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An improvement of J. Rivera-Letelier result on weak hyperbolicity on periodic orbits for polynomials

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Abstract. We prove that for $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ a rational mapping of the Riemann sphere of degree at least 2 and Ω a simply connected immediate basin of attraction to an attracting fixed point, if $|(f^n)'(p)| \geq Cn^{3+\xi}$ for constants $\xi > 0, C > 0$ all positive integers n and all repelling periodic points p of period n in Julia set for f, then a Riemann mapping $R : \mathbb{D} \to \Omega$ extends continuously to $\bar{\mathbb{D}}$ and $\mathrm{Fr}\Omega$ is locally connected. This improves a result proved by J. Rivera-Letelier for Ω the basin of infinity for polynomials, and $5 + \xi$ rather than $3 + \xi$.

We prove the following

Theorem 1. Let f be a polynomial of 1 complex variable of degree at least 2, with connected Julia set. Suppose there are constants $C > 0$ and $\xi > 0$ such that for every repelling periodic point p in the complex plane $\mathbb{C}$ of period n,

$$(*) \qquad |(f^n)'(p)| \geq Cn^{3+\xi}$$

Then a Riemann map $R : \bar{\mathbb{C}} \setminus \bar{\mathbb{D}} \to \bar{\mathbb{C}} \setminus K(f)$ from the complement of the closure of the unit disc $\mathbb{D}$ to the complement of the filled-in Julia set in the Riemann sphere, extends continuously to $\bar{\mathbb{C}} \setminus \mathbb{D}$. In particular Julia set is locally connected and there are no Cremer periodic orbits.

In [R] Juan Rivera-Letelier proved this under the assumption $|(f^n)'(p)| \geq Cn^{5+\xi}$.

The same strategy proves in fact a stronger theorem below, in the setting of [P2], including the case of an arbitrary simply connected basin of immediate attraction to a periodic sink for a rational map of $\bar{\mathbb{C}}$.

Theorem 2. Let f be a rational mapping on the Riemann sphere $\bar{\mathbb{C}}$ of degree at least 2 and let Ω be a simply connected immediate basin of attraction to an attracting fixed point. Suppose that (*) holds for all repelling periodic points p in Julia set for f. Then any Riemann map $R : \mathbb{D} \to \Omega$ extends continuously to $\bar{\mathbb{D}}$ and $\mathrm{Fr}\Omega$ is locally connected.

Most part of our proof of Theorems 1 and 2 follows [R]. The proof of Theorem 1 uses an invariant measure of maximal entropy. However the right measure to use in more general situations, like in Theorem 2, is an f-invariant measure ω equivalent to a harmonic

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measure on $\mathrm{Fr}\Omega$ viewed from Ω; it coincides with the measure of maximal entropy in the case of the basin of ∞ for polynomials.

In the situation of Theorem 2 there is however a technical difficulty, namely proving the existence of an expanding repeller X in $\mathrm{Fr}\Omega$, such that the topological entropy of $f|_X$ is arbitrarily close to the measure theoretic entropy $h_\omega(f)$ and X satisfying some additional technical assumptions, in consequence Hausdorff dimension $\mathrm{HD}(X)$ arbitrarily close to $\mathrm{HD}(\omega) = 1$, see Lemma 3. This fact is a strengthening of the theorem on the density of periodic points in $\mathrm{Fr}\Omega$, see [PZ]. The proof can be obtained as in [PZ] witht the use of Pesin-Katok theory and is omitted here. We devote a separate short paper [P4] to it. In the situation of Theorem 1 the existence of X is also needed in the proof, but this case is easier (see the references in [R]).

Proof of Theorem 1 (and analogously Theorem 2) reduces to checking the summability assumption in the following standard

Lemma 1, see [R]. Let $w_0 \in \mathbb{C} \setminus K(f)$ and $\omega_n, n = 1, 2, ...$ be an increasing sequence of positive real numbers such that $\sum_{n=1}^{\infty} \omega^{-n} < \infty$. If for every $w \in f^{-n}(w_0)$ we have $|(f^n)'(w)| \geq \omega_n$, then the Riemann map R extends continuously to $\bar{\mathbb{C}} \setminus \mathbb{D}$.

