HÖLDER IMPLIES CE

by

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Abstract. — We prove that for every polynomial f if its basin of attraction to ∞ is Hölder and Julia set contains only one critical point c then f is Collet-Eckmann, namely there exists $\lambda > 1, C > 0$ such that for every $n \geq 0$

$$|(f^n)'(f(c)| \geq C\lambda^n.$$

We introduce also *topological Collet-Eckmann rational maps* and *repellers*.

0. Introduction

J. Graczyk and S. Smirnov proved in [GS] that if a rational map is Collet-Eckmann (abbr. CE), then every component of the complement of Julia set J is Hölder. Another proof was provided later in [PR1]. The question whether a converse fact holds remained unanswered. Moreover it has been proved in [PR2] (using an example from [CJY]) that if there are at least two critical points in J, then the converse may occur false, even for polynomials. Namely if the forward trajectory of a critical point c at some times approaches very closely another critical point, but all critical points in J are nonreccurrent, then A_∞ the basin of infinity is John even, but $|(f^n)'(f(c))|$ does not grow exponentially fast.

Here (in Sec.3) we prove that A_∞ Hölder implies CE for polynomials if there is only one critical point in J. In fact we prove this in a more general setting of rational functions. We prove this by using Graczyk and Smirnov's "reversed telescope" idea.

In Section 4 we introduce for rational maps the property *topological Collet-Eckmann* (abbr. TCE). This property means roughly a possibility of going from many small scales around each point to large scale round discs with uniformly bounded criticality under the action of the iterates of f. This property is topological (i.e. it is preserved

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under topological conjugacies) and we prove in Section 4 that it implies CE, provided there is only one critical point in J (or more than one, but none in the ω-limit set of the others). Since by [PR1] CE implies TCE we obtain a new elementary proof that CE is a topological property. The first proof was provided in [PR2]: For f being CE, and g topologically conjugate to it, it was proved that the conjugacy can be improved to a quasiconformal one on a neighbourhood of $J(f)$. This implied CE for g, by a method not much different from presented here (but simpler technically).

In the unimodal maps of the interval case the fact CE is a topological property was proved in [NP] via the same TCE property called there *finite criticality*. The intermediate property used there was *uniform hyperbolicity on periodic orbits* (abbr. UHPer). Here this idea also appears implicitly, though we cannot prove UHPer implies CE (the fact proved for unimodal maps of interval with negative Schwarzian derivative by T. Nowicki and D. Sands in [NS].)

Finally, in Section 5, we introduce and study *holomorphic TCE invariant sets* in particular *repellers* and prove that if a repeller is the boundary of an open connected domain in $\overline{\mathbb{C}}$, then it is TCE iff the domain is Hölder. In consequence, for each domain with repelling boundary, to be Hölder is a topological property. We prove also the analogous rigidity result for Hölder immediate basins of attraction to attracting fixed points.

1. Preliminaries on Hölder basins

Definition 1.1. — Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of the Riemann sphere. We call an f-critical point c (i.e. such that $f'(c) = 0$) exposed if its forward f-trajectory does not meet other critical points)

The map f is called Collet-Eckmann if its every exposed critical point c that belongs to the Julia set $J = J(f)$, or its forward orbit converges to J, satisfies the following Collet-Eckmann condition:

There exists $\lambda > 1$ such that for every $n \geq 0$

$$(CE) \qquad |(f^n)'(c_1)| \geq \mathrm{Const}\lambda^n.$$

Notation. — By Const we denote various positive constants which can change from one formula to another. We use the notation $x_n = f^n(x)$.

The definition of holomorphic Collet-Eckmann map was introduced in [P2] with (CE) assumed only for critical points in J. This allowed parabolic periodic points. Here we modify the definition, in accordance with [GS, Def 1.2].

One calls a simply-connected open hyperbolic domain A *Hölder* if there exists $\alpha > 0$ such that any Riemann mapping from the unit disc D onto A is Hölder continuous. This can be generalized to non-simply connected domains, see [Po] or [GS, Def 5.1]. We shall not rewrite here this definition in absence of dynamics because we do not need this. However if A is an immediate basin of attraction to a sink, for a rational mapping f, $f(A) = A$, Graczyk and Smirnov provided an equivalent definition [GS, Def.1.4, Sec.5 Prop.3] which will be of use for us. Denote $\mathrm{Crit}^+ := \bigcup_{j=1}^\infty f^j(\mathrm{Crit})$, where Crit= Crit$(f)$ means the set of all critical points for f.

Definition 1.2. — We call A Hölder if there exists $\lambda_{\mathrm{Ho}} > 1$ such that for every $z \in A \setminus \mathrm{cl}\,\mathrm{Crit}^+$ there exists $C_1 > 0$ such that for every $y \in f^{-n}(\{z\}) \cap A$

$$|(f^n)'(y)| \geq C_1 \lambda_{\mathrm{Ho}}^n. \tag{1.1}$$

We extend this definition to periodic A, $f^k(A) = A$, by replacing f by f^k above. This replacement allows in proofs to assume $f(A) = A$.

We need also the following

***Notation* (cf. [PUZ]).** — Suppose $f(A) = A$. Let $z^1, ..., z^d$ be all the pre-images of z in A. Consider smooth curves $\gamma^j : [0,1] \to A \setminus \mathrm{cl}\,\mathrm{Crit}^+$, $j = 1, ..., d$, joining z to z^j respectively (i.e. $\gamma^j(0) = z, \gamma^j(1) = z^j$).

Let $\Sigma^d := \{1, ..., d\}^{\mathbb{Z}^+}$ denote the one-sided shift space and σ the shift to the left, i.e. $\sigma((\alpha_n)) = (\alpha_{n+1})$. For every sequence $\alpha = (\alpha_n)_{n=0}^\infty \in \Sigma^d$ we define $\gamma_0(\alpha) := \gamma^{\alpha_0}$. Suppose that for some $n \geq 0$, for every $0 \leq m \leq n$, and all $\alpha \in \Sigma^d$, the curves $\gamma_m(\alpha)$ are already defined. Write $z_n(\alpha) := \gamma_n(\alpha)(1)$.

For each $\alpha \in \Sigma^d$ define the curve $\gamma_{n+1}(\alpha)$ as a lift (image) by $f^{-(n+1)}$ of $\gamma^{\alpha_{n+1}}$ starting at $z_n(\alpha)$.

The graph $\mathcal{T} = \mathcal{T}(z, \gamma^1, ..., \gamma^d)$ with the vertices z and $z_n(\alpha)$ and edges $\gamma_n(\alpha)$ is called a *geometric coding tree* with the root at z. For every $\alpha \in \Sigma^d$ the subgraph composed of $z, z_n(\alpha)$ and $\gamma_n(\alpha)$ for all $n \geq 0$ is called a *geometric branch* and denoted by $b(\alpha)$. Denote by $b_n(\alpha)$ for $n \geq 0$ the subgraph composed of $z_j(\alpha)$ and $\gamma_{j+1}(\alpha)$ for all $j \geq n$.

The branch $b(\alpha)$ is called *convergent* to $x \in \partial A$ if $z_n(\alpha) \to x$.

For an arbitrary basin of attraction A we define the *coding map* $z_\infty : \mathcal{D}(z_\infty) \to \mathrm{cl}\,U$ by $z_\infty(\alpha) := \lim_{n\to\infty} z_n(\alpha)$ on the domain $\mathcal{D} = \mathcal{D}(z_\infty)$ of all such α's for which $b(\alpha)$ is convergent. Notice that for A Hölder $\mathcal{D} = \Sigma^d$ and z_∞ is Hölder.

