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Expanding repellers in limit sets for iteration of holomorphic functions

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Abstract. We prove that for Ω being an immediate basin of attraction to an attracting fixed point for a rational mapping of the Riemann sphere, and for an ergodic invariant measure μ on the boundary $\mathrm{Fr}\Omega$, with positive Lyapunov exponent, there is an invariant subset of $\mathrm{Fr}\Omega$, being an expanding repeller, of Hausdorff dimension arbitrarily close to Hausdorff dimension of μ. We prove also generalizations and geometric coding tree abstract version. The paper is a continuation of a paper in Fund. Math. (1994) by the author and Anna Zdunik, where the density of periodic orbits in $\mathrm{Fr}\Omega$ was proved.

Section 1. Introduction

Let Ω be a simply connected domain in $\bar{\mathbb{C}}$ and f be a holomorphic map defined on a neighbourhood W of $\mathrm{Fr}\Omega$ to $\bar{\mathbb{C}}$. Assume $f(W \cap \Omega) \subset \Omega$, $f(\mathrm{Fr}\Omega) \subset \mathrm{Fr}\Omega$ and $\mathrm{Fr}\Omega$ repells to the side of Ω, that is $\bigcap_{n=0}^{\infty} f^{-n}(W \cap \bar{\Omega}) = \mathrm{Fr}\Omega$. An important special case is where Ω is an immediate basin of attraction of an attracting fixed point for a rational function. This concerns also the case of a component of the immediate basin of attraction to a periodic attracting orbit, as one can consider an iterate of f mapping the component to itself. Distances and derivatives are considered in the Riemann spherical metric on $\bar{\mathbb{C}}$).

Let $R : \mathbb{D} \to \Omega$ be a Riemann mapping from the unit disc to Ω and let g be a holomorphic extension of $R^{-1} \circ f \circ R$ to a neighbourhood of the unit circle $\partial\mathbb{D}$. It exists and it is expanding on $\partial\mathbb{D}$, see [P2, Section 7]. We prove the following

Theorem A. Let ν be an ergodic g-invariant probability measure on $\partial\mathbb{D}$, such that for ν-a.e.$\zeta \in \partial\mathbb{D}$ there exists a radial limit $\hat{R}(\zeta) := \lim_{r \nearrow 1} R(r\zeta)$. Assume that the measure $\mu := \hat{R}_*(\nu)$ has positive Lyapunov exponent $\chi_\mu(f)$. Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a continuous real-valued function.

Then for every $\varepsilon > 0$ there exist $Y \subset \partial\mathbb{D}$ a g-invariant expanding repeller and $C > 0$ such that for all positive integers n and all $\zeta \in Y$

(i) $-\ln C + n(\int \varphi \, d\nu - \varepsilon) \le \sum_{j=0}^{n-1} \varphi(g^j(\zeta)) \le \ln C + n(\int \varphi \, d\nu + \varepsilon)$.

(ii) $X = \hat{R}(Y)$ is an expanding repeller for f contained in $\mathrm{Fr}\Omega$ and $R(r\zeta) \to \hat{R}(\zeta)$ uniformly as $r \nearrow 1$ for $\zeta \in Y$. The mapping $\hat{R}$ is finite-to-one on Y. Both Y and X are Cantor sets.

(iii) $C^{-1} \exp n(\chi_\mu(f) - \varepsilon) \le |(f^n)'(\hat{R}(\zeta))| \le C \exp n(\chi_\mu(f) + \varepsilon)$.

(iv) $\mathrm{HD}(X) \ge \mathrm{HD}(\mu) - \varepsilon$.

The existence of X satisfying (iii) and (iv) holds without the assumption Ω is simply connected.

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Above, X being an *expanding repeller* for f means that $f(X) \subset X$ the map f restricted to X is open, topologically mixing and expanding, that is there exist $C > 0$ and $\lambda > 1$, called *an expanding constant*, such that for every $x \in X$ we have $|(f^n)'(x)| \geq C\lambda^n$. The property that $f|_X$ is open is equivalent to the existence of a neighbourhood U of X in $\mathbb{C}$, called a *repelling neighbourhood*, such that every forward f-trajectory $x, f(x), ...f^n(x), ...$ staying in U must be contained in X. The definition of an expanding repeller $Y \subset \partial\mathbb{D}$ for g is similar. $\mathrm{HD}(X)$ denotes Hausdorff dimension of the set X, $\mathrm{HD}(\mu)$ denotes Hausdorff dimension of the measure μ which is defined as the infimum of Hausdorff dimensions of sets of full measure μ.

The property (iv) is a version of the fact that hyperbolic Hausdorff dimension of Julia set $J(f)$ for a rational mapping (= supremum of Hausdorff dimensions of expanding repellers contained in $J(f)$) is equal to the hyperbolic dynamical dimension (= supremum of Hausdorff dimensions of invariant probability measures on $J(f)$ of positive Lyapunov exponents), see for example [PU].