Definitions. Let $\lambda_n, n = 1, 2, ...$ be an increasing sequence of positive real numbers such that for every n, every repelling periodic point p of period n has the multiplier $(f^n)'(p)$ of absolute value at least λ_n. We call a closed set $X \subset J(f)$ an *expanding repeller* if $f(X) \subset X$ the map f restricted to X is open, topologically mixing and expanding, that is there exist $C > 0$ and $\lambda > 1$, called *an expanding constant*, such that for every $x \in X$ we have $|(f^n)'(x)| \geq C\lambda^n$. The property that $f|_X$ is open is equivalent to the existence of a neighbourhood U of X in $\mathbb{C}$, called a *repelling neighbourhood*, such that every forward f-trajectory $x, f(x), ...f^n(x), ...$ staying in U must be contained in X, see for example [PU1, Ch.5]. This easily implies that if $\{x, f(x), ...f^n(x)\} \subset U$ then $|(f^n)'(x)| \geq C\lambda^n$, maybe for a constant C bigger than before and U a smaller neighbourhood of X. In the sequel C will denote various positive constants which can change even in one consideration.

Lemma 2, see [R]. Let f be a polynomial of 1 complex variable of degree at least 2. Let $X \subset J(f)$ be an expanding repeller of positive Hausdorff dimension, $\mathrm{HD}(X) > 0$, and λ be its expanding constant. Then for every $\varepsilon > 0$ there is $w_0 \in \mathbb{C} \setminus K(f)$ a constant $C > 0$ and an integer $N \geq 0$ such that the following holds.

For every integer $n \geq 1$ there is an integer $\ell = \ell(n)$ satisfying $0 \leq \ell \leq (1/\mathrm{HD}(X) \ln \lambda + \varepsilon) \ln n$, and there is $x = x(n) \in f^{-\ell}(w_0)$ satisfying $x, ..., f^\ell(x) \in U$ a repelling neighbourhood of X, such that for every $z \in f^{-n}(x)$

$$|(f^{n+\ell+N})'(z)| \geq C\lambda_{n+\ell+N}.$$

Sketch of Proof. This Lemma in a slightly different formulation was proved in [R] and in a more rough version in [PRS1]. See also [PRS2, §2] and [P-Kyoto]. The idea is first to find $\hat{x} \in X$, a *safe point*, that is

$$\hat{x} \notin \Big(\bigcap_{k=1}^{\infty} \bigcup_{n \geq k} B(f^{2n}(\mathrm{Crit}(f) \cup f(\mathrm{Crit}(f))), n^{-a})\Big) \cup \bigcup_{n \geq 1} f^n(\mathrm{Crit}(f))$$

for an arbitrarily fixed $a > 1/\mathrm{HD}(X)$. The latter inequality assures the existence of $\hat{x}$. Here $\mathrm{Crit}(f)$ denotes the set of all f-critical points in $\mathbb{C}$. Let ℓ be a minimal time such that $f^\ell(B(\hat{x}, \delta n^{-a}))$ is a large boundedly distorted set B', where $0 < \delta << 1$ is a constant (this makes sense by bounded distortion for iterates of a conformal expanding map). Each branch F_n of f^{-n} on $B(\hat{x}, n^{-a})$, can be composed with branches of f^{-1} a bounded number at most N times to put $B'' := B(\hat{x}, \delta n^{-a})$ deep in B'. Then an appropriate branch of $f^{-(n+\ell+N)}$ maps B' deep in itself, so B'' contains a periodic point $p = p(F_n)$ of period $n+\ell+N$ whose multiplier can be compared with $|(f^{n+\ell+N})'(F_n(\hat{x}))|$. Finally one replaces $\hat{x} \in J(f)$ by a new point x in $B(\hat{x}, \delta n^{-a}) \setminus K(f)$ such that $R^{-1}(w_0)$ for $w_0 := f^\ell(x)$ is far from $K(f)$. A distortion bound allows to find w_0 not depending on n, proving the Lemma. QED

Proof of Theorem 1. Let X and other constants be as in Lemma 2. Let $\beta_2 \geq \beta_1 > 1$ be constants such that for all k large enough and all y such that $y, ..., f^k(y) \in U$ we have $\beta_1^k \leq |(f^k)'(y)| \leq \beta_2^k$.

Consider an arbitrary $w_n \in f^{-n}(w_0)$. Join $x = x(n)$ to w_0 by a hyperbolic geodesic $\gamma = \gamma_n$ in $\mathbb{C} \setminus K(f)$. Let x_n be the end of the component of $f^{-n}(\gamma_n)$ having one end at w_n, different from w_n. Then we write

$$|(f^n)'(w_n)| = |(f^n)'(x_n)| \frac{|(f^n)'(w_n)|}{|(f^n)'(x_n)|}.$$

By Lemma 2 we have

$$|(f^n)'(x_n)| = |(f^{n+\ell+N})'(x_n)| \cdot |(f^{\ell+N})'(x)|^{-1} \geq C\lambda_{n+\ell+N}\beta_2^{-\ell}.$$

Denote $\tilde{w}_0 = R^{-1}(w_0), \tilde{w}_n = R^{-1}(w_n), \tilde{x} = R^{-1}(x)$ and $\tilde{x}_n = R^{-1}(x_n)$.