Finally let $U^1, ..., U^d$ be open topological discs with closures in $A \setminus \mathrm{cl}\,\mathrm{Crit}^+$, containing $\gamma^1, ..., \gamma^d$ respectively. For each α and $n \geq 0$ denote by $U_n(\alpha)$ the component of $f^{-n}(U^{\alpha_n})$ containing $\gamma_n(\alpha)$.

In the subsequent Lemmas A is a Hölder immediate basin of attraction to a periodic sink for a rational function f.

Lemma 1.3. — *There exists $C_2 > 0$ such that for every $\alpha \in \Sigma^d$ and every positive integer m*

$$\mathrm{diam}\, U_m(\alpha) \leq C_2 \lambda_{\mathrm{Ho}}^{-m} \tag{1.2}$$

and

$$U_m(\alpha) \subset B(z_\infty(\alpha), C_2 \lambda_{\mathrm{Ho}}^{-m}/(1 - \lambda_{\mathrm{Ho}}^{-1})). \tag{1.3}$$

Proof. — This follows from (1.1) and uniformly bounded distortion for all the branches of $f^{-n}, n \geq m$ on U^j involved. □

Lemma 1.4. — *For every $x \in \partial A$ there exists $\alpha \in \Sigma^d$ such that $b(\alpha)$ is convergent to x.*

Proof. — Notice that $x = \lim z_{n_k}(\hat{\alpha}^k)$ for a sequence $\hat{\alpha}^k \in \Sigma^d$ and a sequence of integers n_k, see [PZ, proof of (9)]. Now any α a limit of a convergent subsequence of $\hat{\alpha}^k$ satisfies the assertion of the Lemma. The convergence of $b(\alpha)$ is even exponential. This follows from Lemma 1.3 □

Lemma 1.5. — *Let A be a Hölder immediate basin of attraction to a sink for a rational map f. Then for every $\lambda : 1 < \lambda < \lambda_{\mathrm{Ho}}$ there exist $\delta > 0$ and $n_0 > 0$ such that for every $n \geq n_0$ and every $x \in \partial A$, if for every $j = 0, ..., n-1$*

$$\mathrm{dist}(x_j, \mathrm{Crit}) > \exp -\delta n, \tag{1.4}$$

then $|(f^n)'(x)| > \lambda^n$.

Proof. — Consider $\alpha \in \Sigma^d$ such that $b(\alpha)$ converges to x_0. Then for $s = [C_3 + n\delta/(\log \lambda_{\mathrm{Ho}})] + 1$ (the square brackets stand for the integer part), where $C_3 = (\log \frac{C_2}{\varepsilon(1-\lambda_{\mathrm{Ho}})})/\log \lambda_{\mathrm{Ho}}$, for an arbitrary $\varepsilon : 0 < \varepsilon < 1$, one obtains by (1.3)

$$z_s(\sigma^n(\alpha)) \subset B(x_n, C_2\lambda_{\mathrm{Ho}}^{-s}/(1-\lambda_{\mathrm{Ho}}^{-1}) = B(x_n, \varepsilon \exp -\delta n).$$

Moreover for every $0 \leq j \leq n$

$$z_{s+j}(\sigma^{n-j}(\alpha)) \in B(x_{n-j}, \varepsilon \exp -\delta n). \tag{1.5}$$

For $y := z_{s+n}(\alpha)$ we have

$$|(f^n)'(y)| = |(f^{n+s})'(y)| \cdot |(f^s)'(f^n(y))|^{-1} \geq C_1 \lambda_{\mathrm{Ho}}^{n+s} L^{-s}$$

for $L := \sup |f'|$.

Using the definition of s we see that for δ small enough and n large, the latter expression is larger than $\tilde{\lambda}^n$ for an arbitrary $\tilde{\lambda} : 1 < \tilde{\lambda} < \lambda_{\mathrm{Ho}}$, so

$$|(f^n)'(y)| > \tilde{\lambda}^n. \tag{1.6}$$

For ε small enough, in view of (1.4) and (1.5), we can replace y by x in (1.5), changing $\tilde{\lambda}$ by a factor arbitrarily close to 1. □

Definition 1.6. — Let X be an f-forward invariant set. We say that f on X satisfies the property *exponential shrinking of components* if there exist $\xi : 0 < \xi < 1$ and $r > 0$ such that for every $x \in X$ and positive integer n the component of $f^{-n}(B(f^n(x), r))$ containing x has diameter bounded by ξ^n.

Lemma 1.7. — *The property A is Hölder implies the property: exponential shrinking of components, for f on ∂A, with ξ arbitrarily close to $\lambda_{\mathrm{Ho}}^{-1}$.*

In the proof we shall use the following fact, a variant of the *telescope lemma* [P1, Lemma 5]:

Lemma 1.8. — *Let X be a compact set in $\overline{\mathbb{C}}$ and $f : U \to \overline{\mathbb{C}}$ be a holomorphic map on a neighbourhood of X such that $f(X) = X$. Then $(\exists C > 0)(\forall \mu > 1)(\exists \eta > 0)$ such that for every $x \in X$ and positive integer $n > 0$, for every $r > 0$, the disc $B := B(x_n, r)$ and every compact connected set $Y \subset B$ the following holds:*

Denote $W_j := \mathrm{Comp}_{x_{n-j}}(f^{-j}(B))$ for $j = 0, ..., n$. Let Y_n be an arbitrary component of $f^{-n}(Y)$ in W_n. Assume finally that $\operatorname{diam} W_j \le \eta$ for every $j = 0, ..., n-1$. Then

$$\frac{\operatorname{diam} W_n}{\operatorname{diam} B} \le C\mu^n \frac{\operatorname{diam} Y_n}{\operatorname{diam} Y}.$$

Proof of Lemma 1.8. — See [P1]. The idea of the proof is that if W_j is far from Crit then, denoting $Y_j = f^{n-j}(Y_n)$,

$$\frac{\operatorname{diam} W_{j+1}}{\operatorname{diam} W_j} \approx \frac{\operatorname{diam} Y_{j+1}}{\operatorname{diam} Y_j}.$$

If W_{j+1} is close to a critical point of multiplicity ν then, instead of $\approx$, the inequality $\le$ with a constant depending on ν appears on the right hand side. These cases however happen rarely as long as $\operatorname{diam} W_j$ are small. ☐

Proof of Lemma 1.7. — [1] Fix an arbitrary $n > 0$ and $x \in \partial A$. By Lemma 1.4 we can find $\alpha \in \Sigma^d$ such that $b(\alpha) \to x$. By the continuity of f for every $0 \le j \le n$ we have $b(\sigma^j(\alpha)) \to x_j$. Let $m(r)$ be the largest integer such that $\gamma_{m(r)}(\sigma^n(\alpha))$ intersects ∂B for $B := B(x_n, r)$. Denote by b' the curve in $b_{m(r)-1}(\sigma^n(\alpha))$ contained in B and joining ∂B to x_n. Denote by W_j the component of $f^{-j}(B)$ containing x_{n-j}. Denote finally by b'_j the component of $f^{-j}(b')$ contained in $b_{m(r)-1+j}(\sigma^{n-j}(\alpha))$. By Lemma 1.3 we have

$$\operatorname{diam} b'_j \le \mathrm{Const}\lambda_{\mathrm{Ho}}^{-(m(r)+j)}.$$