Theorem A, if we add the property (v') below, extends the main theorem from the paper [PZ], where the density of periodic orbits in $\partial\Omega$ was proved. The idea of the proof, as in [PZ], is to apply Pesin and Katok theories approach, see [HK, Suplement] for a general theory and [PU, Ch.9] for its adaptation in holomorphic iteration. The problem is, as in [PZ], that a standard Katok method to produce a large hyperbolic (here expanding) set does not guarantee that the set is in $\partial\Omega$. It does not give the set Y neither.

We needed this theorem in [P3], applied to $\varphi = \ln|g'|$ and μ in the harmonic measure class, but it is of an independent interest, so we have decided to put it in a separate paper.

Section 2. Additional properties

The following additional properties of suitably constructed X in Theorem A will be proved:

(v) X can be arbitrarily close to the topological support $\mathrm{supp}(\mu)$ in the Hausdorff metric in the space of compact subsets of $\mathrm{Fr}\Omega$.

(vi) For any finite families of real-valued continuous functions $\varphi_1, ..., \varphi_{k'}$ on $\partial\mathbb{D}$ for every $i = 1, ..., k$ and $\psi_1, ..., \psi_{k'}$ on $\partial\Omega$ for every $i = 1, ..., k'$

$$-\ln C + n\Big(\int_{\partial\mathbb{D}} \varphi_i \, d\nu - \varepsilon\Big) \leq \sum_{j=0}^{n-1} \varphi_i(g^j(\zeta)) \leq \ln C + n\Big(\int_{\partial\mathbb{D}} \varphi_i \, d\nu + \varepsilon\Big),$$

$$-\ln C + n\Big(\int_{\mathrm{Fr}\Omega} \psi_i \, d\mu - \varepsilon\Big) \leq \sum_{j=0}^{n-1} \psi_i(f^j(\zeta)) \leq \ln C + n\Big(\int_{\mathrm{Fr}\Omega} \psi_i \, d\mu + \varepsilon\Big)$$

and

(vii) $\mathrm{P}(f|_X, \psi_i) \geq \mathrm{h}_\mu(f) + \int_{\mathrm{Fr}\Omega} \psi_i \, d\mu - \varepsilon$,
$\mathrm{P}(g|_Y, \varphi_i) \geq \mathrm{h}_\nu(g) + \int_{\partial\mathbb{D}} \varphi_i \, d\nu - \varepsilon$,

in particular

(viii) $\mathrm{h_{top}}(f|_X) \geq \mathrm{h}_\mu(f) - \varepsilon$ and
$\mathrm{h_{top}}(g|_Y) \geq \mathrm{h}_\nu(g) - \varepsilon$,

for P topological pressure, $\mathrm{h_{top}}$ topological entropy. Finally, as in (iv),

(ix) $\mathrm{HD}(Y) \geq \mathrm{HD}(\nu) - \varepsilon$.

Remark 1. The property (v) implies

(v') If $\mathrm{supp}\mu = \partial\Omega$ then X is arbitrarily close to $\mathrm{Fr}\Omega$ in the Hausdorff metric.

The assumption $\mathrm{supp}\mu = \partial\Omega$ holds for every $\mu = \hat{R}_*(\nu)$ for ν being a g-invariant Gibbs state (measure) for a Hölder continuous function on $\partial\mathbb{D}$, see [PZ]. In this case ν has positive entropy, hence the existence of radial limit ν-a.e. holds automatically, see [PZ] and references there, in particular [P1]. This automatically implies $\chi_\mu(f) > 0$, since $0 < \mathrm{h}_\nu(g) = \mathrm{h}_\mu(f) \leq 2\chi_\mu(f)$, (Ruelle inequality).

Remark 2. The radial convergence in (ii) automatically implies the non-tangential convergence. This means the following: For every $\zeta \in \partial\mathbb{D}$, $\vartheta : 0 < \vartheta < \pi/2$ and $t > 0$ denote

$$S_{\vartheta,t}(\zeta) = \zeta \cdot (1 + \{x \in \mathbb{C} \setminus \{0\} : \pi - \vartheta \leq \mathrm{Arg}(x) \leq \pi + \vartheta\}, |x| < t).$$

Such a set is called Stolz angle. If we do not mind what is t we skip it and write S_ϑ. Now (ii) writes as

(ii') For every $\vartheta : 0 < \vartheta < \pi/2$ the convergence $R(z) \to \hat{R}(\zeta)$ is uniform for $\zeta \in X$ as $z \to \zeta$ and $z \in S_\vartheta$. The rate of convergence is exponential, more precisely there exists $C > 0$ such that

$$C^{-1}(1-r)^{\chi_\mu(f)/(\chi_\nu(g)-\varepsilon)} \leq \mathrm{dist}(R(r\zeta), \hat{R}(\zeta)) \leq C(1-r)^{\chi_\nu(g)/(\chi_\mu(f)+\varepsilon)}.$$