We have

$$\frac{|(f^n)'(w_n)|}{|(f^n)'(x_n)|} = \frac{|(R^{-1} \circ f^n)'(w_n)|}{|(R^{-1} \circ f^n)'(x_n)|} \frac{|R'(\tilde{w}_0)|}{|R'(\tilde{x})|} = \frac{|(f^{-n} \circ R)'(\tilde{x})|}{|(f^{-n} \circ R)'(\tilde{w}_0)|} \cdot \frac{|R'(\tilde{w}_0)|}{|R'(\tilde{x})|} = I \cdot II,$$

where f^{-n} is the branch leading x_0 to x_n and w_0 to w_n.

Note that $|\tilde{x}| - 1 \geq Cd^{-\ell}$, where C depends only on $|\tilde{w}_0|$. We estimate the fraction I by Koebe Distortion Lemma. Namely there is a constant C_{K} depending only on $\tilde{w}_0$ such that

$$I \geq C_{\mathrm{K}}(|\tilde{x}_0| - 1) = CC_{\mathrm{K}}d^{-\ell}.$$

We have also, denoting $g(z) = z^d$, using $Rg^\ell = f^\ell R$,

$$II \geq |(f^\ell)'(x)| \cdot |(g^\ell)'(\tilde{x})|^{-1} \geq Cd^{-\ell}\beta_1^\ell.$$

In conclusion

$$|(f^n)'(w_n)| \geq C\lambda_n\beta_2^{-\ell}d^{-2\ell}\beta_1^\ell.$$

Invoking the estimate of ℓ we get

$$|(f^n)'(w_n)| \geq C\lambda_n\beta_2^{-\ell}\beta_1^\ell d^{-2(1/(\ln\lambda)\mathrm{HD}(X)+\varepsilon)\ln n} \geq C\lambda_n(\beta_1/\beta_2)^\ell \; n^{-2(1/(\ln\lambda)\mathrm{HD}(X)+\varepsilon)\ln d}$$

By Pesin-Katok theory, applied to the measure of maximal entropy, equal to $\ln d$, there exists X such that $\beta_1 \geq d-\varepsilon$ and $\beta_2 \leq d+\varepsilon$, hence $\lambda \geq d-\varepsilon$. Moreover $\mathrm{HD}(X) \geq 1-\varepsilon$. Hence

$$(1) \qquad |(f^n)'(w_n)| \geq C\lambda_n \frac{(d-\varepsilon)^\ell}{(d+\varepsilon)^\ell}\; n^{-2(\frac{1}{(\ln(d-\varepsilon))(1-\varepsilon)}+\varepsilon)\ln d} \geq \lambda_n n^{-2-\varepsilon'}$$

with ε, hence ε', arbitrarily close to 0. So, if $\lambda_n \geq Cn^{3+\xi}$ the assumptions of Lemma 1 are satisfied and Theorem 1 follows. QED

Remark 1 (corresponding to an observation in [R]). The measure of maximal entropy is optimal in this construction. If μ is any f-invariant ergodic measure on $J(f)$ of positive Lyapunov exponent $\chi_\mu(f) := \int \ln|f'|d\mu$, then $(\ln\lambda)\mathrm{HD}(X) \approx \chi_\mu(f)(\mathrm{h}_\mu(f)/\chi_\mu(f)) = \mathrm{h}_\mu(f)$, where $\mathrm{h}_\mu(f)$ is the measure-theoretic entropy. $\approx$ means that the ratio is arbitrarily close to 1 for appropriate X. Therefore $|(f^n)'(w_n)| \geq \lambda_n n^{-2\ln d/\mathrm{h}_\mu(f)-\varepsilon'}$, which attains maximum at $\mathrm{h}_\mu(f) = \mathrm{h}_{\mathrm{top}}(f) = \ln d$, the topological entropy, where we get (1).

Remark 2. The property (*) excludes an existence of parabolic periodic points in $\mathrm{Fr}\Omega$. Otherwise we would find periodic orbits spending almost all the time close to such a parabolic point q, so its multiplier would about Cn^a, where $a = t/(t-1)$ for $f^m(z) = z + b(z-p)^t + ...$ for some integer m and $b \neq 0$, for z close to q.