So, using Lemma 1.8 and due to $\operatorname{diam} b'$ comparable with $\operatorname{diam} B$, we obtain

$$\operatorname{diam} W_j \le \mu^j \,\mathrm{Const}\lambda_{\mathrm{Ho}}^{-(m(r)+j)}. \tag{1.7}$$

for $\mu > 1$ arbitrarily close to 1, as long as all $\operatorname{diam} W_i$ for all $i < j$ are small. Observe however that if $r \to 0$ then $m(r) \to \infty$ (more precisely $m(r) \ge \frac{\log(1/r)}{\log L} - \mathrm{Const}$. So if r is small enough that $\lambda_{\mathrm{Ho}}^{-m(r)}$ compensates Const, $\operatorname{diam} W_i$ are small and (1.7) holds by induction. ☐

Lemma 1.9. — *Let A be a Hölder domain. Then for every $\vartheta : 0 < \vartheta < \frac{\log \lambda_{\mathrm{Ho}}}{\log L}$ there exists $r(\vartheta) > 0$ such that for every $x \in \partial A$ and $a \le r(\vartheta)$ such that for $W = \mathrm{Comp}_x f^{-n}(B(x_n, a))$*

$$\operatorname{diam} W \le a^{\vartheta}.$$

Proof. — Set $s = [\frac{\log(r/a)}{\log L}]$. Let a be small enough that $s > 0$. We have chosen r and ξ according to Definition 1.6 and Lemma 1.7.

[1] A different proof, for polynomials – using *puzzles*, was obtained jointly by the author and Jacques Carette.

As L is a Lipschitz constant for f, we obtain for $B' := \mathrm{Comp}_{x_n} f^{-s}(B(x_{n+s}, r))$

$$B' \supset B(x_n, a).$$

By Lemma 1.7 diam $\mathrm{Comp}_x f^{-n}(B') \leq \xi^{n+s}$, hence

$$\operatorname{diam} W \leq \xi^{n+s} \leq \xi^s.$$

By $s > \frac{\log(r/a)}{\log L} - 1$ we obtain

$$\operatorname{diam} W \leq (\frac{r}{L})^{-\vartheta} a^{\vartheta} \quad \text{for} \quad \vartheta = \frac{\log(1/\xi)}{\log L}.$$

□

2. A technical lemma

Lemma 2.1. — *For every $\nu \geq 2$ there exist $\varepsilon_1 : 0 < \varepsilon_1 < 1/2$ such that the following holds:*

Write $g(z) = g_u(z) = z^\nu + u$ for an arbitrary u with $|u| < 1$. Consider any $\Phi : g^{-1}(\mathbb{D}) \to \overline{\mathbb{C}}$ univalent and in the spherical metric $\operatorname{diam} \Phi(g^{-1}(\mathbb{D})) \leq \operatorname{diam} \frac{1}{2}\overline{\mathbb{C}}$ *. Here $\mathbb{D}$ is the unit disc in $\mathbb{C}$, considered later with the euclidean metric. Write $F := \Phi^{-1}$ with the domain $\Phi(g^{-1}(\mathbb{D}))$. Assume*

$$u \in \Phi(g^{-1}(\mathbb{D})). \tag{2.1}$$

Moreover assume

$$|u| < \varepsilon_1 \quad \text{and} \quad |(g \circ F)(u)| < \varepsilon_1. \tag{2.2}$$

Then either

$$\mathrm{cl}\ \Phi(g^{-1}(\frac{1}{2}\mathbb{D})) \subset \frac{1}{2}\mathbb{D}, \tag{2.3}$$

or there exists $\varepsilon_2 : 0 < \varepsilon_2 < 1$ such that

$$\Phi(g^{-1}(\varepsilon_2 \mathbb{D})) \supset \varepsilon_2 \mathbb{D}. \tag{2.4}$$

Proof. — Suppose there exist $u_n \searrow 0$ and univalent Φ_n on $g_{u_n}^{-1}(\mathbb{D})$ satisfying (2.1) such that $g_{u_n}(F_n(u_n)) \to 0$ and both (2.3) and (2.4) fail. Then starting from some n the distortion of Φ_n on $V_n := g_{u_n}^{-1}(\frac{1}{2}\mathbb{D})$ is bounded by a constant Q. A reason for this, is for example the existence of a definite geometric annulus in $g_{u_n}^{-1}(\mathbb{D}) \setminus V_n$.

The sequence of the domains V_n converges in Carathéodory's sense, [McM, 5.1], to $V := (\frac{1}{2})^{1/\nu}\mathbb{D}$ and, as all diameters of $\Phi_n(V_n)$ are uniformly bounded by $\frac{1}{2}\operatorname{diam} \overline{\mathbb{C}}$, and one can choose from (Φ_n, V_n) a subsequence convergent to certain (Φ, V).

Now notice that by (2.2) $\Phi(0) = 0$, in particular $0 \in \Phi(V)$. On the other hand by the failure of (2.3) we obtain $\mathrm{cl}\, \Phi(V) \not\subset \frac{1}{2}\mathbb{D}$. Hence $\operatorname{diam} \Phi(V) \geq 1/2$. Hence $\sup_V |\Phi'| \geq \frac{1}{2} 2^{1/\nu}$. So $\inf_V |\Phi'| \geq Q^{-1} 2^{1/\nu - 1}$. A result is that for every $r : 0 < r \leq (\frac{1}{2})^{1/\nu}$ the set $\Phi(r\mathbb{D})$ contains the disc of radius $Q^{-1} 2^{1/\nu - 1} r$ centered at 0.

Thus if $Q^{-1} 2^{1/\nu - 1} \tau^{1/\nu} > \tau$, or after rewriting:

$$\tau < \frac{1}{2} Q^{-\nu/(\nu-1)}, \tag{2.5}$$

we obtain for $g(z) = z^\nu$ the inclusion cl $\Phi(g^{-1}(\tau\mathbb{D})) \supset \tau\mathbb{D}$. This implies the analogous inclusion for n large, what contradicts the assumption that (2.4) fails. $\square$

Remark 2.2. — One could compute $\varepsilon_1, \varepsilon_2$ explicitly, however we have chosen above a more lazy way. In particular ε_2 can be chosen independent of Ψ, i.e. the statement of the Lemma could start with: $(\forall\nu)(\exists\varepsilon_1,\varepsilon_2)$....

3. Hölder implies CE

An important role will be played by the following variant of a lemma proved in [DPU, Lemma 2.3 and (3.2)]

Lemma 3.1. — *Let X be a compact set in $\overline{\mathbb{C}}$ and $f : U \to \overline{\mathbb{C}}$ be a holomorphic map on U a neighbourhood of X such that $f(X) = X$. Fix $c \in \mathrm{Crit}(f) \cap X$. Assume that there is no periodic orbit in X attracting the point c.*

For every $y \in X$ write $k(y) = \max(0, -\log \mathrm{dist}(y, c)$. For $y = c$ write $k(y) = \infty$.

Then there exists a constant C_f such that for each $x \in X$ and $n \geq 1$

$$\sum_{j=0}^{n}{}' k(x_j) \leq nC_f, \tag{3.1}$$

where $\sum'$ denotes summation over all but at most one index j at which $k(x_j)$ is maximal, (∞ is also possible).

