ON HAUSDORFF DIMENSION OF POLYNOMIAL NOT TOTALLY DISCONNECTED JULIA SETS

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Abstract. We prove that for every polynomial of one complex variable of degree at least 2 and Julia set not being totally disconnected nor a circle, nor interval, Hausdorff dimension of this Julia set is larger than 1. Till now this was known only in the connected Julia set case.

We give also an (easy) example of a polynomial with non-connected Julia set and all non one-point components being analytic arcs, thus contradicting Ch. Bishop's conjecture that such components must have Hausdorff dimension larger than 1.

1. Introduction

Christopher Bishop in [1], commenting his result on the existence of an entire transcendental function with one-dimensional Julia set, asked the following question:

The connected components of the Julia set constructed in this paper are all either points or continua of Hausdorff dimension one.(...) However, the situation for polynomials is open. If a polynomial Julia set is connected, then it is either a generalized circle/segment or has Hausdorff dimension strictly greater than 1 (this follows from work of Zdunik [51] and Przytycki [36]). Is this also true of the non-trivial connected components when the Julia set is disconnected? In other words, if J (p) is disconnected, is every connected component either a point or a set of Hausdorff dimension strictly greater than 1?

Let us note that a similar question was asked in [11]. In this article, we answer the above questions and related ones.

We start with the following.

Theorem 1. *Let $f:\mathbb{C}\to\mathbb{C}$ be a polynomial of degree $d\geq 2$. If the Julia set $J(f)$ is disconnected then every non one-point connected component J' of $J(f)$ is eventually periodic, i.e. $f^\ell(J')$ is periodic with period k for some integers ℓ,k. Furthermore, either $f^\ell(J')$ is an analytically embedded interval (i.e. analytic arc), and f^k on it is analytically conjugate to $\pm$ Chebyshev polynomial, or J' has Hausdorff dimension greater than 1.*

This theorem is proved in Section 2. In Section 3 we complete the picture, including possible analytic components and proving the following.

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Theorem 2. *Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$, such that $J(f)$ is not totally disconnected. Assume also that f is not complex affine conjugate to a map $z \mapsto z^d$ or a $\pm$ Chebyshev polynomial. Then $\mathrm{HD}(J(f)) > 1$. Even more, the hyperbolic dimension, $\mathrm{HD}_{\mathrm{hyp}}(J(f))$ is larger than 1.*

So, if $J(f)$ is disconnected but it has a non-trivial connected component (that is, not one-point component), then $\mathrm{HD}_{\mathrm{hyp}}(J(f)) > 1$, even if the dimension of every non-trivial component is equal to one. In the latter case the dimension larger than 1 is achieved due to a Cantor set of other components. To prove it, we follow a strategy by Irene Inoquio-Renteria and Juan Rivera-Letelier in [6], where they proved that geometric pressure $P(t)$ is larger than t times Lyapunov exponent of any probability invariant measure on Julia set of a rational function, provided the exponent is positive. See Proposition 4 for the definition of the geometric pressure and Remark 12.5 in [14].

Theorem 2 strengthens the result of [17], where the analogous theorem was proved for polynomials with connected Julia set.

Recall that hyperbolic dimension of the Julia set $J(f)$, analogously of any other f-invariant subset of $J(f)$, $\mathrm{HD}_{\mathrm{hyp}}(J(f))$ is defined as supremum of Hausdorff dimensions of isolated hyperbolic forward invariant subsets of $J(f)$. Thus, the inequality

$$\mathrm{HD}(J(f)) \geq \mathrm{HD}_{\mathrm{hyp}}(J(f))$$

holds trivially. See [14] for the discussion on equivalent definitions of the hyperbolic dimension.

Finally, we show in Section 4 that the situation we deal with in the proof of Theorem 2 really may happen: in Proposition 10 we provide an example of a family of polynomials of degree 3 with non-trivial components of the Julia set $J(f)$, each of them being an analytically embedded interval.

So, strictly speaking, the answer to Ch. Bishop's question is negative.

