Warsaw, 20 January, 2006,
revised 24 May, 2006

On hyperbolic Hausdorff dimension of the boundary of a basin of attraction for a holomorphic map and of quasirepellers

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Abstract. We prove that hyperbolic Hausdorff dimension of FrΩ, the boundary of a simply connected immediate basin of attraction Ω to an attracting periodic point of a rational mapping of the Riemann sphere, which is not a finite Blaschke product in some holomorphic coordinates, or its 2:1 factor, is larger than 1. We prove a "local version" of this theorem, for a boundary repelling to the side of a domain.

The results extend an analogous fact for polynomials by A. Zdunik and relies on the theory by M. Urbański, A. Zdunik and the author in the late 80-ties. To prove that the dimension is larger than 1, we use expanding repellers in $\partial\Omega$ constructed in [P2].

We deal with a quasi-repeller, namely a limit set for a geometric coding tree, and prove that hyperbolic Hausdorff dimension of the limit set is larger than Hausdorff dimension of the projection by the tree of any Gibbs measure for Hölder potential on the shift space, under a non-cohomology assumption. We consider also Gibbs measures for Hölder potentials on Julia set.

Section 0. Introduction

This paper completes [Z1], [Z2] and [P2]. In particular we prove the following

Theorem A. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of degree $d \geq 2$ of the Riemann sphere and Ω be a simply connected immediate basin of attraction to a periodic attracting orbit. Then, provided f is not a finite Blaschke product in some holomorphic coordinates, or a factor of a Blaschke product by h, a rational function of degree 2, then the hyperbolic Hausdorff dimension of FrΩ is larger than 1.

Immediate basin of attraction to a periodic orbit $O = \{p_1, ..., p_{k-1}\}$ of period k is the union of components intersecting O, of the set of points which converge to O. From now on we shall assume that Ω is a basin of immediate attraction to a fixed point, by considering f^k in the place of f if $k > 1$.

Hyperbolic Hausdorff dimension of an f-invariant set K, $\mathrm{HD}_{\mathrm{hyp}}(K)$, is defined as supremum of Hausdorff dimensions of f-invariant subsets X of K such that $f|_X$ is expanding, i.e. $|(f^N|_X)'| > 1$ for an integer N. One can assume that X are isolated (i.e. $\exists U \supset X$

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[1] supported by Polish KBN grant 2P03A 03425

2000 MSC: Primary 37F35; Secondary 37F15, 37D25

Key words and phrases: boundary of basin of attraction, Gibbs mesure, Hausdorff dimension, hyperbolic dimension, coding tree, iteration of holomorphic function, Central Limit Theorem.

an open set, such that $\bigcap_{n\geq 0}(f|_U)^{-n}(U) = X$. Moreover, we can consider in this definition only X being Cantor sets.

Theorem A in the case f is a polynomial and Ω is the basin of attraction to ∞ was proved in [Z1]. In fact a stronger theorem was proved in [Z1], saying that either a rational map f has a parabolic orbifold (in particular $f(z) = z^d$ or Tchebyshev polynomial, if f is a polynomial), or Hausdorff dimension of Julia set $J(f)$ (which is $\mathrm{Fr}\Omega$ for polynomials) is larger than Hausdorff dimension of the measure of maximal entropy $\mu_{\max}$. In the case of polynomials, $\mu_{\max}$ is equivalent to harmonic measure on $J(f)$ viewed from Ω, whose Hausdorff dimension in the case $J(f)$ is connected, is equal to 1, [Makarov].

To the best of our knowledge the question whether $\mathrm{HD}(J(f)) > 1$ is true for all rational f not being Blaschke or its 2:1 factor and $J(f)$ connected, remains open, though an opposite answer seems improbable. One can ask also about Hauddorff dimension of non one-point components of $J(f)$.

It is written in [Z2] on p. 635, at the end of Introduction, that Theorem A (in the RB version, see below) can be proved by the method of [Z1]. This is indeed true, an invariant expanding set X of dimension larger than 1 can be constructed as in [Z1]. It is not however clear and needs some care in the construction, that $X \subset \mathrm{Fr}\Omega$. We verify this here using a more careful construction originated in [PZ] and carefully described in [P2].

