \in B(x,\delta)$ and each $x \in X$. This is possible since $\log|f'|$ is uniformly continuous in $B(X,\delta)$. (Compare e.g. (4.8) or the proof of Theorem 5.1 for more detailed estimates taking into account distance from $\mathrm{Crit}(f)$.)

Let N be so large that every $P_{N,i} \in \mathscr{R}^N(\mathscr{P})$ has diameter less than δ. Consider now only the puzzle pieces intersecting X. Consider any integer $n \ge 0$ and any (n,δ)-separated set $Y \subset X$ for $f|_X$ (see Definition 2.1). We can assume that $Y \subset \bigcup_{P_{N,\iota} \in \mathscr{R}^N(\mathscr{P})} \mathrm{Int}\, P_{N,\iota}$ by first taking Y 2δ-separated and next correcting it to be in the union of the interiors of puzzle pieces.

Notice that for any distinct $z_1, z_2 \in Y$, the admissible sequences i_k, $k = 0,1,\ldots,n$, such that $f^k(z) \in \mathrm{Int}\, P_{N,i_k} \cap \mathrm{Int}\, f^{-1}(P_{N,i_{k-1}})$ for $z = z_1, z_2$ are different. Use now $\frac{|f'(f^k(z))|}{|f'(y_{N,i_k})|} \approx 1$ (namely $\ge \exp(-\epsilon)$ with $\epsilon > 0$ close to 0 for δ close to 0). Thus each $z \in Y$ contributes

$$\prod_{k=1}^{n} |f'(y_{N,i_k})|^{-t} \ge |(f^n)'(z)|^{-t} \times \exp(-\epsilon n)$$

to the matrix $(\widehat{\mathscr{R}^N(T)}^t)^n$.

Thus, letting $n \to \infty$ and next $\delta \to 0$ with respective $N \to \infty$, finally taking the supremum over X, we obtain (4.4) (with $\limsup_N$), hence $P_{\mathrm{hyp}}(f,t) \le \widehat{P}_{\mathrm{McM}}(f,t)$.

Similarly, considering the puzzle pieces $P_{N,i}$ satisfying (4.5) and distinguished $y_{N,i}$ in them, we prove the equality $\widehat{P}_{\mathrm{McM}}(f,t) = \widehat{P}^0_{\mathrm{McM}}(f,t)$. Clearly only the $\le$ part is non-trivial. Here is the proof:

For any $v \in P = P_{N,i}$ we have, close to a critical point c of multiplicity ν, assuming for simplification that $f(x) = (x-c)^\nu$, and writing y for $y_{N,i}$,

$$
\begin{aligned}
(4.8)\qquad \left|\left(\frac{|f'(v)|}{|f'(y)|}\right)^{1/(\nu-1)} - 1\right| \le \left|\frac{\mathrm{dist}(v,c)}{\mathrm{dist}(y,c)} - 1\right|\\
\le \frac{|\mathrm{dist}(v,c) - \mathrm{dist}(y,c)|}{\mathrm{dist}(y,c)} \le \frac{\mathrm{diam}\, P}{\mathrm{dist}(P,c)} \le A(N)^{-1}
\end{aligned}
$$

(see (4.5)). So $\frac{|f'(v)|}{|f'(y)|} \to 1$ as $N \to \infty$. Without the above simplification we can write $f(z) = a(z-c)^\nu h(z)$ for an analytic map $h(z) = 1 + a_1(z-c) + \cdots$

in a neighbourhood of c, so

$$f'(z) = a\nu(z-c)^{\nu-1}h(z) + a(z-c)^{\nu}h'(z) \tag{4.9}$$
$$= a(h(z)\nu + (z-c)h'(z))(z-c)^{\nu-1}$$
$$= a\big((1 + a_1(z-c) + \cdots)\nu + (z-c)(a_1 + \cdots)\big)(z-c)^{\nu-1}$$
$$= a\nu(z-c)^{\nu-1}(1 + O(z-c)).$$

So for $\mathrm{dist}(P,c)$ small enough, for all N, $|f'(v)/f'(y)| \le 2A(N)^{-1}$. Farther away from c, i.e. in a domain bounded away from $\mathrm{Crit}(f)$, $\log|f'|$ is uniformly continuous, so for large N, $\mathrm{dist}(v,y)$ small implies $\big|\log|f'(v)| - \log|f'(y)|\big|$ small, so $|f'(v)/f'(y)|$ is close to 1, namely it tends to 1 as $N \to \infty$.