2. Repelling boundary domains and proof of Theorem 1

Choose R so large that

$$\overline{f^{-1}(\mathbb{D}(0,R))} \subset \mathbb{D}(0,R). \tag{1}$$

Let C be a connected component of the filled-in Julia set $K(f)$. Denote by $U_n(C)$ the unique connected component of $f^{-n}(\mathbb{D}(0,R))$ containing C. The boundary $\partial U_n(C)$ is disjoint from $K(f)$ since $\partial\mathbb{D}(0,R)$ is, and $f(K(f)) = K(f)$. Notice also that by (1)

$$\overline{U_{n+1}(C)} \subset U_n(C). \tag{2}$$

Finally notice that all U_n are simply-connected (topological discs) by Maximum Principle.

Define $C' := \bigcap_{n=0}^{\infty} U_n(C)$. By $U_n(C) \supset C$ we get $C' \supset C$. On the other hand $C' = \bigcap_{n=0}^{\infty} \overline{U_n(C)}$ the intersection of a decreasing sequence of compact connected sets, hence it is connected, hence as consisting from non-escaping points, i.e. contained in $K(f)$, a subset of C. We conclude with

$$C = \bigcap_{n=0}^{\infty} U_n(C).$$

One fact pivotal for our paper is

Theorem A (Qiu & Yin [9], Kozlovski & van Strien [7])**.** *For a polynomial f of degree at least 2 if a component of filled-in Julia set $K(f)$ is not a point, then its forward orbit contains a periodic component containing a critical point.*

In the proof of Theorem 2 we shall use in fact a weaker version, that if the Julia set of a polynomial is not totally disconnected then there exists a periodic not one-point critical periodic component of $K(f)$.

Theorem A was proved for degree 3 polynomials by Branner and Hubbard [2] and independently by Yoccoz, but for higher degree polynomials stayed unproved for a long time.

Consider any non-trivial component of $K(f)$. We can replace it by a periodic component C which is its image under an iterate of f, i.e., $f^k(C) = C$ for some $k \geq 1$. Since replacing f by f^k does not change the Julia set, we may assume and we do assume from now on that $f(C) = C$.

Thus, consider any C being a component of $K(f)$ such that $f(C) = C$. Notice that for n arbitrary $F := f|_{U_{n+1}(C)}$ maps $U_{n+1}(C)$ onto $U_n(C)$ since $f(C) = C$. So since F is proper and due to (2), it is polynomial-like in the sense of [4].

Lemma 3. *If $C = f(C)$ is a non-trivial component of $K(f)$, then for each n large enough F defined above is polynomial-like with C being its filled-in Julia set, that is $C = \bigcap_{k=0}^{\infty} F^{-k}(U_n(C))$ namely the set of points whose forward trajectories do not escape from $U_{n+1}(C)$. Moreover degree of F is at least 2.*

Proof. Let us write here U_n for $U_n(C)$ for all n. We claim that for n large enough U_{n+1} is the only connected component of $f^{-1}(U_n)$ contained in U_n. Indeed, if there is another such component in U_n then there is a critical point of f in $W := U_n \setminus \overline{f^{-1}(U_n)}$. Otherwise $f : W \to U_{n-1} \setminus \overline{U_n} = W'$ would be a covering map of W to an annulus W'. This contradicts the equality $\chi(W) = \deg(f|_W)\chi(W')$ for Euler's characteristics since $\chi(W)$ is negative and $\chi(W') = 0$. Since the number of critical points of f is finite, the claim follows.

Finally, if degree of F were 1, then the moduli of all $U_n \setminus U_{n+1}$ would be positive equal to each other so C would be a point, contradiction. □

Below we recall the definition of RB-domain, which was introduced in [15].

Definition (RB-domain)**.** *Let Ω be a simply connected domain with $\#(\mathbb{C} \setminus \Omega) > 2$. Assume there exists a holomorphic map defined on a neighborhood U of $\partial\Omega$ such that*

$$f(U \cap \Omega) \subset \Omega, \quad f(\partial\Omega) = \partial\Omega \quad \textit{and} \quad \bigcap_{k=0}^{\infty} f^{-k}(U \cap \overline{\Omega}) = \partial\Omega.$$

Then Ω is called a repelling boundary domain (RB-domain).