As in the other papers we deal with another, local, version of Theorem A, that is we prove

Theorem A' Assume that f is defined and holomorphic on a neighbourhood W of $\mathrm{Fr}\Omega$, where Ω is a simply connected domain in $\overline{\mathbb{C}}$ whose boundary has at least 2 points. We assume that $f(W \cap \Omega) \subset \Omega$, $f(\mathrm{Fr}\Omega) \subset \mathrm{Fr}\Omega$ and $\mathrm{Fr}\Omega$ repells to the side of Ω, that is $\bigcap_{n=0}^{\infty} f^{-n}(W \cap \mathrm{cl}\Omega) = \mathrm{Fr}\Omega$. Then either $\mathrm{HD}_{\mathrm{hyp}}(\mathrm{Fr}(\Omega)) > 1$ or $\mathrm{Fr}\Omega$ is a real-analytic Jordan curve or arc.

Domains Ω with the properties as in Theorem A' were introduced in [P1] and called there RB-domains (repelling boundary to the side of Ω.)

Theorem A' yields Theorem A since if f extends to a rational function on $\overline{\mathbb{C}}$, then $\mathrm{Fr}\Omega$ analytic implies that f is special as in Theorem A by [Brolin].

In Theorems A and A' the proof of the dichotomy saying that either f (or $\mathrm{Fr}\Omega$) is special or $\mathrm{HD}_{\mathrm{hyp}}(\mathrm{Fr}\Omega) > 1$ uses the fact that if $R : \mathbb{D} \to \Omega$ is a univalent holomorphic function from the unit disc to Ω (called also a Riemann mapping) then the radial limit $\hat{R}$ exists l-almost everywhere, for l the length measure (Fatou theorem). Denote by g the extension to $\partial\mathbb{D}$ of the function $R^{-1} \circ f \circ R$ on $\mathbb{D}$. It is analytic, expanding, see [P1]. The function $\psi := \log|g'| - \log|f'| \circ \hat{R}$ composed with iterates of g, i.e. the sequence of random variables $\psi \circ g^n$, satisfies Central Limit Theorem (CLT), see [PUZ]. This sequence is stationary with respect to a g-invariant measure ν on $\partial\mathbb{D}$ equivalent to l.

The case

$$\sigma^2(\psi) := \lim_{n\to\infty} \frac{1}{n} \int \Big(\sum_{k=0}^{n-1} \psi \circ g^k\Big)^2 \, d\nu = 0$$

leads to the real analiticity of $\mathrm{Fr}\Omega$, [Z1], hence in the case f is rational on $\overline{\mathbb{C}}$ to f Blaschke or a 2:1 factor of Blaschke.

The case we need to consider is $\sigma^2 > 0$. Due to deviations of the sums $\sum_{k=0}^{n-1} \psi \circ g^k$ from the expectation value $\int \psi \, d\nu = 0$, see [P1], we construct an iterated function system in $\mathrm{Fr}\Omega$ whose limit set has Hausdorff dimension larger than $\mathrm{HD}(\hat{R}_*(\nu)) = 1$ (the measure $\hat{R}_*(\nu)$ is equivalent to the harmonic measure ω).

As in all the papers mentioned above it is enough to deal with a geometric coding tree version of Theorem A', see Theorem B in Sections 1,2. Namely we consider Gibbs measure for an arbitrary Hölder continuous potential on the shift space, its projection μ via coding given by the tree to the limit set Λ, and prove that usually $\mathrm{HD}_{\mathrm{hyp}}(\Lambda) > \mathrm{HD}(\mu)$. In Theorems A and A', μ is equivalent to harmonic measure. For a more general version see Theorem C in Section 3.

In Section 3, in Theorem D, we consider μ given directly on Julia set $J(f)$, being Gibbs measure for a Hölder potential on $J(f)$, as in [DU], [P3] and [DPU].

Section 1. Geometric coding trees.

We recall the definitions from [PUZ], [PS], [PZ] and [P1]. Let U be an open connected subset of the Riemann sphere $\overline{\mathbb{C}}$. Consider any holomorphic mapping $f : U \to \overline{\mathbb{C}}$ such that $f(U) \supset U$ and $f : U \to f(U)$ is a proper map. Denote $\mathrm{Crit}(f) = \{z : f'(z) = 0\}$. This is called the set of critical points for f. Suppose that $\mathrm{Crit}(f)$ is finite. Consider any $z \in f(U)$. Let $z^1, z^2, ..., z^d$ be some of the f-preimages of z in U, where $d \geq 2$. Consider smooth curves $\gamma^j : [0,1] \to f(U)$, $j = 1, ..., d$, joining z to z^j respectively (i.e. $\gamma^j(0) = z, \gamma^j(1) = z^j$), such that there are no critical values for iterates of f in $\bigcup_{j=1}^d \gamma^j$, i.e. $\gamma^j \cap f^n(\mathrm{Crit}(f)) = \emptyset$ for every j and $n > 0$.