The absence of Cremer periodic orbits follows from the local connectedness, see [R] and the references there. We do not know whether Siegel discs can exist. The proof given in [PRS] under the assumption of the uniform exponential growth of the multipliers of repelling periodic orbits ω_n does not seem to work here. We do not know whether (*) implies a summability condition which would already imply the absence of Siegel discs and Cremer points due to so-called backward asymptotic stability, cf. [GS] or [P3, Th.B and Remark 3.2] and [PU2, Appendix B].

Now we pass to the setting of Theorem 2, where $R : I\!\!D \to \Omega$ is a Riemann mapping. Let g be a holomorphic extension of $R^{-1} \circ f \circ R$ to a neighbourhood of the unit circle $\partial I\!\!D$. It exists and it is expanding on $\partial I\!\!D$, see [P2, §7].

Now we formulate a lemma about the existence of appropriate expanding repellers. As we mentioned in Introduction it follows from Pesin-Katok theory. For the detailed proof see [P4], developing [PZ].

Lemma 3. Let ν be an ergodic g-invariant probability measure on $\partial I\!\!D$, such that for ν-a.e.$\zeta \in \partial I\!\!D$ there exists a radial limit $\hat{R}(\zeta) := \lim_{r\nearrow 1} R(r\zeta)$. Assume that the measure $\mu := \hat{R}_*(\nu)$ has positive Lyapunov exponent $\chi_\mu(f)$. Let $\varphi : \partial I\!\!D \to I\!\!R$ be a continuous real-valued function. Then for every $\varepsilon > 0$ there exist $Y \subset \partial I\!\!D$ a g-invariant expanding repeller and $C > 0$ such that for every $\delta > 0$ small enough there exists $r(\delta) < 1$, such that for all $r : r(\delta) \leq r < 1$ and $\zeta \in Y$ and all positive integers n

(i) $C^{-1}\exp n(\int \varphi\, d\nu - \varepsilon) \leq \exp\sum_{j=0}^{n-1}\varphi(g^j(\zeta)) \leq C\exp n(\int\varphi\, d\nu + \varepsilon)$.

(ii) $X = \hat{R}(Y)$ is an expanding repeller for f and for every $r : r(\delta) < r < 1$ it holds $R(r\zeta) \in B(\hat{R}(\zeta), \delta)$.

(iii) $C^{-1} \exp n(\chi_\mu(f) - \varepsilon) \le |(f^n)'(\hat{R}(\zeta))| \le C \exp n(\chi_\mu(f) + \varepsilon)$.

(iv) $\mathrm{HD}(X) \ge \mathrm{HD}(\mu) - \varepsilon$.

Proof of Theorem 2. For every $\zeta \in \partial D\!\!\!\!I$, $\alpha : 0 < \alpha < \pi/2$ and $t > 0$ denote

$$S_{\alpha,t}(\zeta) = \zeta \cdot (1 + \{x \in \mathbb{C} \setminus \{0\} : \pi - \alpha \le \mathrm{Arg}(x) \le \pi + \alpha\}, |x| < t).$$

Such a set is called Stolz angle. If we do not mind t we skip it and write $S_\alpha(\zeta)$.

By a distortion estimate for iterates of g there exist $t_0 < 1, C > 0$ and $\vartheta : 0 < \vartheta < \pi/2$ such that if for all $j = 0, 1, 2, ..., k$ it holds $1 - |g^j(r\zeta)| \le t_0$ then $g^k(r\zeta) \in S_{\vartheta, Ct_0}(g^k(\zeta))$.

Choose X, Y and all the constants as in Lemma 3, with $\varphi = \ln|g'|$ Consider an arbitrary positive integer n and choose $\hat{x} \in X, \delta > 0$ and ℓ as in Proof of Lemma 2.

Choose $y = s\hat{y}$ for $\hat{y} \in Y$ and $s : 0 < s < 1$, satisfying $\hat{R}(\hat{y}) = \hat{x}$ such that $x := R(y) \in \partial B(\hat{x}, \delta^2 n^{-a})$. (Note that in Proof of Th.1 y was denoted by $\tilde{x}$. It was chosen after x. Here we need to be more careful and first choose y.)

If δ is small enough then all points $g^j(y)$ are close to $\partial D\!\!\!\!I$ for $j = 0, ..., \ell$ by Lemma 3 (ii) since all the distances of $\hat{R}g^j(\hat{y})$ and $Rg^j(y)$ are small, smaller than $C\delta$. (This is what for we consider δ^2 in the choice of x). Otherwise there would be a sequence of points in $D\!\!\!\!I$ with the limit $z \in D\!\!\!\!I$ and $R(z) \in \mathrm{Fr}\Omega$ by the continuity of R what would contradict $R(D\!\!\!\!I) = \Omega$.