Observation: The domain $\Omega := \hat{\mathbb{C}} \setminus C$ together with f restricted to $U := U_{n+1}(C)$ with n sufficiently large, as in Lemma 3, is a *repelling boundary domain (RB–domain)*. Indeed, by this Lemma, $f|_{U_n}^{-1}(C) = C$ which yields $f(U \cap \Omega) \subset \Omega$.

Now we rely on:

Theorem B (See [11] and [16])**.** *If Ω is an RB-domain then either* $\mathrm{HD}_{\mathrm{hyp}}(\partial\Omega) > 1$ *or $\partial\Omega$ is an analytic Jordan curve or an analytically embedded interval, where f is topologically conjugate to z^d or to a $\pm$ Chebyshev polynomial, respectively.*

Theorem B was stated as a conjecture in [15]. A comparison of harmonic measure ω on $\partial\Omega$ to $\mathcal{H}^1$ which is the Hausdorff measure in dimension 1, is the key. For ω absolutely continuous it is was proved in [16] that $\partial\Omega$ were analytic. It was also claimed in [16] that for ω singular $\mathrm{HD}(\partial\Omega) > 1$ can be proved by adapting the methods of [17]. A detailed proof of this appeared in [11]. Analogously to [17], the proof not only showed that $\mathrm{HD}(\partial\Omega) > 1$, but, actually, the hyperbolic dimension of f restricted to $\partial\Omega$ was larger than 1. For a survey see [13].

Combining the above Observation and Theorem B, concludes the proof of Theorem 1, except it does not exclude C being a Jordan domain with an analytic Jordan boundary. In such a case use as Ω the bounded (internal) component Ω' of $C \setminus \partial C$ rather than the external one $\hat{\mathbb{C}} \setminus C$ above. It is forward invariant for f (as we replaced f by its adequate iterate). Moreover, as repelling to the side of Ω, ∂C is repelling to the side of Ω' by symmetry, i.e. Ω' is an RB-domain, so the immediate basin of attraction to a sink. So ∂C is a circle and $f(z) = z^d$ in some complex affine coordinates by [3, Lemma 9.1]. See also [11, Theorem A] or [13, Theorem 8.1]. This contradicts the assumption that $J(f)$ is disconnected.[1]

3. Proof of Theorem 2

We shall use a version of Bowen's formula, which can be found in [10], see also [14, Section 12.5], for a strengthened version.

Proposition 4. *Let f be a rational map of degree $d \geq 2$. There exists an exceptional set $E \subset \hat{\mathbb{C}}$ of Hausdorff dimension 0, such that for every $t \geq 0$ (even every real t) and for every $z \notin E$ the limit*

$$P(t, f; z) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{v \in f^{-n}(z)} \frac{1}{|(f^n)'(v)|^t}$$

exists, and is independent of $z \in \hat{\mathbb{C}} \setminus E$. Denote the common value as $P(t, f)$. It is called the geometric pressure. The function $t \mapsto P(t, f)$ is continuous and non-increasing. Moreover $P(0, f)$ is positive (equal to $\log d > 0$, the topological entropy). The following formula holds:

$$\mathrm{HD}_{\mathrm{hyp}}(J(f)) = \inf\{t > 0 : P(t, f) \leq 0\}.$$

In words, $\mathrm{HD}_{\mathrm{hyp}}(J(f))$ is the first zero of $P(t, f)$.

In view of Theorem 1, to prove Theorem 2 it is sufficient to prove the following

Proposition 5. *Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$ with disconnected Julia set. Suppose there exists a periodic connected component C of $K(f)$ being an analytic arc. Then* $\mathrm{HD}_{\mathrm{hyp}}(J(f)) > 1$.

[1] We are grateful to Fei Yang for paying our attention to this absence of analytic Jordan curves.

Proof. We shall prove that $P(1, f) > 0$, which implies, by Proposition 4, that $\mathrm{HD}_{\mathrm{hyp}}(J(f)) > 1$. Consider $F = f : U_{m+1}(C) \to U_m(C)$, with some fixed m large enough, so that F is a polynomial-like map and C its Julia set, see Lemma 3 (there m was denoted n).