The inequality $\widehat{P}^0_{\mathrm{McM}}(f,t) \le P^0_{\mathrm{McM}}(f,t)$ is obvious because the matrices associated to the former have just zeros replacing some non-zero terms in the latter.

To prove $P^0_{\mathrm{McM}}(f,t) \le P_{\mathrm{tree}}(f,t)$, first select an arbitrary non-exceptional point $z^{N,i}$ in each $P_{N,i} \in \mathscr{R}^N(\mathscr{P})$. Consider an arbitrary admissible sequence of integers $i_0,\dots,i_n$, that is, such that $\mathrm{Int}\, P_{N,i_k} \cap f^{-1}(P_{N,i_{k-1}}) \ne \emptyset$ for all $k = 1,\dots,n$. It contributes to the matrix $(\mathscr{R}^N(T)^t)^n$ because it corresponds to the path in the related directed graph with edges joining consecutively the vertices $i_n,\dots,i_0$.

Choose an arbitrary $z_n \in \bigcap_{k=0}^n f^{-(n-k)}(\mathrm{Int}\, P_{N,i_k})$ (this set may be disconnected!) and $z_k := f^{n-k}(z_n)$ for all $k = n-1,\dots,0$, with common $z = z_0 = z^{N,i_0}$. In fact, the number of possible points z_n is equal to $\prod_{k=0}^{n-1} \deg(f|_{P_{N,k+1} \cap f^{-1}(P_{N,k})})$ (cf. Remark 4.5).

Now $P^0_{\mathrm{McM}}(f,t) \le P_{\mathrm{tree}}(f,t,z)$ follows from the obvious inequality $\inf_{y \in P_{N,i_k}} |f'(y)|^{-1} \le |f'(z_k)|^{-1}$ for each $k = 1,\dots,n$. Indeed, the only issue is that we took care only of the sequences $(i_0,\dots,i_k)$ with common i_0. If we consider all sequences, the constant factor $\#(\mathscr{R}^N(\mathscr{P}))$ appears. However, it disappears as $n \to \infty$ in the definition of the spectral radius.

The last inequality in (4.7) is known if $z \in J(f)$ is hyperbolic and non-exceptional, with a proof via capturing hyperbolic subsets via shadowing; it was mentioned in the proof of Theorem 2.8(1), referring to [PRS2].

Finally, notice that $P^0_{\mathrm{McM}}(f,t) \le P_{\mathrm{McM}}(f,t)$ is obvious. ∎

Remark 4.5 (Fuzzy multiple McMullen pressure). Notice that replacing the sequence y_{N,i_k} for admissible $(i_0,\dots,i_n)$ we do not exploit all appropriate z_n. So in the definition of the matrix $\mathscr{R}^N(T)$ or $(\mathscr{R}^N(T)^t)^{\inf}$ we could consider each entry a_{ij} multiplied by $\deg(f|_{P_{N,i}})$. The related notion of pressure, to be called *fuzzy multiple McMullen pressure*, denoted $P^0_{\mathrm{multMcM}}(f,t)$, is also upper bounded by $P_{\mathrm{tree}}(f,t)$.