Let $\Sigma^d := \{1, ..., d\}^{\mathbb{Z}^+}$ denote the one-sided shift space and σ the shift to the left, i.e. $\sigma((\alpha_n)) = (\alpha_{n+1})$. For every sequence $\alpha = (\alpha_n)_{n=0}^\infty \in \Sigma^d$ we define $\gamma_0(\alpha) := \gamma^{\alpha_0}$. Suppose that for some $n \geq 0$, for every $0 \leq m \leq n$, and all $\alpha \in \Sigma^d$, the curves $\gamma_m(\alpha)$ are already defined. Suppose that for $1 \leq m \leq n$ we have $f \circ \gamma_m(\alpha) = \gamma_{m-1}(\sigma(\alpha))$, and $\gamma_m(\alpha)(0) = \gamma_{m-1}(\alpha)(1)$.

Define the curves $\gamma_{n+1}(\alpha)$ so that the previous equalities hold by taking respective f-preimages of curves γ_n. For every $\alpha \in \Sigma^d$ and $n \geq 0$ denote $z_n(\alpha) := \gamma_n(\alpha)(1)$. Note that $z_n(\alpha)$ and $\gamma_n(\alpha)$ depend only on $(\alpha_0, ..., \alpha_n)$ so sometimes we consider z_n and γ_n as functions on blocks of symbols of length $n+1$. Sometimes it is comfortable to denote z by z_{-1}.

The graph $\mathcal{T}(z, \gamma^1, ..., \gamma^d)$ with the vertices z and $z_n(\alpha)$ and edges $\gamma_n(\alpha)$ is called a *geometric coding tree* with the root at z. For every $\alpha \in \Sigma^d$ the subgraph composed of $z, z_n(\alpha)$ and $\gamma_n(\alpha)$ for all $n \geq 0$ is called a *geometric branch* and denoted by $b(\alpha)$.

For each $j = 1, ..., d$ we define f_j^{-1} on a small neighbourhood of z as the branch of f^{-1} mapping z to z^j. For each $\alpha \in \Sigma^d$ the branch f_j^{-1} has an analytic continuation $f_{j,\alpha}^{-1}$ along the curve $b(\alpha)$. Note that by the construction $f_{j,\alpha}^{-1}(b(\alpha)) = b(j\alpha)$, where $j\alpha$ is the concatenation of the symbol j and the sequence α. By induction, for any v being a block of k symbols in $\{1, ..., d\}$, for f_v^{-k} being the branch of f^{-k} mapping z to $z_{k-1}(v)$ and for

$f_{v,\alpha}^{-k}$ being the analytic continuation along $b(\alpha)$, we get

$$f_{v,\alpha}^{-k}(b(\alpha)) = b(v\alpha). \tag{1}$$

For infinite α the branch $b(\alpha)$ is called *convergent* if the sequence $\gamma_n(\alpha)$ is convergent to a point in cl$U$. We define the *coding map* $z_\infty : \mathcal{D}(z_\infty) \to \mathrm{cl}U$ by $z_\infty(\alpha) := \lim_{n\to\infty} z_n(\alpha)$ on the domain $\mathcal{D} = \mathcal{D}(z_\infty)$ of all such α's for which $b(\alpha)$ is convergent. It was proved in [PS] that $\mathcal{D}$ is the whole Σ^d except a "thin" set. In particular for a Gibbs measure ν as in Theorem B below, $z_\infty(\alpha$ exists for ν-a.e. α, hence the push forward measure $(z_\infty)_*$ makes sense.

For each geometric branch $b(\alpha)$ denote by $b_m(\alpha)$ the part of $b(\alpha)$ starting from $z_m(\alpha)$ i.e. consisting of the vertices $z_k(\alpha), k \geq m$ and of the edges $\gamma_k(\alpha), k > m$.

If the map f extends holomorphically to a neighbourhood of the closure of the limit set Λ of a geometric coding tree, $\Lambda = z_\infty(\mathcal{D}(z_\infty))$, then Λ is called a *quasi-repeller*, see [PUZ].

Theorem B. Let Λ be a quasi-repeller for a geometric coding tree $\mathcal{T}(z, \gamma^1, ..., \gamma^d)$ for a holomorphic map $f : U \to \overline{\mathbb{C}}$. Let ν be a σ-invariant Gibbs probability measure on Σ^d for a Hölder continuous real-valued function u on Σ^d. Consider $\mu := (z_\infty)_*(\nu)$. Assume that for $\psi := -u - \mathrm{HD}(\mu)(\log|f'| \circ z_\infty))$ the asymptotic variance $\sigma^2 := \sigma^2(\psi)$ is positive. Then there exists an f-invariant hyperbolic subset X of Λ such that $\mathrm{HD}(X) > \mathrm{HD}((z_\infty)_*(\nu)$.