For $x \in U_m \setminus C$ denote

$$L_n(F, x) := \log \sum_{F^n(y)=x} \exp(-\log|(F^n)'(y)|) = \log \sum_{F^n(y)=x} |(F^n)'(y)|^{-1}. \tag{3}$$

To continue the proof of Proposition 5 we need two lemmas, which we formulate and prove below. (Lemma 6 and Lemma 7)

Lemma 6. *There exists $C_0 > 0$ such that for all $x \in U_m \setminus C$ and $n > 0$*

$$L_n(F, x) \geq -C_0. \tag{4}$$

Proof. Let $\Omega = \mathbb{C} \setminus C$. We know from the proof of Theorem 1 that Ω is an RB-domain. Recall that $C = \partial\Omega$ is an analytically embedded interval.

We recall the final step of the proof (see [11] and [16]): knowing that C is an analytically embedded interval, with endpoints, say $-1, 1$, we consider the ramified covering map Π onto $\mathbb{C}$, ramified over -1 and 1: $\Pi(z) = \frac{1}{2}(z + \frac{1}{z})$ and the preimage $\gamma = \Pi^{-1}(C)$. Then γ is an analytic Jordan curve, as proved in [16].[2] The curve γ divides the sphere into two disc $\mathcal{D}_1$, $\mathcal{D}_2$, and $\Pi(\mathcal{D}_1) = \Pi(\mathcal{D}_2) = \hat{\mathbb{C}} \setminus C$. The map $F : U_{m+1} \setminus C \to U_m \setminus C$ can be lifted to a holomorphic map G defined on the set $\Pi^{-1}(U_{m+1} \setminus C)$ in a way that for $W_1 := \mathcal{D}_1 \cap \Pi^{-1}(U_{m+1} \setminus C))$, $G(W_1) \subset \mathcal{D}_1$.

By Carathéodory's Theorem (local version), see e.g. [5, Ch.II.3, Theorem 4'], and Schwarz reflection principle, we can extend G holomorphically from $\mathcal{D}_1$ to $\hat{G}$ acting on a neighbourhood of γ. It is expanding on γ because of uniform convergence of $\hat{G}^{-n}$ on a neighbourhood of γ to γ which is a nowhere dense set, which implies that limit functions are constant, hence uniform convergence of derivatives $(\hat{G}^{-n})'$ to 0, by normality, see e.g. [12, Section 7] or [3, Proof of Theorem 6.5]. The domain $\mathcal{D}_1$ bounded by γ is an RB-domain for the action of $\hat{G}$.

Here is a different argument for the existence of the extension of G from W_1 beyond γ: Consider lift of F on say U_{m+1} to G on $W := \Pi^{-1}(U_{m+1})$, a neighbourhood of γ, for the branched covering Π. It can be defined by

$$G := \Pi^{-1} \circ F \circ \Pi. \tag{5}$$

Formally, first define $G : W_1 \to \mathcal{D}_1$ as above. Fix an arbitrary $z \in W_1$ and denote $w = G(z)$. Next extend G along curves in W starting from z mapped to w, using (5), the curves omitting

$$A := \Pi^{-1}(\mathrm{Crit}(F) \cup \{-1, 1\}),$$

where $\mathrm{Crit}(F)$ is the set of F-critical points. Notice that A is precisely the set mapped by $F \circ \Pi$ to 1 or -1, the critical values of Π, namely where Π^{-1} has singularities. This is so because F is topologically conjugate to $\pm$ Chebyshev polynomial. So all the singularities of G defined by (5) are holomorphically removable.

[2]The property of a closed continuous arc joining -1 and 1, that Jordan curve being its preimage under Π is analytic, can be assumed to be a definition of the analyticity of the arc (called therefore an analytic arc or an analytically embedded interval).

Notice finally that W is a topological annulus, with the fundamental group generated by the homotopy class of any curve γ' in W_1 starting and ending at z running once along W_1. The growth of the prolongation G along γ' is 0, because it starts and ends at w. It is so because $\gamma' \subset W_1$, where G has been well-defined.

So G being an analytic continuation along curves is a well-defined (single-valued) holomorphic function on W.

In fact this G coincides with the extension $\hat{G}$ found before, since these maps are holomorphic and coincide in $\mathcal{D}_1$.