Remark 4.6. For a sequence $i_0, \ldots, i_n$ admissible for the matrix $\mathscr{R}^N(\mathscr{P})$, we can prove neither the inequality

$$\liminf_{N\to\infty} \liminf_{n\to\infty} \frac{1}{n} \log\left(\frac{\prod_{k=1}^n |f'(y_{N,i_k})|}{|(f^n)'(z_n)|} \right) \le 0, \tag{4.10}$$

nor the opposite one with liminf replaced by limsup, unlike in e.g. (3.4). The problem is that the expressions $|(f^n)'(z_n)|$ must be replaced by the product $\prod_{k=1}^n |f'(y_{N,i_k})|$, where the domains of f' to which y_{N,i_k} respectively belong are larger than the components of $f^{-k}(B(f^n(z_n), r))$ for large k, so the latter products can be too large or too small. Namely the choices of y_{N,i_k} for P_{N,i_k} close to $\mathrm{Crit}(f)$ are much farther away from $\mathrm{Crit}(f)$ than z_k, or much closer.

A way out would be to replace one telescope in the proof of (3.4) by a sequence of telescopes. But for this, to prove e.g. ≤ 0 in (4.10) we need to know that each z_{n_j} starting a new telescope is non-exceptional (with the same constants) to obtain (3.3), which may be impossible.

So, to avoid $\prod_{k=1}^n |f'(y_{N,i_k})|$ too large, we have chosen to just get rid of the trouble-making backward trajectories, by considering the restricted McMullen pressure.

A way to avoid $\prod_{k=1}^n |f'(y_{N,i_k})|$ too small, that is, $\prod_{k=1}^n |f'(y_{N,i_k})|^{-1}$ too large, is to consider fuzzy McMullen pressure replacing $|f'(y_{N,i})|^{-1}$ by $\inf\{|f'(y)|^{-1} : y \in P_{N,i}\}$ (see Definition 4.2). Without this it can just happen that $P_{\mathrm{McM}}(f,t) > P(f,t)$: see the Warning before Definition 4.1 (or the restricted one as in that definition). For another remedy, replacing the infimum or one point $y_{N,i}$ by pairs of distinguished points, see Section 6.

Remark 4.7. Consider sets of the form

$$X_N = \bigcap_{n=1}^{\infty} \bigcup_{i_0,\ldots i_n} \bigcap_{k=0}^{n} f^{-(n-k)}(P_{N,i_k}), \tag{4.11}$$

with the union over all $(i_0, \ldots, i_n)$ admissible for the restricted matrix, i.e. $\widehat{\mathscr{R}^N(T)}$ (see (4.5)).

Notice that each X_N is hyperbolic for $f|_{X_N}$. Indeed, the inverses of $(f|_{\mathrm{Int}\, P_{N,i_0,\ldots,i_n}})^n$, where $P_{N,i_0,\ldots,i_n} := \bigcap_{k=0}^n f^{-(n-k)}(P_{N,i_k})$, exist by (4.5). For each i_n they form a Montel normal family of holomorphic maps on P_{N,i_n}. This is so, because the ranges omit more than two points in $\overline{\mathbb{C}}$, e.g. a neighbourhood of $\mathrm{Crit}(f)$. So their limits must be points since otherwise a limit domain U would not intersect $J(f)$ (since otherwise all $f^n(U')$ with an open $U' \subset U$ intersecting $J(f)$ and with n large enough are bounded in P_{N,i_n}, contradicting the definition of the Julia set). On the other hand, U must intersect $J(f)$ by its backward invariance and compactness. This implies uniform convergence of $|(f^n)'|^{-n}$ on X_N to 0, hence hyperbolicity.

Notice however that X_N need not be repelling, as in Definition 2.6, or equivalently, the maps $f|_{X_N}$ need not be open. See [PU, Example 4.5.5] where the question of a small extension of X_N to an invariant set on which f is open was discussed.