Section 3. Geometric coding tree version

As in [PZ] we prove a more general, abstract version of these results, in the language of a geometric coding tree. We remind the definitions and notation:

Let U be an open connected subset of the Riemann sphere $\overline{\mathbb{C}}$. Consider any holomorphic mapping $f : U \to \overline{\mathbb{C}}$ such that $f(U) \supset U$ and $f : U \to f(U)$ is a proper map. Denote $\mathrm{Crit}(f) = \{z : f'(z) = 0\}$. This is called the set of critical points for f. Suppose that $\mathrm{Crit}(f)$ is finite. Consider any $z \in f(U)$. Let $z^1, z^2, ..., z^d$ be some of the f-preimages of z in U where $d \geq 2$. Consider smooth curves $\gamma^j : [0,1] \to f(U)$, $j = 1, ..., d$, joining z to z^j respectively (i.e. $\gamma^j(0) = z, \gamma^j(1) = z^j$), such that there are no critical values for iterates of f in $\bigcup_{j=1}^d \gamma^j$, i.e. $\gamma^j \cap f^n(\mathrm{Crit}(f)) = \emptyset$ for every j and $n > 0$.

Let $\Sigma^d := \{1, ..., d\}^{\mathbb{Z}^+}$ denote the one-sided shift space and σ the shift to the left, i.e. $\sigma((\alpha_n)) = (\alpha_{n+1})$. For every sequence $\alpha = (\alpha_n)_{n=0}^\infty \in \Sigma^d$ we define $\gamma_0(\alpha) := \gamma^{\alpha_0}$. Suppose that for some $n \geq 0$, for every $0 \leq m \leq n$, and all $\alpha \in \Sigma^d$, the curves $\gamma_m(\alpha)$

are already defined. Suppose that for $1 \le m \le n$ we have $f \circ \gamma_m(\alpha) = \gamma_{m-1}(\sigma(\alpha))$, and $\gamma_m(\alpha)(0) = \gamma_{m-1}(\alpha)(1)$.

Define the curves $\gamma_{n+1}(\alpha)$ so that the previous equalities hold by taking respective f-preimages of curves γ_n. For every $\alpha \in \Sigma^d$ and $n \ge 0$ denote $z_n(\alpha) := \gamma_n(\alpha)(1)$. Note that $z_n(\alpha)$ and $\gamma_n(\alpha)$ depend only on $(\alpha_0, ..., \alpha_n)$ so sometimes we consider z_n and γ_n as functions on blocks of symbols of length $n+1$. Sometimes it is comfortable to denote z by z_{-1}.

The graph $\mathcal{T}(z, \gamma^1, ..., \gamma^d)$ with the vertices z and $z_n(\alpha)$ and edges $\gamma_n(\alpha)$ is called a *geometric coding tree* with the root at z. For every $\alpha \in \Sigma^d$ the subgraph composed of $z, z_n(\alpha)$ and $\gamma_n(\alpha)$ for all $n \ge 0$ is called a *geometric branch* and denoted by $b(\alpha)$.

For each $j = 1, ..., d$ we define f_j^{-1} on a small neighbourhood of z as the branch of f^{-1} mapping z to z^j. For each $\alpha \in \Sigma^d$ the branch f_j^{-1} has an analytic continuation $f_{j,\alpha}^{-1}$ along the curve $b(\alpha)$. Note that by the construction $f_{j,\alpha}^{-1}(b(\alpha)) = b(j\alpha)$, where $j\alpha$ is the concatenation of the symbol j and the sequence α. By induction, for any w being a block of k symbols in $\{1, ..., d\}$, for f_w^{-k} being the branch of f^{-k} mapping z to $z_{k-1}(w)$ and for $f_{w,\alpha}^{-k}$ being the analytic continuation along $b(\alpha)$, we get

$$f_{w,\alpha}^{-k}(b(\alpha)) = b(w\alpha). \tag{1}$$

Similar notation is used and properties hold for finite sequences α, where for $\alpha = (\alpha_0, ..., \alpha_n)$, $b(\alpha)$ is then the path in $\mathcal{T}$ from z to $z_n(\alpha)$.

For infinite α the branch $b(\alpha)$ is called *convergent* if the sequence $\gamma_n(\alpha)$ is convergent to a point in cl$U$. We define the *coding map* $z_\infty : \mathcal{D}(z_\infty) \to \mathrm{cl}U$ by $z_\infty(\alpha) := \lim_{n\to\infty} z_n(\alpha)$ on the domain $\mathcal{D} = \mathcal{D}(z_\infty)$ of all such α's for which $b(\alpha)$ is convergent.