Let now $x \in U_m \setminus C$. Our aim is to estimate $L_n(F, x)$. Denote by w the unique preimage of x under Π in $\mathcal{D}_1$. The map Π gives a bijection between the set $\{v \in G^{-n}(w)\}$ and $\{y \in F^{-n}(x)\}$. Putting $y = \Pi(v)$ we obtain:

$$(F^n)'(y) = (G^n)'(v) \cdot \frac{\Pi'(w)}{\Pi'(v)}$$

and, consequently,

$$\sum_{y \in F^{-n}(x)} \frac{1}{|(F^n)'(y)|} = \frac{1}{|\Pi'(w)|} \cdot \sum_{v \in G^{-n}(w)} \frac{1}{|(G^n)'(v)|} \cdot |\Pi'(v)|. \tag{6}$$

Let us estimate the expression above, but without Π', that is let us estimate $L_n(G, w)$ (defined for G as (3) for F). First assume $w \in \gamma$. Denote by $\mathtt{Le}$ the length of curves in γ. There exists a constant $c > 0$ such that for each n and every $w \in \gamma$, choosing an arbitrary $w' \in \gamma \setminus \{w\}$, denoting by $\gamma(w)$ the open arc $\gamma \setminus \{w'\}$ we have

$$\sum_{G^n(v)=w} |(G^n)'(v)|^{-1} \geq c \cdot \sum_{G^n(v)=w} \mathtt{Le}(G_v^{-n}(\gamma(w)))/\mathtt{Le}(\gamma(w)) \geq \tag{7}$$

$$c \cdot \mathtt{Le}(\gamma)/\mathtt{Le}(\gamma) = c > 0.$$

This is due to bounded (by c from below) distortion for iterates (or by Koebe distortion lemma), see [14, Section 6.2]. The subscript v at G^{-n} means the component containing v. Due to bounded distortion $w \in \gamma$ can be replaced by an arbitrary point w in a neighbourhood of γ.

We could define so-called transfer operator (Perron-Frobenius-Ruelle) for potential $\psi := -\log |G'|$ acting on continuous φ

$$\mathcal{L}_\psi(\varphi)(w) := \sum_{G(v)=w} (\exp \psi(w)))\varphi(w).$$

Then the expression estimated in (7) could be written as $\mathcal{L}_\psi^n(\mathbb{1})$.

To continue (6) we need to estimate from below the value $\mathcal{L}_\psi^n(\varphi)(w)$ for potential $-\log |G'|$, where $\varphi = |\Pi'|$. The function φ has value zero at two points, but $\mathcal{L}_\psi^2(\varphi)$ is positive, thus bounded from below by some constant $c_1 > 0$. So,

$$\mathcal{L}_\psi^n(\varphi)(w) = \mathcal{L}_\psi^{n-2}(\mathcal{L}_\psi^2(\varphi))(w) \geq \mathcal{L}_\psi^{n-2}(c_1 \cdot \mathbb{1})(w) = c_1 \cdot \mathcal{L}_\psi^{n-2}(\mathbb{1})(w),$$

and the last term is bounded below by $c > 0$ as in (7).

Since $\frac{1}{\Pi'(w)}$ is bounded away from 0 in $\Pi^{-1}(U_m)$, since Π' is upper bounded, we are done.

$\square$

Lemma 7. *There exists an integer $N_1 > 0$ and a connected component V of $f^{-N_1}(U_m)$, such that $\overline{V} \subset U_m \setminus U_{m+1}$.*

Proof. Denote $F_n = f|_{U_n}$ for each $n \in \mathbb{N}$. Since $J(f)$ is not connected, $\deg F < \deg f$. So there exists $k : 0 \le k < m$ such that $\deg F_{m-k+1} < \deg F_{m-k}$. Hence there exists a component $U' \subset U_{m-k}$ of $f^{-1}(U_{m-k})$ in U_{m-k}, different from U_{m-k+1}.

This is so because degree of f on U_{m-k} and on full $f^{-1}(U_{m-k}) \cap U_{m-k}$ (i.e. union of all its components) must be the same.