Notice finally that the sequence of the sets X_N is increasing with respect to inclusion and that the pressures satisfy

$$P(f|_{X_N}, -t\log|f'|) = \log\lambda(\widehat{\mathscr{R}^N(T)}^t).$$

5. Restricted fuzzy tree pressure. One more definition of pressure might be useful for computations, close to the restricted McMullen pressure and to the fuzzy tree pressure as in (2.4) and (2.5), for the potentials $-t\log|f'|$, making sense for all rational maps. Namely define

$$\widehat{P}^0_{\mathrm{tree}}(f,t,z) := \lim_{\Delta\to 0}\widehat{P}^{\Delta}_{\mathrm{tree}}(f,t,z), \tag{5.1}$$

where

$$\widehat{P}^{\Delta}_{\mathrm{tree}}(f,t,z) := \limsup_{n\to\infty}\frac{1}{n}\log\sum_v \widehat{\Pi^\delta}(v), \tag{5.2}$$

where the sum is over all $v\in f^{-n}(z)$ such that $\mathrm{dist}(f^{n-k}(v),\mathrm{Crit}(f)) > \Delta$ for all $1\le k\le n$ with

$$\widehat{\Pi^\delta}(v) := \prod_{k=1}^n |f'(\widehat{v}_k)|^{-t} \tag{5.3}$$

where $\widehat{v}_k$ is a point in $\mathrm{cl}\,B(f^{n-k}(v),\delta)$ where $|f'|^{-1}$ takes the infimum, and

$$\delta = o(\Delta). \tag{5.4}$$

Assume the function $\Delta\mapsto\delta$ is monotone. Then existence of the limit in (5.1) follows by monotonicity, which is obvious since the infima are taken over shrinking sets, whose family is growing. Notice that if we do not care about monotonicity, we can choose $\widehat{v}_k$ arbitrarily (randomly) in the ball.

THEOREM 5.1. *$\widehat{P}^0_{\mathrm{tree}}(f,t,z)$ does not depend on non-exceptional z. Therefore* (*omitting z*) *we have*

$$\widehat{P}^0_{\mathrm{tree}}(f,t) = P(f,t).$$

Proof. The proof is similar to the proof of Theorem 4.4. The inequality $\widehat{P}^0_{\mathrm{tree}}(f,t,z)\le P^0_{\mathrm{tree}}(f,t,z)$ is obvious, just more backward branches of z in the latter pressure are considered. Also $P^0_{\mathrm{tree}}(f,t,z)\le P_{\mathrm{tree}}(f,t,z)$ is obvious, as already mentioned in Theorem 2.8(1).

It remains to prove $P_{\mathrm{hyp}}(f,t)\le\widehat{P}^0_{\mathrm{tree}}(f,t,z)$. For this, consider an arbitrary hyperbolic $X\subset J(f)$. It is enough to consider a non-exceptional point $z\in X$ (or just a preimage under an iterate of f, arbitrarily close to X, of an priori given point z). We prove $P_{\mathrm{tree}}(f|_X,t,z)\le\widehat{P}^0_{\mathrm{tree}}(f,t,z)$.

It is a repetition of the proof of $\widehat{P}_{\mathrm{McM}}(f,t) \le \widehat{P}^0_{\mathrm{McM}}(f,t)$ in Theorem 4.4. For $\varDelta = \operatorname{dist}(X, \operatorname{Crit}(f))$ we use $\delta = o(\varDelta)$ to ensure that for $x \in X$ and $y \in B(x,\delta)$, with x, y close to a critical point $c \in \mathbb{C}$ for f with multiplicity ν, the ratio

$$|f'(x)|/|f'(y)| \le \mathrm{const} \cdot \big(|x-c|/|y-c|)\big)^{\nu-1} \le 1 + \mathrm{const} \cdot \left(\frac{\delta}{\varDelta}\right)^{\nu-1}$$

is close to 1 (cf. (4.8); for more details see (4.9)). In other words, the difference of potentials $-t\log|f'(x)| - (-t\log|f'(y)|)$ is small. ■

Remark 5.2. Introducing $\widehat{P}^0_{\mathrm{tree}}$ between P_{hyp} and P^0_{tree} shows directly how to omit the problem for individual backward trajectories; see the proof of Theorem 2.8(1) in Section 3, Remark 3.4, and compare Remark 4.6.