For each geometric branch $b(\alpha)$ denote by $b_m(\alpha)$ the part of $b(\alpha)$ starting from $z_m(\alpha)$ i.e. consisting of the vertices $z_k(\alpha), k \ge m$ and of the edges $\gamma_k(\alpha), k > m$.

If the map f extends holomorphically to a neighbourhood of the closure of the limit set Λ of a geometric coding tree , $\Lambda = z_\infty(\mathcal{D}(z_\infty))$, then Λ is called a *quasi-repeller*, see [PUZ].

Theorem B. Let Λ be a quasi-repeller for a geometric coding tree $\mathcal{T}(z, \gamma^1, ..., \gamma^d)$ for a holomorphic map $f : U \to \overline{\mathbb{C}}$. Let ν be an ergodic σ-invariant probability measure on Σ^d such that for ν-a.e.$\alpha \in \Sigma^d$ there exists a limit $z_\infty(\alpha)$. Assume that the measure $\mu := \hat{R}_*(\nu)$ has positive Lyapunov exponent $\chi_\mu(f)$. Let φ be a continuous real-valued function on Σ^d. Then all the properties (i)-(ix) hold, with $\hat{R} : \partial \mathbb{D} \to \mathrm{Fr}\Omega$ replaced by $z_\infty : \Sigma^d \to \Lambda$ defined ν-a.e. and $R(r\zeta) \to \hat{R}(\zeta)$ replaced by $\gamma_n(\alpha) \to z_\infty(\alpha)$ as $n \to \infty$.

The assumption that $z_\infty(\alpha)$ exists for ν-a.e. $\alpha \in \Sigma^d$, i.e. $\nu(\mathcal{D}) = 1$, holds for every ν of positive entropy, compare Remark 1, see [PZ, Convergence Theorem], where further references are given. As in the Riemann mapping case, $\chi_\mu(f) > 0$ holds then automatically.

In the setting of Theorem B the property (v') also holds, with $\mathrm{Fr}\Omega$ replaced by clΛ, what immediately follows from (v).

Its assumption $\mathrm{supp}\mu = \mathrm{cl}\Lambda$ holds under the assumptions that ν is a σ-invariant Gibbs state for a Hölder continuous function on Σ^d, compare Remark 1, and additionally the tree $\mathcal{T}$ satisfies $\gamma^j \cap \mathrm{cl}(\bigcup_{n\geq 0} f^n(\mathrm{Crit} f)) = \emptyset$ for all $j = 1, ..., d$ and there exists a neighbourhood $U^j \subset f(U)$ of γ^j such that $\mathrm{Vol}(f^{-n}(U^j)) \to 0$, where Vol denotes the standard Riemann measure on $\overline{\mathbb{C}}$.

For the proof see [PZ, Lemma 3], where $\mathrm{cl}\Lambda$ is replaced by a formally larger set $\hat{\Lambda} := \{$all limit points of the sequences $z_n(\alpha^n), \alpha^n \in \Sigma^d, n \to \infty\}$. It is easy to see that the above conditions about the tree $\mathcal{T}$ hold if $\mathcal{T}$ is in $W \cap \Omega$, close enough to $\mathrm{Fr}\Omega$, as in the situation of Theorem A, see Section 4.

Section 4. Proof of Theorem B

Step 1. Good backward branches and their number

Denote the natural extension of the one-sided shift with the measure ν and Borel σ-algebra σ, i.e. the corresponding two-sided shift, by $(\tilde{\Sigma}^d, \tilde{\sigma}, \tilde{\nu})$. Denote the projection $\tilde{\Sigma}^d \to \Sigma^d$ mapping α to $(\alpha_0, \alpha_1, ...)$ by π_+. For each $\alpha \in \tilde{\Sigma}^d$ denote $\pi_+(\alpha)$ by α^+.

By Pesin theory, see [PZ, Lemma 1] for the version which we apply and by Birkhoff Ergodic Theorem applied to φ, for every $\varepsilon > 0$ we can find a set $K \subset \tilde{\Sigma}^d$, constants $C, \delta > 0$ and a positive integer M such that $\tilde{\nu}(K) > 1 - \varepsilon$ and for all $\alpha \in K$ and $n \geq 0$

B(i) $\quad -\ln C + n(\int \varphi \, d\nu - \varepsilon/2) \leq \sum_{j=0}^{n-1} \varphi(\sigma^j(\alpha^+)) \leq \ln C + n(\int \varphi \, d\nu + \varepsilon/2)$.

B(ii) $\quad b_M(\alpha^+) \subset B(z_\infty(\alpha^+), \delta/3)$.

B(iii) There exist univalent branches f_α^{-n} of f^{-n} on $B(z_\infty(\alpha^+), \delta)$ for all $n = 1, 2, ...$ mapping $z_\infty(\alpha^+)$ to $z_\infty(\tilde{\sigma}^{-n}(\alpha)^+)$.