Define $V' := (f^k|_{U_m})^{-1}(U')$. It is a subset of U_m and $f^{m+1}(V') = U_0 = \mathbb{D}(0, R)$). It is disjoint from U_{m+1}, hence bounded away from C. In consequence V, defined as an arbitrary component of $f^{-(m+1)}(U_m)$ in V', satisfies the assertions of our Lemma. We set $N_1 := m + 1$. □

Notice also that V intersects $J(f)$ because U_0 does and $J(f)$ is completely invariant for f. Moreover, V containes branched holomorphic images of C.

Having Lemmas 6 and 7 at our disposal, we continue the proof of Proposition 5. In order to simplify the notation, we pass again to the iterate of f, replacing now f by f^{N_1} and F by F^{N_1}. Recall again that this modification does not change the Julia set. To simplify further the notation, denote $U_0 := U_m(C)$, $U_1 := U_{m+N_1}(C)$. So, from now on, after this modification, the component V becomes just a component of $f^{-1}(U_0)$ in U_0 different from $U_1 = U_1(C)$. We denote $f|_V$ by F_V. This pairs up with F on U_1 denoted also as F_{U_1}.

First, we build an infinite collection of multivalued maps ϕ_ℓ, $\ell \in \mathbb{N}$, as follows: Let us note that $F_V : V \to U_0$ is a holomorphic proper map onto U_0. Since the degree of this map may be greater than one, the inverse may be not well defined. However, we shall use the notation $h : U_0 \to V$ to denote the multivalued inverse of F_V. Thus, h assigns to a point $x \in U_0$ a collection of its preimages for F_V.

Consider now a family of multivalued maps $\phi_\ell : U_0 \to U_0$, $\ell = 0, 1, \ldots$, defined as multivalued (branched) inverses $F^{-\ell} \circ h$ where $F^{-\ell} : U_0 \to U_\ell$ are multivalued holomorphic maps given by multivalued (branched) inverses of F^ℓ. Since $F^{-\ell}$ is pre-composed by h it is meaningful even when restricted to V.

So, each multivalued map ϕ_ℓ assigns to a point $x \in U_0$ some collection of its preimages under f^n, with $n = n_\ell = \ell + 1$, all of them being in U_0, even in U_ℓ.

For $x \in U_0$ and $\ell \in \mathbb{N}$ we write $\sum |\phi'_\ell(x)|$ to denote the summation of derivatives which runs over all branches of the multivalued map ϕ_ℓ. If a critical point c and its f-image are met then we can put in this sum ∞, as inverse of forward derivative 0. In fact it does not matter since we can restrict to x not post-critical (not in the forward f-trajectory of an f-critical point).

For every $\ell \in \mathbb{N}$ and $x \in U_0$ denote

$$L^*_\ell(f, x) = \log \sum |\phi'_\ell(x)|$$

where the summation runs over all branches of the multivalued map ϕ_ℓ. Denote also by $\underline{\ell}$ a sequence $\underline{\ell} = (\ell_1, \ldots \ell_k)$, and put

$$n_{\underline{\ell}} = n_{\ell_1} + \cdots + n_{\ell_k}$$

Finally, for a sequence $\underline{\ell} = (\ell_1, \ldots \ell_k)$, denote

$$L_{\underline{\ell}}(f,x) := L^*_{\ell_k}(f,y_{k-1}) + ... + L^*_{\ell_2}(f,y_1) + L^*_{\ell_1}(f,x), \tag{8}$$

where $y_1 = \phi_1(x), ..., y_{k-1} = \phi_k(y_{k-2})$ (remember that the maps ϕ_j are multivalued, so we consider sums over their values). The star * means we consider only preimages along first h and next $F^{-\ell_i}$. In other words, more formally,

$$L_{\underline{\ell}}(f,x) = \log \sum |\phi'_{\underline{\ell}}(x)|,$$

where the summation runs over all branches of the multivalued function

$$\phi_{\underline{\ell}} = \phi_{\ell_k} \circ \phi_{l_{k-1}} \circ \cdots \circ \phi_{\ell_1}.$$

It is important to note that all the multivalued functions $\phi_{\underline{\ell}}$ are mutually distinct, compare *free Iterated Function System* in [6]. Indeed, let $\underline{\ell} \neq \underline{\ell}'$, but $n_{\underline{\ell}} = n_{\underline{\ell}'}$. Let i be the first integer such that $\ell_i \neq \ell'_i$. Then, supposing that $n_{\ell_i} > n_{\ell'_i}$ we compose in $\underline{\ell}'$ after ℓ'_i with h with range in V, whereas in $\underline{\ell}$ still within ℓ_i with F^{-1} having range U_1. Further compositions by branches of f^{-1} preserve distinction.