6. Final remarks, more geometric pressures and examples

6.1. On convergence

Remark 6.1. It is clear that the sequence of functions $t \mapsto \widehat{P}^{\varDelta}_{\mathrm{tree}}(f,t,z)$ in (5.1) converges uniformly (locally) as $\varDelta \to 0$ (cf. Section 5) and the limit $P(f,t)$ is non-increasing, e.g. by the definition of $P_{\mathrm{hyp}}(f,t)$). So calculating these functions and their first zeros we obtain as the limit the first zero of $P(f,t)$, which is $\mathrm{HD}_{\mathrm{hyp}}(J(f))$ (see Proposition 2.7). Unfortunately, we do not know the speed of convergence for general f. For some classes of maps f the situation is better, e.g. for topological Collet–Eckmann maps, see Remark 6.3.

Remark 6.2. It might be worthy to use instead the functions

$$t \mapsto \log \lambda(\widehat{\mathscr{R}^N(T)^t}),$$

as $N \to \infty$, and their zeros.

Note that the zeros can be calculated as solutions of the equation $\lambda(\varLambda^t) = 1$ if all the entries of a primitive matrix $\varLambda$ are non-negative, here for $\varLambda = \widehat{\mathscr{R}^N(T)}$ [McM, Practical considerations]. See also Remark 4.7.

Remark 6.3. It may happen that $\mathrm{HD}_{\mathrm{hyp}}(J(f)) < \mathrm{HD}(J(f))$ [Lyu, Section 2.13.2], so the methods here are unsuitable to estimate $\mathrm{HD}(J(f))$, unless e.g. f is *topological Collet–Eckmann* (see e.g. [P3]), where $P(f,t)$ has only one zero, say t_0, and $\mathrm{HD}_{\mathrm{hyp}}(J(f)) = \mathrm{HD}(J(f))$.

Notice that in this case $(dP/dt)(t_0) < 0$ so the convergence of approximations, say $\mathrm{HD}(X_N) \to t_0$, is faster than if the left derivative of $t \mapsto P(f,t)$ at t_0 were 0.

Conclusions and comments. As noted in Section 1, our aim is to approximate the geometric pressure $P(f,t)$ from below by quantities depending on a parameter δ for tree pressures, or on N for McMullen's pressures. If approximating quantities exceed $P(f,t)$, and if we do not know how far they are from the limit, we do not know how big an error might be in our estimates of $P(f,t)$ from below. In these estimates we use $P(f,t) = P_{\mathrm{tree}}(f,t,z)$. To be on the safe side we want also the numbers under the $\limsup_n$ (see (2.6)) to be as small as possible. To this end we can choose any z close to $J(f)$ but outside it (cf. [DGT]), hence not only non-exceptional, but not accumulated by forward trajectories of critical points at all. Notice that if z_1, z_2 are like z and belong to the same component B of the Fatou set, then all the ratios $\Pi_n(t,v_1)/\Pi_n(t,v_2)$ for corresponding $v_i \in f^{-n}(z_i)$, $i = 1,2$ are uniformly bounded (corresponding in the sense that for a curve γ joining z_1 to z_2 in $B \setminus \mathrm{PC}(f)$ the points v_1 and v_2 are the end points of a lift of γ for f^n). This happens for polynomials, where both z_i belong to the basin of ∞.

As noted in Section 1, among the notions of geometric pressures we introduced to calculate $\mathrm{HD}(J(f))$, or approximate it from below, the infimum (fuzzy) pressures and/or restricted pressures are appropriate, in particular $\widehat{P}^0_{\mathrm{McM}}(f,t)$, $P^0_{\mathrm{McM}}(f,t)$, $P^0_{\mathrm{multMcM}}(f,t)$, $\widehat{P}^0_{\mathrm{tree}}(f,t,z)$ and $P^0_{\mathrm{tree}}(f,t,z)$ might turn out to be useful, since elements of the sequences defining them, depending on δ or N, do not exceed $P(f,t)$.

The pressure $\widehat{P}_{\mathrm{McM}}(f,t)$ is "almost" increasing, because of bounded distortion in the puzzle pieces satisfying (4.5). This distortion, responsible for possible decreasing, shrinks to 0 as $N \to \infty$ with speed depending on $A(N)$.