In particular $g^j(y) \in S_{\vartheta, Ct_0}(g^j(\hat{y}))$. So all the distances $|g^j(\hat{y}) - g^j(y)|$ are small hence by Lemma 3 (i) for $\zeta = \hat{y}$ and by the continuity of $\ln|g'|$ we get

$$|(g^\ell)'(\hat{R}(y))| \le C \exp \ell(\chi_\nu(f) + 2\varepsilon)$$

On the other hand the point $g^\ell(y) \in S_{\vartheta, Ct_0}(g^j(\hat{y}))$ is well inside $D\!\!\!\!I$ by Lemma 3 (ii). This follows from the assumption that $w_0 := R(g^\ell(y)) = f^\ell(x) \in f^\ell(\partial B(\hat{x}, \delta^2 n^{-a}))$ is far from $f^\ell(\hat{x})$, namely within the distance at least $C\delta$. The uniform radial continuity of $\hat{R}$ at Y in Lemma 3 (ii), has been replaced by the uniform nontangential continuity, where $z \to \zeta$ for $z \in S_\vartheta$.

Then the final estimate in Proof of Theorem 1 replaces by

$$|(f^n)'(w_n)| \ge \lambda_n n^{-2\chi_\nu(g)/\chi_\mu(f)\mathrm{HD}(\mu) - \varepsilon'}.$$

Now we apply $\mathrm{HD}(\mu) = \mathrm{h}_\mu(f)/\chi_\mu(f)$ see [PU1, Ch.9] and $\mathrm{h}_\nu(g) = \mathrm{h}_\mu(f)$, see [P1] and [P2, §4]. We get

$$|(f^n)'(w_n)| \ge \lambda_n n^{-2\chi_\nu(g)/\mathrm{h}_\mu(f) - \varepsilon'} = \lambda_n n^{-2\chi_\nu(g)/\mathrm{h}_\nu(g) - \varepsilon'} = \lambda_n n^{-2(1+\varepsilon')}$$

the latter equality for ν equivalent to length (harmonic) measure, where $\chi_\nu(g)/\mathrm{h}_\nu(g) = \mathrm{HD}(\nu) = 1$. QED

Remark 3. As in Remark 1 note that the measure ν equivalent to the length is optimal in the sense that for any other g-invariant probability measure of positive Lyapunov exponent (which implies that $\mu = \hat{R}_*(\nu)$ also has positive Lyapunov exponent, see [P2]), as $\mathrm{HD}(\nu) \le \mathrm{HD}(\partial \mathbb{D}) = 1$, we obtain $|(f^n)'(w_n)| \ge \lambda_n n^{-2(\mathrm{HD}(\nu)+\varepsilon')}$, the estimate which is not better.

Remark 4. It would be natural to prove a local version of Theorem 2, in the setting of [P2], assuming (*) only for periodic orbits in $\mathrm{Fr}\Omega$. More precisely the question is whether the following holds:

Let Ω be a simply connected domain in $\bar{\mathbb{C}}$ and f be a holomorphic map defined on a neighbourhood W of $\mathrm{Fr}\Omega$ to $\bar{\mathbb{C}}$. Assume $f(W \cap \Omega) \subset \Omega$, $f(\mathrm{Fr}\Omega) \subset \mathrm{Fr}\Omega$ and $\mathrm{Fr}\Omega$ repells to the side of Ω, that is $\bigcap_{n=0}^{\infty} f^{-n}(W \cap \bar{\Omega}) = \mathrm{Fr}\Omega$. Suppose that (*) holds for all repelling periodic points p in $\mathrm{Fr}\Omega$. Then any Riemann map $R : \mathbb{D} \to \Omega$ extends continuously to $\bar{\mathbb{D}}$ and $\mathrm{Fr}\Omega$ is locally connected.

We do not know how to overcome troubles with finding N consecutive branches of f^{-1} whose composition maps $F_n(B'')$ deep in B' (in the notation in Proof of Lemma 2). Even if we succeed we do not know whether the periodic point p belongs to $\mathrm{Fr}\Omega$. The problem is that we want to control every backward branch of f^{-n} leading x into Ω, rather than measure typical as in [PZ], or in accordance to some invariant hyperbolic subset of $\mathrm{Fr}\Omega$.

Note that Lemma 3 holds in this setting, see [P4].

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