Proof. — To proceed as in [DPU] extend f to $\overline{\mathbb{C}}$ in a differentiable way. The observation used in [DPU] is that $f^n(U) \subset U$ for U small intersecting X is not possible. In case f is a rational map and X contained in Julia set, the family f^{jn} on U for $j = 1, 2, ...$ would be normal, what contradicts a property of Julia set. In general case we use also the additional property of f^n on U if (3.1) fails: $|(f^n)'| < 1/2$. This yields an attracting periodic orbit in X, what, together with the property $U \ni c$ also following from the construction in [DPU], contradicts the assumptions. $\square$

Definition 3.2. — We call A *regular* for f if for every small $r > 0$, for every $x \in \partial A$, every positive integer n and every component W of $f^{-n}(B(x, r))$ if $W \cap A \neq \emptyset$ then $f^n(W \cap A) = B(x, r) \cap A$.

The notion of *regular* is introduced *ad hoc* because we do not know how to prove our main theorem below without assuming this. Of course if A is completely invariant, i.e. $f^{-1}(A) = A$, then A is regular.

The reader will see that in the proof instead of $B = B(x, r)$ for all r it is sufficient to consider B boundedly distorted in many scales. To have such B satisfying Definition 3.2 it is sufficient to assume that A is Hölder and Jordan. The idea is that if B is large and $B \cap A$ is connected, then for pullbacks W_j (components of $f^{-j}(B)$) $W_j \cap A$ are also connected. If a critical value is met in ∂A then only one component of $f^{-1}(W_j \cap A) \cap W_{j+1}$ can intersect A. Otherwise their boundaries would be glued at a critical point, contradicting Jordan property. Bounded distortion and many scales are due to TCE property (see Sec.4).

Theorem 3.3. — *Let A be a Hölder immediate basin of attraction to a periodic sink for a rational map f. Assume A to be regular. Let $c \in \partial A$ be a critical point whose closure of the forward orbit is disjoint from* $\mathrm{Crit} \setminus \{c\}$. *Then c satisfies (CE), with λ arbitrarily close to λ_{Ho}.*

Corollary 3.4. — *Let f be a polynomial and A_∞ be Hölder. Suppose there is only one critical point in $J(f)$. Then f is CE, with λ arbitrarily close to λ_{Ho}.*

Proof of Theorem 3.3.. — The proof uses the procedure of the "reversed telescope" invented by Graczyk and Smirnov [GS, Appendix] to prove that CE2 (plus the so-called R-expansion property) implies CE. CE2 means $|(f^n)'(y)| \geq \mathrm{Const}\lambda^n$ for every $y \in J$ and n such that n is the smallest positive integer for which $f^n(y) \in \mathrm{Crit}$, $\lambda > 1$. Here instead of CE2 we shall use the definition of Hölder domain, the property (1.1).

Step 1. The block preceding the telescope. Fix an arbitrary, large, n. Let $0 \leq m \leq n$ be the last time $\mathrm{dist}(x_m, \mathrm{Crit}) \leq \exp -n\delta\varepsilon$ for an arbitrary constant $\varepsilon : 0 < \varepsilon < 1$ and for δ from Lemma 1.5. Here $x = x_0 := c$. (We use the symbol x for c to distinguish the trajectory c_n of c from c it passes by.) A critical point c' such that $\mathrm{dist}(x_m, c') \leq \exp -n\delta\varepsilon$ must be c, supposed that n is large enough that $\exp -n\delta\varepsilon < \mathrm{dist}(O^+(c), \mathrm{Crit} \setminus (\{c\}))$, where $O^+(c)$ stands for the forward orbit of c (c included).

1. If $n - m - 1 \geq \varepsilon n$ then $|(f^{n-m-1})'(x_{m+1})| \geq \lambda^{n-m-1}$ by Lemma 1.5.
2. If $n - m - 1 < \varepsilon n$ then by Lemma 3.1 we have $\sum_{m<j<n} k(x_j) \leq (n-m-1)C_f$ the function k considered with respect to c. Hence $|(f^{n-m-1})'(x_{m+1})| \geq \exp\big(-(n-m-1)C'_f\big)$ for a constant $C'_f > 0$.

Notice that by $k(x_m) > k(x_j)$ for every $j : m < j \leq n$, there is no need to exclude an exceptional j from the above sum, because the exceptional index in $\sum_{m \leq j < n}$ might be only $j = m$.

Suppose we know that there exists a constant $\lambda > 1$ such that

$$|(f^m)'(x_1)| \geq \lambda^m \tag{3.2}$$

Then in the case 1. (CE) for c is proved. In the case 2. we obtain

$$|(f^n)'(x_1)| \geq \lambda^m \exp\big(-(n-m-1)C'_f\big) \geq \tilde{\lambda}^n$$

for $1 < \tilde{\lambda} < \lambda$ with $\tilde{\lambda}$ arbitrarily close to λ for ε appropriately small, in particular we also obtain (CE).

Thus, we need to prove (3.2), provided

$$\mathrm{dist}(x_m, c) \leq \exp -n\delta\varepsilon \leq \exp -m\delta\varepsilon. \tag{3.3}$$

We shall prove this with an arbitrary $\lambda : 1 < \lambda < \lambda_{\mathrm{Ho}}$ and for m large enough. More precisely, we shall prove (3.2) with the lower bound $\mathrm{Const}\lambda_{\mathrm{Ho}}^m$ where Const depends only on δ, ε.

Note that n large implies m large by the first inequality of (3.3) (c cannot too soon approach itself).

Step 2. Telescope: the first tube. Define first some constants. Let $T = [2(\delta\varepsilon\vartheta)^{-1}C_f] + 2$ for ϑ from Lemma 1.9. Let $C_4 = (\frac{1}{2}\varepsilon_1)^{-T}$ for ε_1 from Lemma 2.1.

Consider now $B := B(x_{m+1}, C_4 \mathrm{dist}(x_{m+1}, c_1)$. For every $j = 0, 1, ...$ define $W_j := \mathrm{Comp}_{x_{m+1-j}} f^{-j}(B)$.

Fix $j = j_0$ the first time W_{j+1} intersects Crit, at c' say. This can happen only with $c' = c$. Indeed, otherwise, using Lemma 1.9, we obtain

$$\mathrm{dist}(O^+(c), \mathrm{Crit} \setminus \{c\}) \le \mathrm{dist}(c_m, c') < \mathrm{diam}\, W_{j+1} \le C_4 \exp -m\delta\varepsilon\vartheta,$$

what for m large enough is not possible.

We conclude with $W_{j+1} \ni c$.

We have two cases:

Case 1^o $f^j(c_1) \notin \frac{1}{2}\varepsilon_1 B$;

Case 2^o $f^j(c_1) \in \frac{1}{2}\varepsilon_1 B$.

Consider the case 2^o. (Then we call $f^j : W_j \to B$ the first tube of our telescope.) In appropriate charts, in particular for B identified to $\mathbb{D}$, we can decompose f^j into $g \circ F$, the decomposition in the language of Lemma 2.1, where g corresponds to the $z \mapsto z^\nu + u$ part of f and F takes care of the rest, in particular it includes f^{j-1}.

We have also $c_1 \in \frac{1}{2}\varepsilon_1 B$ because $\frac{1}{2}\varepsilon_1 > C_4^{-1}$. Finally $\mathrm{diam}\, W_j \le \frac{1}{2}\mathrm{diam}\, \hat{\mathbb{C}}$ by Lemma 1.9. Thus we can apply Lemma 2.1. We multiplied c_1 by $\frac{1}{2}$ because here we consider the spherical metric whereas in Sec.2 we considered discs $\tau\mathbb{D}$ with respect to the euclidean metric on $\mathbb{C}$.

