THERMODYNAMIC FORMALISM METHODS
IN THE THEORY OF ITERATION OF MAPPINGS
IN DIMENSION ONE, REAL AND COMPLEX

FELIKS PRZYTYCKI

The XIII Annual Lecture dedicated to the memory of Professor Andrzej Lasota

CONTENTS

1. Introduction

In equilibrium statistical physics, originated by Boltzmann (1877) and Gibbs (1902), the Ising model of ferromagnetism is considered. Let Ω be the *configuration space* of functions $\mathbb{Z}^n \to A$ on the integer lattice $\mathbb{Z}^n$ with interacting values in A over its sites, e.g. "spin" values $+$ or $-$, assigning the resulting energy (potential) for each configuration. One considers probability distributions on Ω, invariant under translation, called *equilibrium states* depending of this potential functions on Ω and on "temperature".

In 1960/70 Yakov Sinai, David Ruelle and Rufus Bowen applied this theory to investigate invariant sets in dynamics distributing measures on them, see [29], [28] and [1].

Let us start with the following important

LEMMA 1.1 (Finite variational principle). *For given real numbers $\phi_1, \ldots, \phi_d$, the function*

$$F(p_1, \ldots p_d) := \underbrace{\sum_{i=1}^{d} -p_i \log p_i}_{entropy} + \underbrace{\sum_{i=1}^{d} p_i \phi_i}_{average\ potential}$$

on the simplex $\{(p_1, \ldots, p_d) : p_i \geq 0, \sum_{i=1}^{d} p_i = 1\}$ attains its maximum, called pressure *or* free energy, *equal to $P(\phi) = \log \sum_{i=1}^{d} e^{\phi_i}$, at the only element of the simplex, called* equilibrium state,

$$\hat{p}_j = e^{\phi_j} / \sum_{i=1}^{d} e^{\phi_i}.$$

Hint: $\sum_{i=1}^{d} -p_i \log p_i + \sum_{i=1}^{d} p_i \phi_i = \sum_{i=1}^{d} p_i \log(e^{\phi_i}/p_i)$.

Introduction: corresponding dynamics notions

Let $f \colon X \to X$ be a continuous map for a compact metric space (X, ρ) and $\phi \colon X \to \mathbb{R}$ be a continuous function (*potential*).

DEFINITION 1.2 (Variational topological pressure).

$$P_{\mathrm{var}}(f, \phi) := \sup_{\mu \in \mathcal{M}(f)} \left(h_\mu(f) + \int_X \phi \, d\mu \right),$$

where $\mathcal{M}(f)$ is the set of all f-invariant Borel probability measures on X and $h_\mu(f)$ is measure-theoretical entropy.

Any measure where supremum above is attained is called *equilibrium state* or *equilibrium measure*. Let us recall the definition

$$h_\mu(f) := \sup_{\mathscr{A}} \lim_{n \to \infty} \frac{1}{n+1} \sum_{A \in \mathscr{A}^n} -\mu(A) \log \mu(A),$$

supremum over finite partitions $\mathscr{A}$ of X, where $\mathscr{A}^n := \bigvee_{j=0,\ldots,n} f^{-j}\mathscr{A}$.

DEFINITION 1.3 (Topological pressure via separated sets).

$$P_{\text{sep}}(f, \phi) := \lim_{\varepsilon \to 0} \overline{\lim}_{n \to \infty} \frac{1}{n} \log \Big(\sup_Y \sum_{y \in Y} \exp S_n \phi(y) \Big),$$

supremum over all $Y \subset X$ such that for distinct $x, y \in Y$, $\rho_n(x, y) := \max\{\rho(f^i(x), f^i(y)), 0 \le i \le n\} \ge \varepsilon$.

THEOREM 1.4 (Variational principle: Ruelle, Walters, Misiurewicz, Denker, ...). $P_{\text{var}}(f, \phi) = P_{\text{sep}}(f, \phi)$.

For this and related theory see e.g. [30] or [25]. In view of Theorem 1.4 we can omit subscripts and write $P(f, \phi)$.

Call $f: X \to X$ *distance expanding* if there exist $\lambda > 1, C > 0$ such that for all $x, y \in X$, sufficiently close to each other, then

$$\rho(f^n(x), f^n(y)) \ge C\lambda^n \rho(x, y) \quad \text{for all } n \in \mathbb{N}.$$

Sometimes we use the word *hyperbolic*.