Recall that a probability measure ν is called a Gibbs measure for a Hölder continuous function $u : \Sigma^d \to \mathbb{R}$ if there is a constant $C_1 > 0$ such that for every cylinder

$$\mathcal{C}_{n-1}(\beta) := \{\alpha \in \Sigma^d : \alpha_0 = \beta_0, ..., \alpha_{n-1} = \beta_{n-1}\}$$

for a sequence $\beta = (\beta_0, ..., \beta_{n-1}) \in \{1, ..., d\}^n$,

$$C_1^{-1} \leq \nu(\mathcal{C}_{n-1}) / \exp(S_n(u)(\alpha) - nP(\sigma, u)) \leq C_1, \tag{2}$$

where $P(\sigma, u)$ is topological pressure for the shift mapping σ and potential function u, see [Bowen] or [PU].

We can assume that $P(\sigma, u) = 0$ by replacing u by $u - P(\sigma, u)$. Then we have $\int \psi \, d\nu = 0$, see [PUZ, Sec.4, Lemma 7]

Section 2. Proof of Theorem B

We proceed as in [P2]. Denote the natural extension of the one-sided shift σ with the measure ν and Borel σ-algebra σ by $(\tilde{\Sigma}^d, \tilde{\sigma}, \tilde{\nu})$. This is a two-sided shift space with the shift $\tilde{\sigma}$ to the left. Denote the projection $\tilde{\Sigma}^d \to \Sigma^d$ mapping α to $(\alpha_0, \alpha_1, ...)$ by π_+. For each $\alpha \in \tilde{\Sigma}^d$ denote $\pi_+(\alpha)$ by α^+.

We have Lyapunov exponent $\chi(\mu) := \int \log|f'| \, d\mu > 0$ since $2\chi(\mu) \geq \mathrm{h}_\mu(f) = \mathrm{h}_\nu(\sigma) > 0$, compare [P1] and [PUZ].

By the existence of z_∞ ν-a.e. and by Pesin theory, see [PZ, Lemma 1] for the version which we apply, for every $\varepsilon > 0$ we find a set $K \subset \tilde{\Sigma}^d$, $\delta > 0$, a positive integer M and a constant $C > 0$ such that $\tilde{\nu}(K) > 1 - \varepsilon$ and for all $\alpha \in K$ and $n \geq 0$

(i) $b_M(\alpha^+) \subset B(z_\infty(\alpha^+), \delta/3)$.

(ii) There exist univalent branches f_α^{-n} of f^{-n} on $B(z_\infty(\alpha^+), \delta)$ for all $n = 1, 2, ...$ mapping $z_\infty(\alpha^+)$ to $z_\infty(\tilde{\sigma}^{-n}(\alpha)^+)$, of distortion bounded by C, i.e.

$$\sup_{x \in B(z_\infty(\alpha^+),\delta)} |(f_\alpha^{-n})'(x)| / \inf_{x \in B(z_\infty(\alpha^+),\delta)} |(f_\alpha^{-n})'(x)| \leq C.$$

In the notation accompanying the property (1) these branches are the continuations along $b(\alpha^+)$ of $f^{-n}_{(\alpha_{-n},\dots,\alpha_{-1})}$, i.e. the branches $f^{-n}_{(\alpha_{-n},\dots,\alpha_{-1}),\alpha^+}$.

The assumption $\sigma^2(\psi) > 0$ implies $\sigma^2(\psi \circ \pi_+, \tilde{\sigma}^{-1}) = \sigma^2(\psi) > 0$ for the asymptotic variance of the sequence of $\psi \circ \pi_+$ composed with the iterates of the mapping $\tilde{\sigma}^{-1}$. Due to CLT, verified in [PUZ], we know that there exists $\varepsilon_1 > 0$ such that for

$$K_n := \{\alpha \in \tilde{\Sigma}^d : S_n(\psi)(\alpha) \geq \sqrt{n}\}$$

we have

$$\tilde{\nu}(K_n) \geq \varepsilon_1$$

for all n large enough. Here we denote $S_n\psi := \sum_{j=0}^{n-1} \psi \circ \pi_+ \circ \tilde{\sigma}^{-j}$.