We obtain two possibilities:

1. The closure of $W'_j := \mathrm{Comp}_{x_{m+1-j}} f^{-j}(\tau_0 B)$ is contained in $\tau_0 B$.

τ_0 replaces here $\frac{1}{2}$ resulting from the difference between the spherical metric on B and the euclidean on $\mathbb{D}$.

2. There exists $\varepsilon_2 : 0 < \varepsilon_2 < 1$ such that

$$W''_j := \mathrm{Comp}_{x_{m+1-j}} f^{-j}(\varepsilon_2 B) \supset \varepsilon_2 B \tag{3.4}$$

(Here is an explanation of the existence of $\varepsilon_1, \varepsilon_2$ that yield this alternative, for a reader who does not wish to decipher Section 2: If 1. does not hold we shrink B to $\varepsilon_2 B$ so that $\mathrm{diam}\, W''_j >> \mathrm{diam}\, \varepsilon_2 B$ and consider ε_1 small enough that $f^j(c_1)$ is close to x_{m+1}, hence c_1 is a "center" of the boundedly distorted W''_j. ε_1 small means also that c_1 is close to the center of $\varepsilon_2 B$. This gives (3.4).)

Notice now that (3.4) contradicts $x_{m+1-j} \in J(f)$. Thus, we can suppose that

$$\mathrm{cl}\, W'_j \subset \tau_0 B.$$

Step 3. The capture of expansion. $f^j : W'_j \to \tau_0 B$ is polynomial-like, hence W'_j contains an f^j-fixed point p. In the case f is a polynomial (Corollary 3.4) $A = A_\infty$ is completely invariant, hence $p \in \partial A$ of course. Hence if we had UHPer on ∂A we would obtain for a constant $\tilde{\lambda} > 1$

$$|(f^j)'(p)| \geq \tilde{\lambda}^j. \tag{3.5}$$

(We shall come back to this discussion in Section 4. In particular UHPer on ∂A will be deduced from A Hölder, with $\tilde{\lambda} = \lambda_{\rm Ho}$.)

In the general case we do not know whether $p \in \partial A$, unfortunately. So, instead, we use the assumption A is regular. Denote f^j by F. As W'_j itersects ∂A at x_{m+1-j}, it intersects also A at, say, y^1. Write $F(y^1) = y^0$. Since y^0 has an F-preimage in $W'_j \cap A$, by the regularity of A also y^1 has an F-preimage y^2 in $W'_j \cap A$, next y^2 has, etc. . Hence $|(F^i)'(y^i)| \geq R\lambda_{\rm Ho}^{ji} = (R^{1/i}\lambda_{\rm Ho}^j)^i$, where R is a constant dependent on $\operatorname{diam} B$. (If $\operatorname{diam} B$ is small then R is large. It arises from the ratio of derivatives of f^{-k} at y^0 and z in Def.1.2., resulting from a distortion bound.) Hence for an arbitrary $\tilde{\lambda} < \lambda_{\rm Ho}$ one can take i large enough and find $s : 0 \leq s < i$ such that

$$|(f^j)'(p)| \geq \tilde{\lambda}^j \quad \text{for} \quad p = F^s(y^i). \tag{3.5$'$}$$

By construction the distortion of f^{j-1} on W'_j is bounded by a constant. Hence we have

$$|(f^{j-1})'(p)|/|(f^{j-1})'(x_{m+1-j})| \leq \text{Const}.$$

We have also $|f'(f^{j-1}(p))|/|f'(x_m)| \leq \text{Const} C_4^{(\nu-1)/\nu}$ for ν the multiplicity of f at c.

(We use the assumption that c_1 is peripheric in B with the factor C_4, i.e. $\operatorname{dist}(c_1, x_{m+1}) \geq C_4^{-1} \operatorname{diam} B$. Notice that earlier, in Step 2., we used the opposite inequality, that c_1 is close to the center.)

We conclude using (3.5) or (3.5') that

$$|(f^j)'(x_{m+1-j})| \geq \lambda_{\rm Ho}^j \text{Const} C_4^{(\nu-1)/\nu}. \tag{3.6}$$

Step 4. Longer first tube. Consider now the case 1^o. In this case instead of taking W_j for $j = j_0 + 1, ...$ we replace B by $\frac{1}{2}\varepsilon_1 B$ in the definition. Denote the resulting sets by $\hat{W}_j$. We stop at $j = j_1$ such that for the first time $\hat{W}_{j+1} \ni c$. Either we have the case 2^o now or again the case 1^o in which we continue with preimages of $\varepsilon_1^2 B$, etc. . Notice that we finally arrive at the case 2^o because if we stop at $j = m$ we have $x_{m+1-j} = c_1$.

The conclusion (3.6), in the case 2^o, ending this procedure at some j_t, holds if $(\frac{1}{2}\varepsilon_1)^{t+1} < C_4^{-1}$.

We have fortunately, by Lemma 1.9, (compare Step 2.)

$$\operatorname{diam} \hat{W}_j \leq C_4^{\vartheta} \exp -m\delta\varepsilon\vartheta.$$

Hence, by Lemma 3.1,

$$t \leq C_f(m+1)(-\vartheta \log C_4 + m\delta\varepsilon\vartheta)^{-1}$$

that is less than $T-1$ defined in Step 2. if m is large enough. The estimate $(\frac{1}{2}\varepsilon_1)^{t+1} > C_4^{-1}$ follows now from the definition of C_4.

Step 5. The number of tubes. Conclusion. Thus, we have (3.6) for $j = j_t$. Denote this integer by k_1. We consider now $B := B(x_{m+1-k_1}, C_4 \mathrm{dist}(x_{m+1-k_1}, c_1))$ and repeat the above construction. We obtain the inequality (3.6) for x_{m+1-k_2} instead of x_{m+1-k_1} and for $j = k_2$. We continue until $\sum_{i=1}^{I} k_i = m$. We have constructed a reversed telescope [GS, Appendix]. Setting $\gamma := \mathrm{Const} C_4^{(\nu-1)/\nu}$ we conclude with

$$|(f^m)'(c_1)| \geq \gamma^I \lambda_{\mathrm{Ho}}^m. \tag{3.7}$$

Indeed, at each step $\gamma\tilde{\lambda}^{k_i} >> 1$ for m large enough, because k_i is large, c cannot too soon approach itself. So, by (3.6), using bounded distortion for the appropriate branch of $f^{-(k_i-1)}$ on the appropriate $B' = \mathrm{Comp} f^{-1}(\frac{1}{2}\varepsilon_1 B)$, we obtain

$$... < \mathrm{dist}(x_{m-k_1-k_2}, c) < \mathrm{dist}(x_{m-k_1}, c) < \exp -m\delta\varepsilon, \tag{3.8}$$

resulting from the related inequalities concerning $\mathrm{dist}(x_{m-k_1-...-k_i+1}, c_1)$.

Formally, (3.8), the construction of the i-th tube and k_i large, are proved alternately by induction over i. In particular $m - k_1 - ... - k_i \geq k_I$ for $i < I$ is also large. Here $\mathrm{dist}(x_{m-k_1-...-k_i}, c) \leq \exp -m\delta\varepsilon \leq \exp -(m - k_1 - ... - k_i)\delta\varepsilon$ replaces (3.3)

By Lemma 3.1 we obtain a bound for I:

$$I \leq (\delta\varepsilon)^{-1} C_f + 1.$$

Thus (3.7) yields (3.2), with λ arbitrarily close to λ_{Ho} for m large enough. Therefore as noticed at the beginning we obtain (CE).