Lemma 1.1 becomes in the infinite (dynamical, expanding) setting:

THEOREM 1.5 (Gibbs measure – uniform case). *Let $f: X \to X$ be a distance expanding, topologically transitive continuous open map of a compact metric space X and $\phi: X \to \mathbb{R}$ be a Hölder continuous potential. Then, there exists exactly one $\mu_\phi \in \mathcal{M}(f, X)$, called a Gibbs measure, such that for constants $C, r_0 > 0$, all $x \in X$ and all $n \in \mathbb{N}$*

$$C^{-1} < \frac{\mu_\phi(f_x^{-n}(B(f^n(x), r_0)))}{\exp(S_n\phi(x) - nP)} < C,$$

called the Gibbs property, where f_x^{-n} is the local branch of f^{-n} mapping $f^n(x)$ to x and $S_n\phi(x) := \sum_{j=0}^{n-1} \phi(f^j(x))$.

- μ_ϕ *is the unique equilibrium state for* ϕ, *and is ergodic. It is equivalent to the unique* $\exp-(\phi - P)$-*conformal measure* m_ϕ, *that is an* f-*quasi-invariant measure with Jacobian* $\exp-(\phi - P)$ *for a constant* P.
- $P = P(f, \phi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in f^{-n}(x_0)} \exp S_n\phi(x)$. *This normalizing limit exists and is equal* $P(f, \phi)$ *for every* $x \in X$.

An important example of a distance expanding map is $\varsigma\colon \Sigma^d \to \Sigma^d$, where Σ^d is the space of all sequences $(\alpha_n)_{n=0,1,\dots}$ with $\alpha_n \in \{1, \dots, d\}$, and ς is the left shift $\varsigma((\alpha_n)) = (\alpha_{n+1})$, used for 'coding' other maps, see e.g. Section 7.

2. Introduction to dimension one

Thermodynamic formalism is useful for studying properties of the underlying space X. In dimension one, for f real of class $C^{1+\varepsilon}$ or f holomorphic (conformal) for an expanding repeller X, considering $\phi = \phi_t := -t \log|f'|$ for $t \in \mathbb{R}$, the Gibbs property gives, as $\exp S_n(\phi_t) = |(f^n)'|^{-t}$,

$$\mu_{\phi_t}(f_x^{-n}(B(f^n(x), r_0))) \approx \exp(S_n\phi(x) - nP(\phi_t))$$

$$\approx (\operatorname{diam} f_x^{-n}(B(f^n(x), r_0)))^t \exp(-nP(\phi_t)).$$

The latter follows from a comparison of the diameter with the inverse of the absolute value of the derivative of f^n at x, due to *bounded distortion.*

All this is not literally true if f has critical points in X, i.e. points where the derivative f' is zero. Then the "escalator" f^n to large scale deforms shapes when passing close to critical points. Also ϕ is not Hölder at these points. Some correctness of Theorem 1.5 depends then on recurrence of critical points and on t where $1/t$ mimics temperature for $t > 0$.

When $t = t_0$ is a zero of the function $t \mapsto P(\phi_t)$, this gives (for expanding (f, X))

$$(2.1) \qquad\qquad \mu_{\phi_{t_0}}(B) \approx (\operatorname{diam} B)^{t_0}$$

for all small balls B, hence $\mathrm{HD}(X) = t_0$. Moreover, the Hausdorff measure H_{t_0} of X in this dimension is finite and nonzero.

The potentials $-t\log|f'|$, their pressure and equilibria are called *geometric* since they provide a tool for a local geometrical insight in the space.

A model application

THEOREM 2.1 (Bowen, Series, Sullivan). *For $f_c(z) := z^2 + c$ for an arbitrary complex number $c \neq 0$ sufficiently close to 0, the invariant Jordan curve J (Julia set for f_c) is fractal, i.e. has Hausdorff dimension bigger than 1.*

If $\mathrm{HD}(J) = 1$, then $0 < H_1(J) < \infty$ by Theorem 1.5 and $h = R_2^{-1} \circ R_1$ on S^1 is absolutely continuous (F. & M. Riesz' theorem), where R_i are Riemann maps, $R_1(0)$ is the f fixed point in $\mathbb{C}$ and $R_2(\infty) = \infty$, see Fig. 1. Then $g_i := R_i^{-1} \circ f_c \circ R_i$ for $i = 1, 2$ preserve length ℓ on S^1 and are ergodic. Hence h preserves ℓ so it is a rotation, identity for appropriate R_1, R_2. Hence R_1 and R_2 glue together to a holomorphic automorphism R of the Riemann sphere, a homography. (Compare Mostov rigidity theorem.) Therefore $R^{-1} \circ f_c \circ R(z) = \lambda z^2$ for λ with $|\lambda| = 1$ and in consequence $c = 0$.
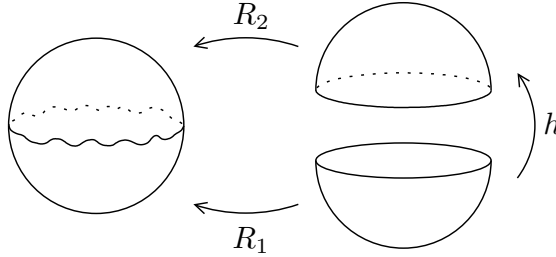