Consider $K_n' := K \cap K_n$.

For each n large enough we choose a cylinder $\mathcal{C}_M = \mathcal{C}_M(\beta) := \{\alpha \in \tilde{\Sigma}^d : \alpha_0 = \beta_0, ..., \alpha_M = \beta_M\}$ for a sequence $\beta = (\beta_0, ..., \beta_M) \in \{1, ..., d\}^{M+1}$, such that

$$\tilde{\nu}(\mathcal{C}_M \cap K_n') \geq \frac{\varepsilon_1}{2} \tilde{\nu}(\mathcal{C}_M) := \xi,$$

which is possible if ε is small enough. Restricting to a subsequence $\mathcal{N}$ of n's we can fix one $\mathcal{C}_M$, in particular ξ is fixed, independent of $n \in \mathcal{N}$.

Consider $L_n := \tilde{\sigma}^{-n}(\mathcal{C}_M \cap K_n')$.

By Birkhoff Ergodic Theorem there exists $N \geq 0$ independent of n such that for all $n \in \mathcal{N}$

$$\tilde{\nu}(\{\alpha \in L_n : \exists i : 0 \leq i < N, \tilde{\sigma}^{-i}(\alpha) \in K \cap \mathcal{C}_M\}) \geq \xi/2.$$

Therefore for every $n \geq 0$ there exists $N' : 0 \leq N' < N$ such that, denoting $n' := n + N'$, for $A(n') := \{\alpha \in \mathcal{C}_M \cap K_n' : \tilde{\sigma}^{-n'}(\alpha) \in K \cap \mathcal{C}_M\}$ we have

$$\tilde{\nu}(A(n')) \geq \xi/2N. \tag{3}$$

As in [P2] let $G_{n'} \subset \{1, ..., d\}^{n'}$ be the set of sequences $v = (v_{-n'}, ..., v_{-1})$ of length n' being truncations of $\alpha \in A(n')$ to the coordinates $\{-n', ..., -1\}$. For each v we choose one $\alpha = \alpha_v$ whose truncation is v.

We consider an Iterated Function System consisting of the branches $F_v := f^{-n'}_{v,\alpha^+}$ on $B := B(z_M(\beta), 2\delta/3)$, for $v \in G_{n'}$. This does not depend on α^+, see [P2]. It is proved in [P2] that indeed F_v map B to B and the limit set

$$X' := \bigcap_{j \geq 0} \bigcup_{v^1, \dots, v^j \in G_{n'}} F_{v^j} \circ ... \circ F_{v^1}(\mathrm{cl} B)$$

is in the quasi-repeller Λ. This set is invariant under $\Phi := f^{n'}$.

As in [Z1, Section 9] we estimate the topological pressure on X' for $t = \mathrm{HD}(\mu)$ using preimages of $y = z_M(\beta)$ (we could use any point in B here), remembering that C bounds distortion for all F_v on B, finally denoting

$$Q_{v,n,\alpha}(u) := u(v_{-n}...v_{-1}\alpha^+) + ... + u(v_{-1}\alpha^+),$$

as follows

$$P(\Phi, -t\log|\Phi'|) = \lim_{j\to\infty}\frac{1}{j}\log\sum_{v^1,\dots,v^j\in G_{n'}}|F'_{v^j}(F_{v^{j-1}}\circ...\circ F_{v^1}(y))|^t\cdot...\cdot|F'_{v^1}(y)|^t$$

$$\geq -t\log C + \log\sum_{v=(v_{-n'},\dots,v_{-1})\in G_{n'}}\exp\big(\sqrt{n} + Q_{v,n,\alpha_v}(u) - (n'-n)t\sup\log|f'|\big)$$

$$\geq -t\log(C\sup|f'|^{N'}) + \sqrt{n} + \log\sum_{v=(v_{-n'},\dots,v_{-1})}\exp Q_{v,n,\alpha_v}(u)$$

$$\geq -t\log(C\sup|f'|^{N'}) + \sqrt{n}\ + C_1^{-1}\xi/2N, \tag{4}$$

where the latter estimate of the sum follows from (3) and from the definition of Gibbs measure (2).

Indeed, for each $v\in G_{n'}$ we have

$$Q_{v,n,\alpha_v}(u) \geq C_1^{-1}\tilde{\nu}\{\alpha\in A(n') : \alpha_{-n} = v_{-n}, \alpha_{-1} = v_{-1}\}, \tag{5}$$

which, summed over v, was estimated from below in (3).