□

It is possible to prove Theorem 3.3 without refering to Lemmas 1.7, 1.9[(2)]. The method relies on the following fact (weaker than Lemma 1.9)

Lemma 3.5. — *For A Hölder, there exists $C > 0$ such that for every $Q \geq 1, r > 0$, positive integer n and $x \in \partial A$, for B a topological disc of diameter r boundedly distorted around $f^n(x)$, say containing $B(x_n, r/4)$, for W the component of $f^{-n}(B(x_n, r))$ containing x, if $f^n|_W$ is univalent and the distortion of f^{-n} on $B = B(x_n, r)$ is bounded by Q (namely $\sup |(f^{-n})'| / \inf |(f^{-n})'| \leq Q$), then*

$$\mathrm{diam}\, W \leq CQr^{\vartheta}, \quad \textit{for} \quad \vartheta = \log \lambda_{\mathrm{Ho}} / \log L.$$

Proof. — Consider $\alpha \in \Sigma^d$ such that $b(\alpha)$ converges to x. W contains a round disc centered at x of diameter equal to $Q^{-1} \cdot \frac{1}{4} \mathrm{diam}\, W$. By Lemma 1.3 $\lambda_{\mathrm{Ho}}^{-t} < \mathrm{Const} \cdot Q^{-1} \mathrm{diam}\, W$, in particular $t = [(\mathrm{Const} + \log Q + \log(-\mathrm{diam}\, W)) / \log \lambda_{\mathrm{Ho}}] + 1$, implies $z_t(\alpha) \in W$. Hence $z_{t-n}(\sigma^n(\alpha)) \in B$. So $t - n \geq (\log \frac{1}{r}) / \log L - \mathrm{Const}$.

Now $t \geq t - n$ implies

$$[(\mathrm{Const} + \log Q + \log(-\mathrm{diam}\, W)) / \log \lambda_{\mathrm{Ho}}] + 1 \geq (\log \frac{1}{r}) / \log L - \mathrm{Const},$$

(2) We followed that way in the first distributed version of the paper.

hence after exponentiating the both sides, $\operatorname{diam} W \le \mathrm{Const} Q r^{\vartheta}$. $\square$

Now, in Proof of Theorem 3.3, in Step 2, one can define W_j with the use of the "shrinking neighbourhoods" procedure, see [P2, Sec.2]:

For $B := B(x_{m+1}, C_4 \mathrm{dist}(x_{m+1}, c_1))$, for every $j = 0, 1, \ldots$ write $B_{[j} := \beta_j B$ for $\beta_j = \Pi_{i=1}^{j}(1 - b_i)$ for $b_i := \exp -\kappa i$ for an arbitrary $\kappa > 0$, close to 0.

Write $B' = \mathrm{Comp}_{x_m} f^{-1}(B)$ and $B'_{[j} = \mathrm{Comp}_{x_m} f^{-1}(B_{[j})$. Finally define $W_j := \mathrm{Comp}_{x_{m+1-j}} f^{-(j-1)}(B'_{[j})$

For $j \le j_0 - 1$ we have by construction f^{j-1} univalent on $f(W_{j+1})$ with distortion bounded by $\exp(-\mathrm{Const}\kappa j)$ (using Koebe distortion theorem). Hence $\operatorname{diam} W_j$ can be estimated due to Lemma 3.5 by $\mathrm{Const}\, r^{\vartheta}$ for ϑ arbitrarily close to $\log \lambda_{\mathrm{Ho}} / \log L$.

We do the same trick in Step 4. in the definition of $\hat{W}_j$.

4. TCE rational maps and the topological invariance of CE

In this section we shall provide a new proof of the theorem proved first in [PR2], that CE is a topological condition provided the following holds:

Condition (*). — For every exposed critical point $c \in J(f)$ it holds $\mathrm{cl} \bigcup_{n>0} f^n(c) \cap (\mathrm{Crit} \setminus \{c\}) = \emptyset$.

In other words for no critical point in $J(f)$ its ω-limit set contains another critical point. This condition was already present in Theorem 3.3. (Recall that *exposed* means the forward trajectory of c does not meet other critical points.)

Let us introduce some properties of a rational map f related to CE and to *exponential shrinking of components*, compare Def. 1.6. (We follow the numeration and terminology from [NP].)

2^o (*exponential shrinking of components*) There exist $0 < \xi_2 < 1$ and $r_2 > 0$ such that for every $x \in J$ every $n > 0$ and $W = \mathrm{Comp}_x f^{-n}(B(f^n(x), r_2))$ one has $\operatorname{diam} W \le \xi_2^n$.

3^o (*exponential shrinking of components at critical points*) The same as above, but only for W containing a critical point.

4^o (*finite criticality* or *topological Collet-Eckmann*, abbr. TCE) There exist $M > 0, P > 1$ and $r > 0$ such that for every $x \in J$ there exists an increasing sequence of positive integers $n_j, j = 1, 2, \ldots$ such that $n_j \le Pj$ and for each j

$$\sharp\{i : 0 \le i < n_j, \mathrm{Comp}_{f^i(x)} f^{-(n_j - i)} B(f^{n_j}(x), r) \cap \mathrm{Crit}\} \le M.$$

5^o (*mean exponential shrinking of components*) There exist $P > 1, 0 < \xi_5 < 1$ and $r_5 > 0$ such that for every $x \in J$ there exists an increasing sequence of positive integers $n_j = n_j(x), j = 1, 2, \ldots$ such that $n_j \le Pj$ and for each j one has
$\operatorname{diam} \mathrm{Comp}_x f^{-n_j}(B(f^{n_j}(x), r_5)) \le \xi_5^{n_j}$.

Another interesting condition is *uniform hyperbolicity on periodic orbits* (abbr: UHPer): There exists $\lambda_{\mathrm{Per}} > 1$ such that every periodic $p \in J(f)$ satisfies

$$|(f^k)'(p)| \geq \lambda_{\mathrm{Per}}^k.$$

where k is a period of p.

Formally we do not restrict 3^o to critical points in J, but this condition implies there are no critical points outside J attracted to J (which is equivalent to the absence of parabolic periodic orbits).

Notice that 4^o is a topological condition, i.e. if it is satisfied by f and there exists a homeomorphism h from a neighbourhood of $J(f)$ to a neighbourhood of $J(g)$ such that $h(J(f)) = J(g)$ and $hf = gh$ then 4^o holds also for g.

The implications CE$\Rightarrow$ $2^o \Rightarrow 3^o \Rightarrow 4^o$ have been proven in [PR1] (see also [NP]). $4^o \Rightarrow 5^o$ has also been proven in [PR1]. Here we shall prove $5^o \Rightarrow 2^o$ and next $2^o \Rightarrow$CE provided (*). Thus we shall prove:

Theorem 4.1. — *If f is topological Collet-Eckmann and satisfies (*) (a particular case is that there is only one critical point in J), then f is CE.*

In view of the above discussion we shall obtain

Corollary 4.2. — *(CE & (*)) is a topological property.*

Remark that it is straightforward to prove $5^o \Rightarrow$ UHPer, see Lemma 4.7. below. Unfortunately we cannot prove UHPer$\Rightarrow$ CE, to mimic the interval case [NP], [NS].