Some notions may increase the speed of approximation, but may lead to results exceeding $P(f,t)$, as it may happen with $P_{\mathrm{McM}}(f,t)$ (see (4.3)). This is so because of the use of distinguished points where $|f'|$ can be too small (its inverse too large). See Remark 4.6. On the other hand, considering $P^{\mathrm{pullinf}}_{\mathrm{tree}}(f,t)$ requires finding infima in sets shrinking with the number of iterations, which might be computationally awkward.

6.2. Double sampling pressures. A remedy to avoid the quantities exceeding $P(f,t)$ and an excessive complexity of calculations would be something in between, e.g. double (or multiple) sampling variants.

Definition 6.4. Define the *double sampling tree pressure* $P^*_{\mathrm{tree}}(f,t,z)$ similarly to $P^0_{\mathrm{tree}}(f,t,z)$ but replacing infima by minima over two points:

$$P^*_{\mathrm{tree}}(f,t,z) := \limsup_{\delta\to 0} P^{*,\delta}_{\mathrm{tree}}(f,t,z), \quad \text{where} \tag{6.1}$$

$$P^{*,\delta}_{\mathrm{tree}}(f,t,z) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{v\in f^{-n}(z)} \Pi^{*\delta}_n(v),$$

where

$$\Pi^{*\delta}_n(v) := \prod_{k=1}^{n} \min(|f'(v_{k,1})|^{-t}, |f'(v_{k,2})|^{-t}),$$

where $v_{k,1}, v_{k,2} \in B(f^{n-k}(v), \delta)$ are symmetric to each other with respect to $v_k = f^{n-k}(v)$ and $\mathrm{dist}(v_{k,1}, v_{k,2}) \geq \delta$. Compare (5.3). Thus in place of one distinguished point in $B(v_k, \delta)$, we choose two.

If there is a critical point c close to v_k, then at least one $v_{k,i}$ is farther away from c than v_k. Even $|f'(v_{k,i})| \geq |f'(v_k)|$, hence $|f'(v_{k,i})|^{-t} \leq f'(v_k)|^{-t}$. At other points v_k, for which $\delta = o(\mathrm{dist}(v_k, \mathrm{Crit}(f)))$, this inequality holds up to a factor $1 + \epsilon$, where $\epsilon = O(\delta)$. Then $|f'(v_{k,i})|^{-t}/|f'(v_k)|^{-t} \approx 1$ for $i = 1, 2$. So

$$\frac{\Pi_n^{*\delta}(v)}{|(f^n)'(v)|^{-t}} \leq 1 + \epsilon. \tag{6.2}$$

So, taking $n \to \infty$, then summing over v and letting $\delta \to 0$ we obtain

Proposition 6.5. *For every non-exceptional z,*

$$P_{\mathrm{hyp}}(f, t) \leq P^*_{\mathrm{tree}}(f, t, z) \leq P_{\mathrm{tree}}(f, t, z).$$

Unfortunately, we cannot prove the monotonicity of $P^{*,\delta}_{\mathrm{tree}}(f, t, z)$ as $\delta \to 0$, nor $P^{*,\delta}_{\mathrm{tree}}(f, t, z) \leq P^*_{\mathrm{tree}}(f, t, z)$. For v_k close to a critical point c, if $\mathrm{dist}(v_k, c) = C\delta$, we have

$$\mathrm{dist}(v_{k,i}, c) \geq \delta\sqrt{C^2 + 1/2} = \mathrm{dist}(v_k, c)\frac{\sqrt{C^2 + 1/4}}{C}$$

for $i = 1$ or 2. Hence

$$\frac{|f'(v_{k,i})|^{-t}}{f'(v_k)|^{-t}} \leq \left(\frac{C}{\sqrt{C^2 + 1/4}}\right)^{t\nu(c)} (1 + O(\delta)).$$