The latter expression in (4) is positive for n large enough. Hence $\mathrm{HD}(X) > t$, as $\mathrm{HD}(X)$ is (the only) zero of $P(\Phi, -t\log|\Phi'|)$.

Finally we define an f-invariant isolated Cantor set in Λ of Hausdorff dimension larger than $\mathrm{HD}(\mu)$ as in [P2] by $X := \bigcup\{f^j(X') : j = 0, ..., n'-1\}$ QED

Section 3. Generalizations. Gibbs measures for Hölder functions on $J(f)$.

By the same considerations as in Proof of Theorem B one can prove the following

Theorem C. Suppose that for a quasi-repeller Λ as in Theorem B, ν is a σ-invariant probability measure on Σ^d, of positive entropy, and $u = \log(\mathrm{Jac}_\nu(\sigma^{-1}))\circ\sigma : \Sigma^d \to I\!R$, where $\mathrm{Jac}_\nu(\sigma^{-1})$ is Jacobian of σ^{-1} with respect to ν (the branch of σ^{-1} inversing σ). (We assume that σ^{-1} is absolutely continuous with respect to ν.) Suppose there exists $\varepsilon > 0$ and a sequence $a_n \to \infty$ such that for $K_n := \{\alpha\in\tilde{\Sigma}^d : S_n(\psi)(\alpha) \geq a_n\}$ with $S_n(\psi)$ being the "backward sum" defined in Proof of Theorem B, we have $\tilde{\nu}(K_n) \geq \varepsilon$. Then, there exists an f-invariant hyperbolic subset X of Λ such that $\mathrm{HD}(X) > \mathrm{HD}((z_\infty)_*(\nu)$.

To prove Theorem C we choose for each v in (5) $\alpha_v \in A(n')$ such that for $B_v := \{v\alpha^+ \in \Sigma^d : \alpha \in A(n'),\ \alpha_j = v_j \quad \forall j = -n, ..., -1\}$

$$\exp Q_{v,n,\alpha_v}(u) \geq \int_{B_v} \exp Q_{v,n,\alpha}(u)\, d(\nu \circ \sigma^n)(v\alpha^+) = \nu(B_v)$$

and next, as in Section 2, we use $\sum \nu(B_v) = \tilde{\nu}(A(n')) \geq \text{Const} > 0$.

We do not need to assume Hölder continuity of u.

We can apply Theorem C to a class of measures discussed in the following

Theorem D. Let f be a rational mapping of degree $d \geq 2$ on the Riemann sphere. Let μ be a Gibbs measure for Hölder function $\varphi : J(f) \to I\!R$ such that the pressure satisfies $P(f, \varphi) > \sup \varphi$, see [P3] and [DU]. Then, if φ is not cohomologous to $-\mathrm{HD}(\mu) \log |f'|$, we get

$$\mathrm{HD}(\mu) < \mathrm{HD}_{\mathrm{hyp}}(J(f)).$$

Theorem D follows from Theorem C under an additional assumption that there exists a geometric coding tree $\mathcal{T}(z, \gamma^1, ..., \gamma^d)$, with d being the degree of f, and there exists a probability measure ν on Σ^d, such that

$$\mu = (z_\infty)_*(\nu). \tag{$*$}$$

This holds for example if $\mathrm{diam}(\gamma_n(\alpha)) \to 0$ as $n \to \infty$, uniformly over α, see [P4], in particular if z is taken in the basin of attraction to ∞ for f a polynomial.

We do not know whether $\mu = (z_\infty)_*(\nu)$ in a general situation for every μ of positive Lyapunov exponent.

Let us assume that (*) holds. Since the entropy $\mathrm{h}_\mu(f) > 0$, by $P(f, \varphi) > \sup \varphi$, by Variational Principle, see [P3] or [DU], we get $\mathrm{h}_\mu(f) > 0$.

Now, to apply Theorem C, it is sufficient to prove that $\psi = -\varphi - \mathrm{HD}(\mu) \log |f'|$ satisfies some assumptions sufficient for CLT to hold. Namely it is sufficient to prove that

$$\sum_{n=0}^{\infty} ||\mathcal{L}^n(\psi/h) - \mu(\psi/h)||_2 < \infty, \tag{6}$$

see [DPU] and [Gordin]. Here the norm is in $L_2(\mu)$ (square integrable functions) and $\mathcal{L}$ is the normalized transfer operator, namely the transfer operator for the function $\varphi_0 := \varphi - P(f, \varphi) + h \circ f - h$, where h is the eigenfunction for the transfer operator with potential φ corresponding to the leading eigenvalue equal to $\exp P(f, \varphi)$. By [DPU] h is Hölder.