Let us start the proofs with $5^o \Rightarrow 2^o$ which is surprisingly easy.

Lemma 4.3. — *Mean exponential shrinking of components implies exponential shrinking of components.*

Proof. — Fix an arbitrary $x \in J(f)$ and $n \geq 0$. Write $B_j := B(x_j, r_5)$ for $j = 0, 1, ..., n$. Set $t_0 := 0$ and define the increasing sequence of integers $0 < t_1 < t_2, ...$ by induction as follows: Given t_i take t_{i+1} such that $t_i + \frac{n-t_i}{2P} \leq t_{i+1} \leq n$ and for

$$K_{i+1} := \mathrm{Comp}_{x_{t_i}} f^{-(t_{i+1}-t_i)}(B_{t_{i+1}}),$$

$$\operatorname{diam} K_{i+1} \leq \xi_5^{t_{i+1}-t_i}. \tag{4.1}$$

This is possible by the definitions of the constants in 5^o. 5^o implies that the number of $n_j = n_j(x_{t_i})$'s not exceeding $m = Pk$ is at least k for every $k = 1, 2, ...$ So for every $m \geq 0$ we obtain $\sharp\{n_j : n_j \leq m\} \geq [m/P]$ (the integer part of m/P). In particular for $m \geq 2P$ we obtain $\sharp\{n_j : n_j \leq m\} \geq m/2P$, hence $\{n_j : m/2P \leq n_j \leq m\} \neq \emptyset$. Finally apply this to $m = n - t_i$ and choose as t_{i+1} any n_j from the latter nonempty set.

If $\xi_5^{t_{i+1}-t_i} \leq r_5$, i.e.

$$t_{i+1} - t_i \geq \frac{\log r_5}{\log \xi_5} \tag{4.2}$$

then by (4.1)

$$K_{i+1} \subset B_{x_{t_i}}. \tag{4.3}$$

Suppose $i = I$ is the smallest integer such that either $n - t_I < 2P$ so we may not find t_{I+1} satisfying (4.1), or (4.2) does not hold. The latter: $t_{I+1} - t_I < \log r_5 / \log \xi_5$, together with $t_{I+1} - t_I \geq (n - t_I)/2P$ imply $n - t_I \leq 2P \log r_5 / \log \xi_5$. Denote the maximum of this constant and $2P$ by C.

Due to (4.3) for every $i = 1, ..., I-1$ we have a "telescope" so we obtain

$$\mathrm{Comp}_{x_{t_1}} f^{-(t_I - t_1)}(B_{t_I}) \subset B_{t_1}.$$

Hence, applying also (4.1) for $i = 0$,

$$\operatorname{diam} \mathrm{Comp}_x(f^{-t_I}(B_{t_I})) \leq \operatorname{diam} \mathrm{Comp}_x f^{-t_1}(B_{t_1}) \leq \xi_5^{t_1} \leq \xi_5^{n/2P} \tag{4.4}$$

provided $n \geq 2P$ (otherwise $I = 0$, i.e. there is no t_1).

Finally, due to $n - t_I$ bounded by C , we can replace in (4.4) B_{t_I} by $\mathrm{Comp}_{x_{t_I}} f^{-(n-t_I)}(B(x_n, r_2))$ for a constant r_2 small enough. We conclude with

$$\operatorname{diam} \mathrm{Comp}_x f^{-n}(B(x_n, r_2)) \leq \xi_5^{n/2P}$$

which proves 2^o with $\xi_2 = \xi_5^{1/2P}$. The case $n < 2P$ is trivial, r_2 small enough does the
job. □

Lemma 4.4. — *Assume 2^o. Then there exist $\vartheta : 0 < \vartheta < 1$ such that for every $a > 0$ small enough, for every $x \in J(f)$ and every $n \geq 0$*

$$\operatorname{diam} \mathrm{Comp}_x f^{-n}(B(f^n(x), a)) \leq a^{\vartheta}. \tag{4.5}$$

Proof. — See Proof of Lemma 1.9. □

Analogously to Lemma 1.5 we have the following

Lemma 4.5. — *Assume 2^o. Then there exist $\lambda > 1$, $\delta > 0$ and an integer $n_0 > 0$ such that for every $n \geq n_0$ and $x \in J(f)$, if for every $j = 0, ..., n-1$*

$$\operatorname{dist}(x_j, \mathrm{Crit}) > \exp -\delta n,$$

then $|(f^n)'(x)| > \lambda^n$.

Remark 4.6. — This Lemma will be proved with λ arbitrarily close to ξ_2^{-1}. Notice that this, for ∂A rather than $J(f)$, together with Lemma 1.7, give a new proof of Lemma 1.5.

Proof of Lemma 4.5. — Consider arbitrary $\delta, \varepsilon > 0$. Let
$$s := [\frac{\log \varepsilon}{\log \xi_2} + \frac{\delta n}{-\log \xi_2}] + 1.$$
Then, by 2^o, for all $0 \le j \le n$

$$\text{diam Comp}_{x_{n-j}} f^{-s-j}(B(x_{n+s}, r_2)) \le \xi_2^{s+j} \le \xi_2^s \le \varepsilon \exp -\delta n. \tag{4.6}$$

Now let $B = B(x_n, r_2 \exp -\delta M n)$ for $M = [\frac{\log L}{-\log \xi_2}] + 1$. Then for n large enough we obtain, using (4.6) for $j = 0$ and the definition of s,
$$\text{Comp}_{x_n} f^{-s}(B(x_{n+s}, r_2)) \supset B.$$
Let $W_n = \text{Comp}_x f^{-n}(B)$. Then there exists $y \in W_n$ such that
$$|(f^n)'(y)| \ge \frac{\text{diam}\, B}{\text{diam}\, W_n} \ge (2r_2 \exp -\delta M n)\xi_2^n,$$
where the second inequality follows from 2^o. Now, as in Proof of Lemma 1.5, for ε small, with the use of (4.6), we can switch from y to x, hence $|(f^n)'(x)| \ge \lambda^n$, for λ arbitrarily close to ξ_2^{-1} if δ, ε are appropriately small. □

Lemma 4.7. — *2^o implies UHPer on $J(f)$. Moreover $\lambda_{\text{Per}} = \xi_2^{-1}$.*

Proof. — For each periodic point $x \in J(f)$, with $f^k(x) = x$, we consider the backward trajectory $x^j = f^{Nk-j}(x)$, N such that $Nk - j > 0$. By 2^o, for j large enough, W_j are so small that shrinking of W_j is comparable to decreasing of derivatives of the respective branches of f^{-j} (critical points are far away, so there is almost no distortion). Moreover we obtain $|(f^{Nk})'(x)| \ge \text{Const}\xi_2^{-Nk}$ for every positive N with the same Const, that implies $|(f^k)'(x)| \ge \xi_2^{-k}$. Compare [NP, section 2]. □

Proof of Theorem 4.1. — It is sufficient to prove that 2^o and (*) imply (CE) for every exposed critical point in J. The proof is the same as the proof of Theorem 3.3. Having obtained a mapping corresponding to the polynomial-like $f^j|_{W_j'} \to \tau_0 B$ we find a periodic point p and we refer to UHPer, compare (3.5). (We do not have the harder case (3.5')).