Denote by Σ^* the set of all finite sequences $\underline{\ell} = (\ell_1, \dots \ell_k)$, $k \geq 1$. For an arbitrary $x \in U_0$ denote

$$\Lambda_N(x) = \sum_{\underline{\ell} \in \Sigma^* : n_{\underline{\ell}} = N} \exp L_{\underline{\ell}}(f,x).$$

The star * again means we consider only preimages along first h next F^{-1}.

Proposition 8. *For every $N \geq 1$ and non-exceptional, in particular not postcritical, $x \in U_0$,*

$$P(1,f;x) \geq \liminf_{N\to\infty} \frac{1}{N} \log \Lambda_N(x).$$

Proof. In view of Proposition 4 it is sufficient to prove

$$\sum_{y \in f^{-N}(x)} \frac{1}{|(f^N)'(y)|} \geq \Lambda_N(x)$$

This is however obvious, because on the left hand side all $y \in f^{-N}(x)$ appear, whereas on the right hand side only selected ones. □

Proposition 9. *Let $\Lambda_N := \inf_{x\in U_0} \Lambda_N(x)$. Then*

$$\liminf_{N\to\infty} \frac{1}{N} \log \Lambda_N > 0.$$

Proof. Put $a := \inf_{x\in U_0} \left(\sum |h'(x)|\right)$, and $b := \exp(-C_0)$ where C_0 comes from Lemma 6.

Consider all sequences $\underline{\ell} = (\ell_1, \dots \ell_k)$ such that $n_{\underline{\ell}} = N$, for each $k \leq N$. The number of such sequences can be calculated in the following way:

For each $k \leq N$, there are $\binom{N-1}{k-1}$ ways of choosing $k-1$ positions $m_1, \dots m_{k-1}$ from the set $\{1, \dots N-1\}$. Having these positions chosen, we assign to them the values $\ell_1 = m_1 - 1$, $\ell_2 = (m_2 - m_1) - 1$, $\ell_k = (N - m_{k-1}) - 1$ and the (multivalued) map $\phi_{\underline{\ell}}$ with $\underline{\ell} = (\ell_1, \dots \ell_k)$ and $n_{\underline{\ell}} = (\ell_1 + 1) + (\ell_2 + 1) + \cdots + (\ell_k + 1) = N$.

Then we have the estimate

$$\exp L_{\underline{\ell}}(f,x) \geq a^k b^k$$

and, consequently,

$$\sum_{\underline{\ell}\in\Sigma^*:n_{\underline{\ell}}=N} \exp L_{\underline{\ell}}(f,x) \geq \sum_{k=1}^{N} \binom{N-1}{k-1} (ab)^k = ab(1+ab)^{N-1}.$$

□

Remark. *Let us note that a calculation in a similar spirit appeared in* [6] *and in* [8]*. The authors used a method of generating functions (some power series with coefficients related to a value similar to Λ_N). Our case is simpler, in a sense that the set of admissible values n_ℓ forms an arithmetic sequence. So, the straightforward calculation provided above is sufficient.*

Obviously, combining Proposition 8 and Proposition 9 we conclude the proof of Proposition 5.

□

4. Example

To complete the answer to the question of Ch Bishop, it remains to ask whether there exists a polynomial with disconnected Julia set, and a connected component of $J(f)$ being an analytically embedded interval.

In the following proposition we provide a family of maps with this property.

Proposition 10. *Consider a family of cubic polynomials*

$$f_{\varepsilon,\beta}(z) = \varepsilon z^3 + z^2 - \beta, \tag{9}$$

with ε and β real. Then there exists a smooth curve Γ of parameters (ε,β), passing through the point $(0,2)$ such that for every $(\varepsilon,\beta)\in\Gamma$ the Julia set of $f_{\varepsilon,\beta}$ is disconnected, and contains infinitely many components being analytic arcs.