Figure 1. Broken egg argument

Complex case

In the complex case we consider f a rational mapping of degree at least 2 of the Riemann sphere $\overline{\mathbb{C}}$. We consider f acting on its Julia set $K = J(f)$ (generalizing the $z^2 + c$ model), see Fig. 2. Formally the Julia set is the complement in the sphere of the Fatou set which is the set where the family of the iterates f^n is locally equicontinuous. $J(f)$ is compact completely invariant and f on it acts in a "chaotic" way.

Real case

DEFINITION 2.2 (Real case, [20]). $f \in C^2$ is called a *generalized multimodal map* if it is defined on a neighbourhood of a compact invariant set K, critical points are not infinitely flat, bounded distortion (BD) property for iterates holds, is topologically transitive, and has positive topological entropy on K.
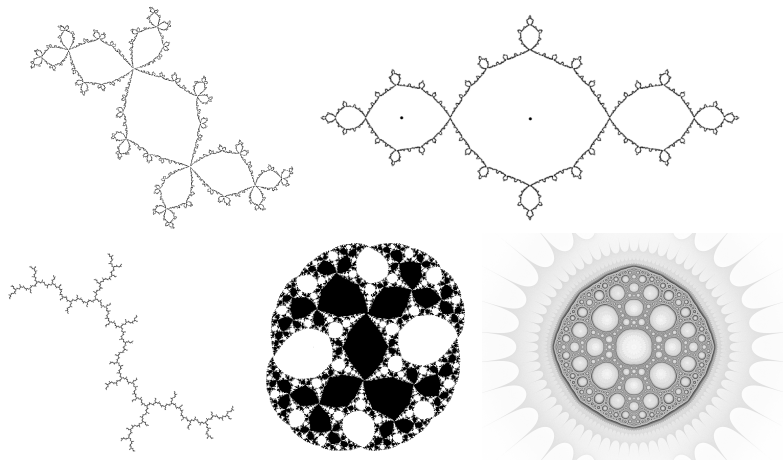
Figure 2. Julia sets zoo: rabbit $f(z) = z^2 - 0.123 + 0.745i$, basilica $f(z) = z^2 - 1$, dendrite $f(z) = z^2 + i$, basilica mated with rabbit $f(z) = \frac{z^2+c}{z^2-1}$ for $c = \frac{1+\sqrt{-3}}{2}$ with $J(f)$ being the boundary between white (basilica) and black (rabbit), Sierpiński-Julia carpet $f(z) = z^2 - 1/16z^2$ i.e. boundaries of Fatou set components do not touch each other (the corona-like shapes are lines of the same speed of escape to ∞).

Also K is a maximal forward invariant subset of a finite union $\hat{I}$ of pairwise disjoint closed intervals, whose endpoints are in K.

This maximality corresponds to the Darboux property. We write $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$, where $+$ marks positive entropy. In place of BD one can assume C^3 (and write $(f, K) \in \mathscr{A}_+^3$) and assume that all periodic orbits in K are hyperbolic repelling. Then changing f outside K allows to get $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$.

EXAMPLES:. Basic sets in spectral decomposition via renormalizations [3, Theorem III.4.2].

3. Hyperbolic potentials

For continuous f and ϕ as in Definitions 1.2 and 1.3 call $\phi \colon K \to \mathbb{R}$ satisfying $P(f, \phi) > \sup_{\nu \in \mathcal{M}(f)} \int \phi \, d\nu$ a *hyperbolic potential*. Equivalently $P(f, \phi) > \sup_K \frac{1}{n} S_n \phi$ for some n. See [9].

THEOREM 3.1 (Complex and real: Denker, Urbański, Przytycki, Haydn, Rivera-Letelier, Zdunik, Szostakiewicz, H. Li, Bruin, Todd). *If ϕ is a Hölder continuous hyperbolic potential, then there exists a unique equilibrium state μ_ϕ. For every Hölder $u\colon K \to \mathbb{R}$, the Central Limit Theorem (CLT) and Law of Iterated Logarithm (LIL) for the sequence of random variables $u \circ f^n$ and μ_ϕ hold.*

The CLT follows from sufficiently fast convergence of iteration of transfer operator (spectral gap). The LIL is proved via LIL for a return map (inducing) to a nice domain related to μ_ϕ (Mañé, Denker, Urbański) providing a Markov structure (Infinite Iterated Function System) avoiding critical points, satisfying BD.