In the notation accompanying the property (1) these branches are the continuations along $b(\alpha^+)$ of $f^{-n}_{(\alpha_{-n}, \ldots, \alpha_{-1})}$, i.e. the branches $f^{-n}_{(\alpha_{-n}, \ldots, \alpha_{-1}), \alpha^+}$.

Moreover

B(iv) $\quad C^{-1} \exp n(\chi_\mu(f) - \varepsilon/2) \leq |(f^n)'(z_\infty(\tilde{\sigma}^{-n}(\alpha)^+)| \leq C \exp n(\chi_\mu(f) + \varepsilon/2)$.

and

B(v) $\quad \frac{|(f_\alpha^{-n})'(x)|}{|(f_\alpha^{-n})'(y)|} < C \;$ for all $x, y \in B(z_\infty(\alpha^+), \delta)$.

For any $-\infty \leq r \leq s \leq \infty$, for every $\alpha \in \tilde{\Sigma}^d$ or $\alpha \in \Sigma_{r,s} = \{1, ..., d\}^{\{r, r+1, ..., s\}}$ we denote by $C_{r,s}(\alpha)$ the cylinder $\{w \in \tilde{\Sigma}^d : w_j = \alpha_j, \;\; \forall j : r \leq j \leq s\}$. The projection $\tilde{\Sigma}^d \ni (...\alpha_j, ...) \mapsto (\alpha_r, ..., \alpha_s) \ni \Sigma_{r,s}$ will be denoted by $\pi_{r,s}$. Note that $C_{r,s}(\alpha) = \pi_{r,s}^{-1}\pi_{r,s}(\alpha)$.

Choose an arbitrary cylinder $C_M := C_{0,M}(\beta)$ for a fixed sequence $\beta = (\beta_0, ..., \beta_M) \in \Sigma_M := \Sigma_{0,M}$, such that $\tilde{\nu}(C_M \cap K) \geq \tilde{\nu}(C_M)/2$, which is possible provided $\varepsilon \leq 1/2$.

Denote $C_M \cap K$ by K'. For all $n \geq 0$ consider $K_n := \tilde{\sigma}^{-n}(K')$. By the invariance of $\tilde{\nu}$ we have $\tilde{\nu}(K_n) \geq \tilde{\nu}(C_M)/2 := \xi$.

By Birkhoff Ergodic Theorem there exists $N \geq 0$ such that

$$\nu(\{\alpha \in K_n : \exists i : 0 \leq i \leq N, \tilde{\sigma}^{-i}(\alpha) \in K'\} \geq \xi/2.$$

Therefore for every $n \geq 0$ there exists $N' : 0 \leq N' \leq N$ such that, denoting $n' := n + N'$, for $A(n') := \{\alpha \in K' : \tilde{\sigma}^{-n'}(\alpha) \in K'\}$ we have

$$\tilde{\nu}(A(n')) \geq \xi/2N. \tag{2}$$

For every $\alpha \in A(n')$ we obtain $b_M(\tilde{\sigma}^{-n'}(\alpha)^+) \subset B(z_M(\alpha^+), \delta/3)$. Indeed, for $\alpha' = \sigma^{-n'}(\alpha)$ we have $\pi_{0,M}(\alpha') = \beta$, as we have landed with α' in C_M. The length of $b_M(\alpha'^+)$ is at most $\delta/3$ as $\alpha' \in K$.

Hence

$$f_\alpha^{-n'} \mathrm{cl}(B(z_M(\beta), \frac{2}{3}\delta) \subset B(z_M(\beta), \frac{2}{3}\delta) \tag{3}$$

for all n large enough, more precisely for n such that

$$|(f_\alpha^{-n'})'(x)| < 1/2 \;\text{ for all }\; x \in B(z_M(\beta), \frac{2}{3}\delta). \tag{4}$$

By B(ii)-B(iv) this holds for $n \geq (2\ln C + \ln 2)/(\chi_\mu(f) - \varepsilon)$.

Claim. The branches $f_\alpha^{-n'}$ on $B(z_\infty(\alpha), \delta)$ depend only on $\pi_{-n',M}(\alpha)$, more precisely on $\pi_{-n',-1}(\alpha)$ as $\pi_{0,M}(\alpha) = \beta$ has been fixed, on the common domain $B(z_M(\beta), \frac{2}{3}\delta)$.

This is so since if two of α's in $a(N')$ have the same block $(\alpha_{-n'}, ..., \alpha_{-1})$, then they are continuations of the same branch at z along curves coinciding till $z_M(\beta)$ and next contained in the common domain.

We shall not use this claim directly, but we put it to help the reader to understand the proof and to simplify notation later on.