Proof. We start with the Chebyshev polynomial $f(z) = z^2-2$. Its Julia set is just the interval $I := [-2,2]$.

Consider now the polynomials $f_{\varepsilon,\beta}$ with ε real and close to 0, and β real and close to 2. The map $f_{\varepsilon,\beta}$ has a real repelling fixed point $p_{\varepsilon,\beta}$ close to $p_{0,2}=2$ and 0 is a (not moving) critical point of $f_{\varepsilon,\beta}$.

Lemma 11. *There exists a smooth curve Γ of parameters (ε,β), passing through the initial parameters point $(0,2)$ for which*

$$f^2_{\varepsilon,\beta}(0) = p_{\varepsilon,\beta}.$$

Proof. This is a straightforward calculation. Denoting $\gamma(\varepsilon,\beta) = f^2_{\varepsilon,\beta}(0)$, we have $\gamma(\varepsilon,\beta) = -\varepsilon\beta^3+\beta^2-\beta$, so

$$\mathrm{grad}\gamma(0,2) = [-8,3].$$

Denoting by $p(\varepsilon,\beta)$ the fixed point of $f_{\varepsilon,\beta}$ close to 2, we calculate (differentiating the implicit function)

$$\mathrm{grad}p(0,2) = [-8/3,1/3].$$

Thus,

$$\mathrm{grad}(\gamma-p)(0,2) \neq (0,0)$$

and therefore, there exists a smooth curve of parameters (ε, β), passing through the initial parameters point $(0, 2)$ for which

$$f_{\varepsilon,\beta}^2(0) = p_{\varepsilon,\beta}.$$

$\square$

Notice that $f_{\varepsilon,\beta}(0) = -\beta$. The interval $I_{\varepsilon,\beta} = [-\beta, p_{\varepsilon,\beta}]$ is thus invariant under $f_{\varepsilon,\beta}$ and the map $f : I_{\varepsilon,\beta} \to I_{\varepsilon,\beta}$ is two-to-one, with critical point at 0. Let us note that the Julia set $J(f_{\varepsilon,\beta})$ is not connected, as the trajectory of the second critical point $c = -\frac{2}{3\varepsilon}$ tends to infinity. Denote by C the connected component of $J(f_{\varepsilon,\beta})$ containing $I_{\varepsilon,\beta}$, so in particular it is fixed under $f_{\varepsilon,\beta}$ and is not one-point.

By Lemma 3, there exist connected neighbourhoods of C,

$$C \subset U_1 \subset \overline{U}_1 \subset U_0,$$

such that the map $F := (f_{\varepsilon,\beta})_{|U_1} : U_1 \to U_0$ is a polynomial-like map, and C is its filled-in Julia set. Since the Julia set of f is disconnected, the degree of F

is not maximal, so it is equal to 2. By [4] F is quasiconformally conjugate to a true quadratic polynomial. Therefore, preimages of every point in C are dense in C. But the preimages under F of points from $I_{\varepsilon,\beta}$ remain in $I_{\varepsilon,\beta}$, which implies that

$$C = I_{\varepsilon,\beta}.$$

So, each map $f_{\varepsilon,\beta}$ with $(\varepsilon, \beta) \in \Gamma$ is a polynomial of degree 3 with disconnected Julia set, for which the Julia set has an invariant component being a true interval on which the degree of the map is equal to 2. So, for each such map the filled-in Julia set, here equal to the Julia set, has a collection of countably many non-trivial components, each of them being an analytic arc; this collection is formed by the invariant analytic arc and all its preimages under the iterates of $f_{\varepsilon,\beta}$.

Note also that these are the only non-trivial components of the filled-in Julia set $K(f_{\varepsilon,\beta})$. Indeed, by Theorem A all non-trivial components are eventually periodic and by Lemma 3 every non-trivial periodic component of the filled-in Julia set has to contain a critical point in its orbit. In our situation, there are two critical points; one of them is escaping, and the other one is already contained in the invariant interval $I_{\varepsilon,\beta}$.

$\square$

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