However, we cannot achieve this far away from $\mathrm{Crit}(f)$. A remedy would be to consider *triple sampling tree pressures*, with $v_{k,i}$, $i = 1, 2, 3$, at the vertices of an equilateral triangle centered at v_k. Then f' in a small neighbourhood of v_k is almost affine, so the inequality $|f'(v_{k,i})| \geq f'(v_k)|$ holds for some i provided there is no point x_0 where $f''(x) = 0$. If the latter occurs, assume $f'(x) = a(x - x_0)^{m(x)} + \cdots$ for $a \neq 0$ and an integer $m(x) > 1$. To cope with this case, consider *m-sampling tree pressure*, with $v_{k,i}$, $i = 1, \ldots, m$, at the vertices of a regular m-gon centered at v_i. Then, for $m = 3\max\{m(x) : f''(x) = 0\}$, there exists i such that $|f'(v_{k,i})| \geq |f'(v_k)|$. So we can skip ϵ in (6.2) and then conclude with $P^{*,\delta}_{\mathrm{tree}}(f, t, z) \leq P(f, t)$ for each δ for m-sampling tree pressure.

Definition 6.6 (Double sampling McMullen pressure). Similarly we define the *double sampling McMullen pressure* $P^*_{\mathrm{McM}}(f, t)$. For $P = P_{N,i}$ for which $\mathrm{Int}\, P_{N,i} \cap f^{-1}(\mathrm{Int}\, P_{N,j}) \neq \emptyset$ we consider two points $v_{P,1}, v_{P,2} \in B(P, r_{N,P})$ where $r_{N,P} := A(N)\,\mathrm{diam}\, P$ for an arbitrary sequence $A(N) \to \infty$ as $N \to \infty$, but $A(N)\,\mathrm{diam}\, \mathscr{R}^N(\mathscr{P}) \to 0$ (cf. (4.5)). The points $v_{P,1}, v_{P,2}$ are chosen symmetric with respect to an arbitrary point z^*_P in P

and $\mathrm{dist}(v_{P,1}, v_{P,2}) \approx r_{N,P}$ (i.e. far from P compared to its diameter). We distinguish such a pair only for P such that $\mathrm{dist}(P, \mathrm{Crit}(f)) \le r_{N,P}$. In this case we need to do so because of the arbitrariness of the choices of z_P^*.

Now define the matrices $\mathscr{R}^N(T)^*$ by changing in $\mathscr{R}^N(T)$ defined at the beginning of Section 4 the entries $|f'(y_{N,i})|^{-1}$ with distinguished points $y_{N,i}$ to $a_{ij} = \min(|f'(v_{P,1})|^{-1}, |f'(v_{P,2})|^{-1})$ for P close to $\mathrm{Crit}(f)$ as above. Finally, define

$$P^*_{\mathrm{McM}}(f,t) = \lim_{N\to\infty} \log \lambda((\mathscr{R}^N(T)^*)^t). \tag{6.3}$$

Consider now an arbitrary non-exceptional $z \in J(f)$ not belonging to the boundary of any puzzle piece of any generation. Then for each $z_k \in f^{-k}(z) \in P_{N,i}$, we have $a_{ij}^t \le |f'(z_k)|^{-t}(1+\epsilon)$, where $\epsilon \to 0$ as $N \to \infty$ (cf. (6.2)). So indeed we have

Proposition 6.7. $P_{\mathrm{hyp}}(f,t) \le P^*_{\mathrm{McM}}(f,t) \le P_{\mathrm{tree}}(f,t,z) = P_{\mathrm{tree}}(f,t)$.