We have by (2) $\tilde{\nu}\big(\pi_{-n',M}^{-1}\pi_{-n',M}(A(n'))\big) \geq \xi/2N$. By Shennon-McMillan-Breiman theorem for every $\eta > 0$ and all integers k large enough

$$\nu\Big(\bigcup\{C_{0,k}(w) : w \in \Sigma_{0,k}, \quad \nu(C_{0,k}(w)) \leq \exp k(\mathrm{h}_\nu(\sigma) + \varepsilon/2)\Big) \geq 1 - \eta.$$

Setting $\eta = \xi/4N$ and $k = M + n'$ we get

$$\tilde{\nu}\Big(\bigcup\{C_{-n',M}(w) : w \in A(n')), \quad \tilde{\nu}(C_{-n',M}(w)) \leq \exp(n' + M)(\mathrm{h}_\nu(\sigma) + \varepsilon/2)\}\Big)$$

$$\geq \xi/2N - \xi/4N = \xi/2N.$$

Therefore for n large enough, the estimate of the number of "good backward trajectories" of time n' is

$$\#\Big(\pi_{-n',-1}(A(n'))\Big) \geq \exp n'(\mathrm{h}_\nu(\sigma) - \varepsilon). \tag{5}$$

Step 2. The sets X, Y and IFS

Now define $Y' \subset \Sigma^d$ as the set of one-sided sequences being concatenations of blocks v^k belonging to $G_{n'} := \pi_{0,n'-1}\tilde{\sigma}^{-n'}(A(n'))$, that is

$$Y' = \{\alpha = v^0 v^1 ... \in \Sigma^d : v^k = \pi_{0,n'-1}\sigma^{kn'}(\alpha) \in G_{n'} \;\; \forall k = 0, 1, ...\}$$

and

$$X' = z_\infty(Y').$$

Finally define

$$Y = \bigcup \{\sigma^j(Y') : j = 0, ..., n'-1\} \quad \text{and}$$

$$X = \bigcup \{f^j(X') : j = 0, ..., n'-1\} = z_\infty(Y).$$

For each $\alpha \in \Sigma^d$ and $r \leq s$ denote by $b_{r,s}$ the part of the branch $b(\alpha)$ starting from $z_{r-1}(\alpha)$, ending at $z_s(\alpha)$.

Now, saying briefly, by (3) and (4) for every $\alpha \in Y'$ the length of $b_{kn',(k+1)n'-1}(\alpha)$ is less than $C2^{-k}$ for a constant $C > 0$. Hence $z_n(\alpha) \to z_\infty(\alpha)$ uniformly (even exponentially) fast, which proves (ii) on Y', hence on Y by the unifom continuity of f. By (3) and (4) X', hence X, are expanding repellers for $f^{n'}$ and f respectively.

Let us be now precise. Let $\alpha \in Y'$ be a concatenation of $v^k = \pi_{0,n'-1}\tilde{\sigma}^{-n'}(w^k)$, for $w^k \in A(n')$, for $k = 0, 1, ...$ We want to analyse $b(\alpha)$. Note that by (1)

$$b_{(k-1)n',kn'-1}(\alpha) = f^{-n'}_{v^0,v^1v^2...v^k}(...(f^{-n'}_{v^{k-1},v^k}(b_{0,n'-1}(v^k)))). \tag{6}$$

Assume that $n \geq M$. Then all $b(\alpha)$ for $\alpha = v^0 v^1 ... \in Y'$ pass through $z_M(\beta)$ since $v^0 \in G_{n'}$ implies that $b_{0,n'}(\alpha)$ depending only on v^0 passes through $z_M(\beta)$. (There is no reason for α to belong to $A(n')$, which would yield passing through $z_M(\beta)$ by definition, as in Claim. So for the first time in the proof we need to use $n' \geq M$.)

Let us denote $B := B(z_M(\beta), \frac{2}{3}\delta)$.

Now we apply induction on k. Suppose that for all $\alpha \in \Sigma_{0,kn'-1}$ being a concatenation $v^0 v^1, ... v^k$ of blocks v^j in $G_{n'}$ we have $b_{M,kn'}(\alpha) \subset B$. Take an arbitrary $v \in G_{n'}$ being the truncation of $w \in A(n')$, more precisely $v = \pi_{0,n'-1}\tilde{\sigma}^{-n'}(w)$. Then $f^{-n'}_{v,\alpha}$ and $f^{-n'}_w$ coincide on B, in particular on $b_{M,kn'}(\alpha)$, since also $b_{M,kn'}(w^+)$ is contained in B, as $w \in K$ yielding the path in $\mathcal{T}$ joining $z_{kn'}(\alpha)$ to $z_\infty(w^+)$ being entirely contained in B, compare the proof of Claim. Hence, by (3) applied to $f^{-n'}_w$ we get $b_{M,(k+1)n'}(v\alpha) \subset B$, which finishes induction.