By [DPU] $\mathcal{L}^n(\varphi/h - \mu(\varphi/h))$ converges to 0 with the speed of order $\exp -\sqrt{n}$, even in L_∞. (N. Haydn proved in [Haydn] that the speed is in fact exponential.)

So it is sufficient to discuss $||\mathcal{L}^n(\log |f'|/h - \mu(\log |f'|)/h)||_2$.

Note that for any $\eta \in L_2(\mu)$, using $\sum_{y\in f^{-1}(x)} \exp\varphi_0(y) \equiv 1$ and the convexity of the square function, we get

$$||\mathcal{L}(\eta)||_2^2 = \int |\mathcal{L}(\eta)|^2 \, d\mu = \int |\sum_{y\in f^{-1}(x)} (\exp\varphi_0(y))\eta(y)|^2 \, d\mu(x)$$

$$\leq \int \sum_{y\in f^{-1}(x)} (\exp\varphi_0(y)|\eta|^2(y)| \, d\mu(x) \leq \int \mathcal{L}(|\eta|^2) \, d\mu = \int |\eta|^2 \, d\mu = ||\eta||_2^2 \tag{7}$$

We shall use also the following fact, see [DPU, (3.1)]:
For every $0 < \alpha \leq 1$ there exists $\tau > 0$ such that

$$\sup_{n\geq 0} ||\mathcal{L}^n(\eta)||_{\mathcal{H}_\tau} < \infty \quad \forall \, \eta \in \mathcal{H}_\alpha,$$

where $||\cdot||_{\mathcal{H}_a}$ denotes Hölder norm in the Banach space of real-valued Hölder continuous functions $\mathcal{H}_a$ for $a = \alpha$ or τ (Lipschitz for $a = 1$).

The constant τ has been chosen in [DPU] so that the operators $\mathcal{L}^n$ are continuous. So, by Banach-Steinhaus Theorem (or direct analysis of the proof of the above estimate in [DPU]), there is a common bound A for the norms of the family of the operators $\mathcal{L}^n : \mathcal{H}_\alpha \to \mathcal{H}_\tau$.

Having this, we can rewrite in a uniform form, on $\mathcal{H}_\alpha$, Lemma 4.4 from [DPU], as follows

Lemma 1. There exist $0 < \lambda < 1, 0 < \gamma < 1, C > 0$ such that for every $C_1 > 0$ and $\Theta > 0$, and for $n = n(\Theta)$ the least positive integer bigger than $(\log\Theta - \log CC_1)/\log\lambda$, if $\mu(\eta) = 0$, $||\mathcal{L}^n(\eta)||_{\mathcal{H}_\tau} < C_1$ and $||\eta||_\infty < \Theta$, then

$$||\mathcal{L}^n(\eta)||_\infty < \gamma\Theta.$$

Now we can prove (6). In the sequel C will denote various positive constants, depending only on f.

Let $\eta := -(\log|f'|)/h - \mu(-(\log|f'|)/h)$ and $\eta_n := \min(\eta, \sqrt{n})$.

Then, summing over $c \in \mathrm{Crit}(f) \cap J(f)$, using $\mu(B(z,r)) \leq r^\kappa$ for a constant $\kappa > 0$ and for all discs $B(z,r)$ small enough (true by the consideration as in the proof of Lemma 4 in [PUZ]), we get for all n large enough,

$$||\eta - \eta_n||_2^2 \leq \sum_c \sum_{j=C^{-1}n}^{\infty} C(\sqrt{j})^2 \mu(B(c, \exp-\sqrt{j}))$$

$$\leq \sum_c \sum_{j=C^{-1}n}^{\infty} Cj \exp -\kappa\sqrt{j} \leq \exp -(\kappa/2)\sqrt{C^{-1}n}.$$

(C^{-1} under the sums depends also on $\inf h$, i.e. on φ.)

Hence, with the use of (7), we get

$$||\mathcal{L}^n(\eta-\eta_n)-\mu(\eta-\eta_n)||_2 \le 2\exp -(\kappa/2)\sqrt{C^{-1}n}.$$

Now, let us consider η_n. Its Lipschitz norm, hence α-Hölder norm, is bounded by $C(\sqrt{(n+1)}-\sqrt{n})/(\exp-\sqrt{n}-\exp-\sqrt{(n+1)}) \le C\exp\sqrt{n}$. Hence $||\mathcal{L}^n(\eta_n)-\mu(\eta_n)||_{\mathcal{H}_\tau} \le AC\exp\sqrt{n}$.