□

5. TCE repellers

Call a pair (X, f) a *holomorphic invariant set* if $X \subset \overline{\mathbb{C}}$ is compact and $f : X \to \overline{\mathbb{C}}$ is defined on a neighbourhood of X and $f(X) = X$. (Recall that in Lemmas 1.8 and 3.1 we considered already such pairs.) We say that holomorphic invariant sets (X, f) and (Y, g) are *topologically conjugate* if there exist neighbourhoods U_X, U_Y of X, Y respectively, and a homeomorphism $h : U_X \to U_V$ such that $hf = gh$. Recall that a property of holomorphic invariant sets is called *topological* if for every (X, f) and (Y, g) topologically conjugate, if (X, f) satisfies this property then (Y, g) satisfies this too. Sometimes we restrict the space of holomorphic invariant sets under consideration to those that satisfy certain property (not necessarily topological, for example to those f's that extend to rational functions).

For example it is easy to see (and is well-known) that the *expanding property* (namely $|(f^k)'| > 1$ for a positive integer k), is topological. An argument is that expanding is equivalent to 4^o with n_j being the sequence of all positive integers and $M = 0$, that is of course a topological condition.

Here we consider (X, f) with properties weaker than expanding, namely with: $2^o - 5^o$ with J replaced by X, W intersecting X and f not necessarily extendable to a rational function on the Riemann sphere.

One can define also CE as (CE) for every exposed $c \in W$ for W intersecting X.

We call a holomorphic invariant set (X, f) a *holomorphic repeller* if there exists a neighbourhood V of X in the domain of f such that $(\forall x \in V \setminus X)(\exists n > 0)$ such that $f^n(x) \notin V$.

We have the following

Proposition 5.1. — *For (X, f) holomorphic invariant sets, $5^o \Rightarrow 2^o \Rightarrow 3^o \Rightarrow 4^o$. Moreover, for (X, f) holomorphic repellers, or holomorphic invariant sets such that f extends to a rational function and $X \subset J(f)$, the properties 2^o, 3^o, 4^o and 5^o are equivalent. Then all of them are topological properties, CE implies each of $2^o - 5^o$ and conversely, provided (*) from Section 4.*

Proof of Proposition 5.1. — As mentioned in Sec.4 the proof of $3^o \Rightarrow 4^o$ has been done in [PR1, Lemma 2.2] for f rational ($X = J$ has been considered there, but for X strictly in J the proof is the same). For (X, f) an arbitrary holomorphic invariant set we need to refer to [DPU] as it is stated here in Lemma 3.1. The assumptions are satisfied: if there existed a periodic point $p \in X$ whose periodic orbit attracts a critical point $c \in X$, then for $x = c$ the condition 2^o would not hold.

As mentioned in Sec.4 the implication $4^o \Rightarrow 5^o$ for f rational has been proven in [PR1]. For (X, f) holomorphic repeller we refer to the following fact proved in [PR2, Appendix]:

If $X \neq \hat{\mathbb{C}}$ then 4^o implies that X is nowhere dense.

Now repeating [PR1, (2.6)] one uses the repelling property to know that all the maps $f^n : W \to B$, for every $B(x, r), x \in X$, r small and every W a component of $f^{-n}(B)$ intersecting X, are proper. This is needed in the proof that if f^n have uniformly bounded criticalities on W, then the respective preimages of $\frac{1}{2}B$ have diameters shrinking to 0 as $n \to \infty$.

To prove the latter fact we find a little disc D in $\frac{1}{2}B \setminus X$, so that the components W' of f^n-preimages of D in W have diameters shrinking to 0. Such D exists due to X nowhere dense and $W' \to X$. Finally we use a bounded distortion lemma in a bounded criticality setting (for example [PR1, Lemma 2.1].

$5^o \Rightarrow 2^o$ is automatic, see Lemma 4.3. The proof that CE implies 4^o is the same as in [PR1] and the proof of the opposite implication is the same as in Section 4. ☐

We call a holomorphic repeller satisfying any of the properties $2^o - 5^o$ a *topological Collet-Eckmann repeller*, abbr. TCE repeller.

Proposition 5.2. — *Let $X = \partial A$ for a connected open domain $A \subset \overline{\mathbb{C}}$. Let f be a holomorphic map defined on a neighbourhood U of X such that $f(U \cap A) \subset A$, $f(X) = X$ Then A Hölder implies (X, f) is TCE (i.e. it satisfies 4^o). Coversely, if (X, f) is TCE and additionally it is a repeller or f extends to a rational map on $\overline{\mathbb{C}}$, then A is Hölder.*

Proof. — Assume that A is Hölder. Definition 1.2 is still valid except that one considers only $z \in A$ close to ∂A. Observe now that Hölder implies 2^o. The proof is similar to Proof of Lemma 1.7., except that in this situation one needs *Markov geometric coding tree*. Instead of one point z as in Sec.1, choose a finite family $Z \subset A$ in a small Hausdorff distance from the whole X and join each $z^j \in f^{-1}(Z)$ by a curve γ^j to a point in Z, so that γ^j is close to ∂A, in particular in the domain of f^{-1}.

Next 2^o implies TCE by Proposition 5.1.

In the opposite direction TCE implies 2^o by Proposition 5.1. Next we prove that A is Hölder: Consider a disc $B := B(x, r)$ for $x \in X$ such that $\operatorname{diam} W \le \xi^n$ for W components of $f^{-n}(B)$ intersecting X (compare property 2^o) and consider $D = B(z, \delta) \subset A \cap B$. Let W' be a component of $f^{-n}(D)$ in W. Hence $\operatorname{diam} W' \le \xi^n$ so by bounded distortion $|(f^n)'(y)| \ge \mathrm{Const}\xi^{-n}$ for $y \in W'$, $f^n(y) = z$. Compare [GS, Sec.5] and [PR1, Sec.3]. □

We obtain an immediate

Corollary 5.3. — *(Rigidity of Hölder domains.) Let A be a connected open domain $A \subset \overline{\mathbb{C}}$. Let f be a holomorphic map defined on a neighbourhood U of ∂A such that $f(U \cap A) \subset A$, $f(\partial A) = \partial A$. Suppose A is Hölder. Let g be a holomorphic map on a neighbourhood of a compact set $Y \subset \overline{\mathbb{C}}$ such that f on a neighbourhood of $X = \partial A$ is conjugated by a homeorphism h to g and $h(X) = Y$. Assume that g extends to a rational function on $\overline{\mathbb{C}}$ or assume that (X, f) (hence (Y, g)) are repellers. Then the component of $\overline{\mathbb{C}} \setminus Y$ intersecting $h(A)$ is Hölder.*

Let us underline that we allow above critical points in X to be in the ω-limit set of other critical points. Proposition 5.2 and Corollary 5.3 are much easier than the corresponding Theorem 3.3 and Corollary 4.2.

Remark finally that in between *holomorphic expanding repellers* (i.e. holomorphic repellers with expanding property) and *holomorphic TCE repellers* there lies the class of *holomorphic semihyperbolic repellers*, that is satisfying the property 4^o with n_j being the sequence of all positive integers. Semihyperbolicity is of course a topological condition.

Notice that this semihyperbolicity is equivalent to 2^o with all $f^n|_W$ of uniformly bounded criticality. Notice also that for compact nowhere dense repellers semihyperbolicity is equivalent to the assumption that critical points in X are nonrecurrent, see [CJY]. Parabolic points cannot happen for repellers.

If $X = \partial A$ for A a basin of a sink, I believe that f semihyperbolic is equivalent to A John. This has been proven in the case $A = A_\infty$ for polynomial f in [CJY].

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