Therefore in (6) we can replace $f^{-n'}_{v^j,v^{j+1}v^{j+2}...v^k}$ by $f^{-n'}_{w^j}$ for all $j = 0, 1, ..., k-1$, in particular these branches of $f^{-n'}$ act on branches of the tree $\mathcal{T}$ in the common domain B (except $b_{0,n'-1}(v^k)$).

One can view the family of branches $F_v := f^{-n'}_v$ for $v \in G_{n'}$ as an iterated function system (IFS) on B. It satisfies so-called Strong Open Set Condition, i.e. all $F_v(B)$ have pairwise disjoint closures. The Claim allows us to write v rather than w, where v is the truncation of w. These branches act also (extend to) $b_{0,M}(\beta)$ (the line in the tree joining z to $z_M(v)$ which need not be contained in B. So F_v need not contract it. But the further iteration contracts them exponentially since $F_v(b_{0,M}(\beta))$ lie already in B.

The limit set is contained in $z_\infty(\Sigma^d)$, since F_v preserve the tree $\mathcal{T}$.

Step 3. Proving the properties (i)-(ix) in Theorem B

To prove (i) consider an arbitrary $\alpha = v^0 v^1 ... \in Y'$ for $v^k = \pi_{0,n'-1}\tilde{\sigma}^{-n'}(w^k)$, where $w^k \in A(n')$. Then, for each $k = 1, 2, ...$

$$\sum_{j=0}^{kn'-1} \varphi(\sigma^j(\alpha^+)) - \sum_{i=0}^{k-1}\sum_{j=0}^{n'-1} \varphi(\sigma^j((\tilde{\sigma}^{-n'}(w^i))^+)) \leq kn'\varepsilon/2 \tag{7}$$

for n large enough. This is so by by the continuity of φ and by the fact that $\sigma^{in'+j}(\alpha)$ and $\sigma^j((\tilde{\sigma}^{-n'}(w^i))^+)$ are very close to each other for all i and $0 \leq j << n'$. This is so because the both one-sided sequences have the same beginning of length $n' - j$. Now (i) follows from the estimate B(i) on $\sum_{j=0}^{n'-1} \varphi(\sigma^j((\tilde{\sigma}^{n'}(w^i))^+)$. Passing from Y' to Y changes only the constant C in (i).

This consideration proves also (vi). Indeed, in the case of ψ one assures for this the property of K analogous to B(i), namely

$$\text{B(vi)} \quad -\ln C + n(\textstyle\int \psi \, d\mu - \varepsilon/2) \leq \sum_{j=0}^{n-1} \psi(f^j(z_\infty(\alpha^+))) \leq \ln C + n(\int \psi \, d\mu + \varepsilon/2),$$

following from the ν-integrability of $\psi \circ z_\infty$ and Birkhoff Ergodic Theorem. Use also the property analogous to (7), for ψ and f, rather than φ and σ. It follows from the continuity of ψ and the fact that the preimages of points in B under the same branch $f^j f_{v^i}^{-n'}$ of $f^{-(n'-j)}$ are very close to each other for $0 \leq j << n'$.

The uniform (exponential) convergence in (ii) has been already proven. The injectivity and the property of X' of being a Cantor set, follows from the Strong Open Set Condition of the IFS $\{F_v\}$. This implies that z_∞ is finite-to-one on Y and X is also a Cantor set

By (5) and (i) and by the definition of pressure

$$\mathrm{P}(\sigma^{n'}|_{Y'}, \sum_{j=0}^{n'-1} \varphi \circ \sigma^j) \geq \mathrm{h}_\nu(\sigma^{n'}) + n' \int_{Y'} \varphi \, d\nu - \varepsilon$$

hence easily $\mathrm{P}(\sigma|_Y, \varphi) \geq \mathrm{h}_\nu(f) + \int_Y \varphi \, d\nu - \varepsilon$, hence (vii) for $\mathrm{P}(\sigma|_Y, \varphi)$. The consideration to prove (vii) for $\mathrm{P}(f|_x, \psi)$ is similar, using (vi) for ψ.

Note that one cannot pullback to Σ^d to refer to (vii) for $\mathrm{P}(\sigma, \psi \circ z_\infty)$ on Y since $\psi \circ z_\infty$ need not be continuous on $\partial I\!D$, even not defined, so we might not have (7).