Let us iterate Lemma 1 s times starting from $\xi_n := \eta_n - \mu(\eta_n)$. We get for $\Theta := ||\xi_n||_\infty \le 2\sqrt{n}$ and all s

$$||\mathcal{L}^{N(s)}(\xi_n)||_\infty \le 2\gamma^s\sqrt{n}.$$

where $N(s) := n(\Theta)+n(\gamma\Theta)+...+n(\gamma^{s-1}\Theta)$. We have

$$N(s) \le \sum_{j=0}^{s-1}(\log(\gamma^j\ 2\sqrt{n}) - \log AC - \sqrt{n})/\log\lambda$$

$$\le \frac{s(s-1)}{2}\frac{\log\gamma}{\log\lambda} + +s\frac{\log n}{\log(1/\lambda)} + s\frac{\log AC}{\log(1/\lambda)} + s\frac{\sqrt{n}}{\log 1/\lambda}. \tag{8}$$

If we want that $N \le n$, it is sufficient that the latter sum in (8) is less than n, which holds if s satisfies

$$A_1 s^2 + A_2 s\log n + A_3 s + A_4 s\sqrt{n} \le n$$

for positive constants A_1, A_2, A_3, A_4 resulting from (8), i.e. if s satisfies $s \le C\sqrt{n}$. Then,

$$||\mathcal{L}^n(\xi_n)||_\infty \le 2\gamma^{C\sqrt{n}} \le \exp -\vartheta\sqrt{n}.$$

for $\vartheta = -(C/2)\log\gamma$. Thus we proved the convergence of the series in (6). Moreover the sequence under the sum in (6) converges at least with the speed $\exp -\vartheta\sqrt{n}$ for a constant $\vartheta > 0$. Hence, denoting $\psi_0 := \psi/h - \mu(\psi/h)$, we get

$$\sum_{n=1}^\infty n\int \psi_0(\psi_0\circ f^n\, d\mu = \sum_{n=1}^\infty n\int \mathcal{L}^n(\psi_0)\psi_0\, d\mu \le \sum_{n=1}^\infty n||\mathcal{L}^n(\psi_0)||_2||\psi_0||_2 < \infty.$$

So by the assumption that ψ is not cohomologous to 0, we get $\sigma^2(\psi/h) > 0$, see [PUZ, Lemma 1].

If we do not assume (*) we cannot use Theorem C. Fortunately lifting the problem to the shift space, in the case we look for a hyperbolic X in $J(f)$ rather than in a proper subset of $J(f)$, is purely formal. So we can use a counterpart of Theorem C directly on $J(f)$, for u being logarithm of Jacobian of f^{-1} with respect to μ on $J(f)$ of positive Lyapunov exponent, for $\psi := -u - \mathrm{HD}(\mu)\log|f'|$. This ends Proof of Theorem D. QED

Conjecture 1. For any rational map f there is at most one f-invariant probability measure μ on Julia set $J(f)$, of positive Lyapunov exponent, such that $\mathrm{HD}(\mu) = \mathrm{HD}_{\mathrm{hyp}}(J(f))$.

In other words Conjecture 1 says that there is at most one measure μ which is an equilibrium measure for the potential function $\beta := -\mathrm{HD}_{\mathrm{hyp}}(J(f))\log|f'|$. Indeed, $\mathrm{HD}(\mu) = \mathrm{h}_\mu(f)/\chi(\mu)$, [Mañé], hence $\mathrm{h}_\mu(f) - \mathrm{HD}(\mu)\chi(\mu) = 0$. Since $P(f|_{J(f)}, \beta) = 0$, see [P5] or [PRS], μ is equilibrium for β iff $\mathrm{HD}(\mu) = \mathrm{HD}_{\mathrm{hyp}}(J(f))$.

A stronger conjecture would be if we omit the positive Lyapunov exponent assumption.

Let us note that such a measure μ exists (and is indeed unique) for f Topological Collet-Eckmann, see [PR]. In this case $\mathrm{HD}_{\mathrm{hyp}}(J(f)) = \mathrm{HD}(J(f))$, see [P5], and $\mathrm{Jac}_\mu(f)$ is cohomologous in measurable functions to $\mathrm{HD}(J(f))\log|f'|$ on $J(f)$. The function giving the cohomology is $\log d\mu/dm$, where m is a unique conformal measure with exponent $\mathrm{HD}_{\mathrm{hyp}}(J(f))$, see [PR].

Acknowledgements. We wish to thank Juan Rivera-Letelier for several useful remarks.

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