Definition 6.8. In fact, we could distinguish $v_{P,1}, v_{P,2} \in \mathrm{Int}\, P$. Indeed, assume $d := \mathrm{dist}(v_{P,1}, v_{P,2}) \ge \frac{1}{2} \mathrm{diam}\, P$. So $\mathrm{dist}(z_k, v_{P,\iota}) \le 2d$ for $z_k \in P$ for both $\iota = 1$ and $\iota = 2$. But $\mathrm{dist}(c, v_{P,\iota}) \ge d/2$ for $\iota = 1$ or $\iota = 2$ for each point c close to P, in particular a critical one. So for such ι we get due to the triangle inequality, skipping indices, $\mathrm{dist}(c,z) \le \mathrm{dist}(c,v) + \mathrm{dist}(v,z)$, hence

$$\frac{\mathrm{dist}(c,z)}{\mathrm{dist}(c,v)} \le 1 + \frac{\mathrm{dist}(v,z)}{\mathrm{dist}(c,v)} \le 1 + \frac{2d}{d/2} \le 5.$$

So, for ν denoting the multiplicity of f at c,

$$\frac{|f'(z_k)|^{-1}}{|f'(v_{P,\iota})|^{-1}} \ge \mathrm{const}\cdot 5^{-(\nu-1)}, \quad \text{hence} \quad |f'(v_{P,\iota})|^{-1} \le \mathrm{const}^{-1} 5^{\nu-1} |f'(z_k)|^{-1}.$$

Since this happens rarely with N large the constant $\mathrm{const}^{-1}5^{\nu-1}$ does not matter.

Note that the shape of P can be very distorted, making finding $v_{P,1}, v_{P,2} \in \mathrm{Int}\, P$ difficult. So instead we can consider taking $v_{P,1}, v_{P,2} \in B(P, \mathrm{diam}\, P)$.

Unfortunately, these constructions allow one to prove neither monotonicity nor that the elements of the sequence in (6.3) do not exceed $P(f,t)$ (though discrepancies seem low), unless we modify the definition to an *m-sampling McM-pressure*, as in the tree pressure case in Definition 6.4.

6.3. Examples

Example 6.9. For each non-renormalizable polynomial f, say with connected Julia set and all periodic orbits repelling, one considers Yoccoz's puzzle construction, namely a covering $\mathscr{P}$ of a neighbourhood of $J(f)$ whose pieces have boundaries consisting of equipotential lines for the Green's function in the basin of ∞ and closures of external rays to fixed points dissecting

$J(f)$. Assume these points are not post-critical. The diameters of the consecutive pullbacks of these pieces shrink uniformly to 0, so the assumptions of Theorem 2.8(2) are satisfied. See e.g. [KvS] for this and more general cases.

REMARK 6.10. There is no need in the definition of McMullen's pressures that the consecutive puzzle structures are of the form $\mathscr{R}^N(\mathscr{P})$. One can just take any sequence $\mathscr{P}_N$ of puzzle structure coverings such that the diameters of their elements tend uniformly to 0. Then in Example 1 one may allow the infinitely renormalizable case, with the so-called *a priori complex bounds* condition, guaranteeing the existence of such a sequence (see [Lyu] and references therein).

EXAMPLE 6.11. For the Feigenbaum map $f_{\mathrm{Feig}}(z) = z^2 + c_{\mathrm{Feig}}$ where $c \approx -1.40155$, infinitely renormalizable, where c_{Feig} is the limit of the decreasing sequence of the period doubling real parameters, a different puzzle structure is used (see [DS] and references therein). The critical point 0 is in the boundary of four first generation puzzle pieces adjacent to it, so all restrictions of f_{Feig} to interiors of all generations puzzle pieces are injective, and $s(i,j) = 0$ or 1 (see beginning of Section 4). In [DGT] it is announced for $f = f_{\mathrm{Feig}}$ that $\mathrm{HD}_{\mathrm{hyp}}(J(f)) = \lim_{N\to\infty} \delta_N$, where δ_N is the first zero t of $\log \lambda^{\inf}_{N,t}$ (see Definition 4.2), is equal to $\delta_{\mathrm{cr}}(f)$ denoting the critical exponent of the Poincaré series, equal to the Minkowski dimension (box dimension) of $J(f)$ provided the area of $J(f)$ is zero (which is the case by [DS] for f_{Feig}); for this see [Bish]. (There, Whitney's critical exponent appears, but it is straightforward that it is the Minkowski dimension of $J(f)$, by Koebe's 1/4 lemma. In [Bish], actions by Kleinian groups were considered.)

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