By [M], or [P1, Sec.3] where further references are provided, we have $\mathrm{HD}(\mu) = \mathrm{h}_\mu(f)/\chi_\mu(f)$. Consider an arbitrary $\varepsilon' > 0$ and set $t' := \mathrm{HD}(\mu) - \varepsilon'$. Then $t' = \mathrm{h}_\mu(f)/\chi_\mu(f) - \varepsilon'$. By (iii) and (5)

$$\mathrm{P}(f|_X, -t' \ln |f'|_X|) \geq \mathrm{h}_{\mathrm{top}}(f|_X) - t'(\chi_\mu(f)+\varepsilon) \geq \mathrm{h}_\mu(f) - \varepsilon - (\mathrm{h}_\mu(f)/\chi_\mu(f) - \varepsilon')(\chi_\mu(f)+\varepsilon)$$

$$\geq -\varepsilon - \varepsilon \mathrm{h}_\mu(f)/\chi_\mu(f) + \varepsilon' \chi_\mu(f) + \varepsilon\varepsilon'$$

which is positive if $\varepsilon' > \frac{\varepsilon(1+\mathrm{h}_\mu(f))/\chi_\mu(f)}{\chi_\mu(f)+\varepsilon}$.

Hence $\mathrm{HD}(X) > t'$ as it is not smaller than the first zero of pressure function $t \mapsto \mathrm{P}(f|_X, -t \ln |f'|_X|)$, by Bowen theorem, see for example [PU]. If we choose ε small we obtain ε' small, hence $\mathrm{HD}(X)$ arbitrarily close to $\mathrm{HD}(\mu)$, which proves (iv).

We prove (ix) similarly.

To prove (v) consider the cylinder $C_M = C_{0,M}(\beta)$ for β being the truncation of a sequence α dense in $\mathrm{supp}\nu$ and M large. The proof of Theorem B is finished. QED

Section 5. Conclusions

Theorem B easily implies Theorem A. One builds the tree $\mathcal{T}$ in the basin of attraction. It is only sufficient to note that the branches of the tree $R^{-1}(\mathcal{T})$ converge to $\partial I\!D$ nontangentially, so the convergence of each branch $b(\alpha$ in $\mathcal{T}$ implies the nontangential in particular radial convergence of R at $\lim R^{-1}(b(\alpha)) \in \partial I\!D$, with the same limit. One considers the pull-back $\varphi \circ (R^{-1}(z) : \Sigma^d \to I\!R$, finds Y in Σ^d maps it by $(R^{-1}(z))_\infty$ with the use of $R^{-1}(\mathcal{T})$ into $I\!D$ and with the use of $\mathcal{T}$ to $X \subset \mathrm{Fr}\Omega$ as in Theorem B. The map $\hat{R}$ is finite-to-one on Y since z_∞ is. The rate of the exponential convergence in (ii) and more precisely in (ii') in Remark 2 follows easily from (iii), (i) applied to $\varphi = \ln g'$ and the chain rule $R'(z) = (f^{-n})'(R(g^n(z)))R'(g^n(z)) \cdot (g^n)'(z)$ for $z = r\zeta$ the integer n such that for the first time $g^n(z)$ is far from $\partial I\!D$ and the appropriate branch of f^{-n}, for details of this consideration see [P2]. See also [P3].

Remark 3. If ν is mixing, which is the case for Gibbs ν as in Remark 1, then one can assure f on X topologically mixing, that is $\forall U, V$ open subsets of X there exists n_0 such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq n_0$.

Indeed for n large we have by mixing $\tilde{\nu}(\tilde{\sigma}^{-n}(C_M) \cap C_M) \sim \nu(C_M)^2$. Hence, if $\nu(K) \approx 1$, then $\tilde{\nu}(\tilde{\sigma}^{-n}(A(n)) \geq \mathrm{Const} > 0$ for all n large. We can repeat the previous construction by taking instead of one n' two different mutually prime integers.

Remark 4. Theorem A holds in the case Ω is an immediate connected simply connected basin of attraction to a parabolic fixed point p, i.e. such $\partial \in \mathrm{Fr}\Omega$ that $f(p) = p$ and $f'(p)$ is a root of unity.

Indeed, in this case $R^{-1} \circ f \circ R$ extends to $\bar{C\!\!\!\!I}$ to g being a Blaschke product, such that $I\!D$ (and $\bar{C\!\!\!\!I} \setminus \mathrm{cl} I\!D$) is a basin of a parabolic fixed point for g in $\partial I\!D$. As in the conclusion that Th.B implies Th.A, we consider the trees $\mathcal{T}$ and $R^{-1}(\mathcal{T})$. All the branches of $R^{-1}(\mathcal{T})$ converge (polynomially fast, but not necessarily non-tangentially) and at each limit point

$\zeta = (R^{-1}(z))_\infty(\alpha)$ for $\alpha \in \mathcal{D}(z_\infty)$, in particular in Y, the radial limit $\hat{R}(\zeta)$ coincides with $z_\infty(\alpha)$ by Lindelöf's theorem. Hence $z_\infty = \hat{R} \circ (R^{-1}z)_\infty$ on Y and all the maps involved are finite-to-one since z_∞ is finite-to-one on Y.

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