4. Non-uniform hyperbolicity

Define the following conditions, both for real and complex (rational) cases:

(a) *Collet–Eckmann condition* (CE). There exist $\lambda > 1, C > 0$

$$|(f^n)'(f(c))| \geq C\lambda^n$$

for all critical points $c \in K$ whose forward orbit is disjoint from the set $\mathrm{Crit}(f)$ of all critical points of f. Moreover there are no indifferent periodic orbits in K.

(b) *Backward Collet–Eckmann condition at $z_0 \in K$* (CE2(z_0)). There exist $\lambda > 1$ and $C > 0$ such that for every $n \geq 1$ and every $w \in f^{-n}(z_0)$ (in a neighbourhood of K in the real case)

$$|(f^n)'(w)| \geq C\lambda^n.$$

(c) *Topological Collet–Eckmann condition* (TCE), [23]. There exist $M \geq 0$, $P \geq 1$, $r > 0$ such that for every $x \in K$ there exist increasing n_j, $j = 1, 2, \ldots$, such that $n_j \leq P \cdot j$ and for each j and discs $B(\cdot)$ below, understood in $\overline{\mathbb{C}}$ or $\mathbb{R}$,

$$\#\{0 \leq i < n_j : (\mathrm{Comp}_{f^i(x)}\, f^{-(n_j - i)}B(f^{n_j}(x), r)) \cap \mathrm{Crit}(f) \neq \varnothing\} \leq M.$$

(d) *Exponential shrinking of components* (ExpShrink). There exist $\lambda > 1$ and $r > 0$ such that for every $x \in K$, every $n > 0$ and every connected

component W_n of $f^{-n}(B(x, r))$ for the disc (interval) $B(x, r)$ in $\overline{\mathbb{C}}$ (or $\mathbb{R}$), intersecting K

$$\mathrm{diam}(W_n) \leq \lambda^{-n}.$$

(e) *Lyapunov hyperbolicity* (LyapHyp). There is $\lambda > 1$ such that the Lyapunov exponent $\chi(\mu) := \int_K \log |f'| \, d\mu$ of any ergodic measure $\mu \in \mathcal{M}(f, K)$ satisfies $\chi(\mu) \geq \log \lambda$.

(f) *Uniform hyperbolicity on periodic orbits* (UHP). There exists $\lambda > 1$ such that every periodic point $p \in K$ of period $k \geq 1$ satisfies

$$|(f^k)'(p)| \geq \lambda^k.$$

Note that whereas in the complex case for a ball $B = B(f(x), \tau)$ and its pullback $B' = \mathrm{Comp}_x f^{-1}(B)$ (the component of the preimage containing x) we have $f(B') = B$, in the real case it may be false, because of "folds". Therefore in the real case additional difficulties in this theory appear, in particular in TCE it is not equivalent to write that degrees of all f^{n_j} on $\mathrm{Comp}_x f^{-(n_j)} B(f^{n_j}(x), r)$ are uniformly bounded.

THEOREM 4.1 (..., Keller, Nowicki, Sands, Przytycki, Rohde, Rivera-Letelier, Graczyk, Smirnov). *Assume there are no indifferent periodic orbits in K. Then*

1. *The conditions* (c)–(f), *and* (b) *for some z_0, are equivalent (in the real case under the assumption of weak isolation: any periodic orbit close to K must be in K).*
2. *CE implies* (b)–(f).
3. *If there is only one critical point in the Julia set in the complex case or if f is S-unimodal on $K = I$ in the real case, then all conditions above are equivalent to each other.*
4. *TCE is topologically invariant; therefore all other conditions equivalent to it are topologically invariant.*

See e.g. [21]. For polynomials (b)–(f) are equivalent for $K = J(f) = \mathrm{Fr}\,\Omega_\infty(f)$, to Ω_∞ the basin of ∞, being Hölder (Graczyk, Smirnov). Note that for rational maps f satisfying TCE, if $J(f) \neq \overline{\mathbb{C}}$, then it is *mean porous* hence $\mathrm{HD}(J(f)) < 2$, see [23].

An order of proving the equivalences in Theorem 4.1 is, for z_0 safe (defined below),

$$\mathrm{CE2}(z_0) \Rightarrow \mathrm{ExpShrink} \Rightarrow \mathrm{LyapHyp} \Rightarrow \mathrm{UHP} \Rightarrow \mathrm{CE2}(z_0).$$

Separately one proves ExpShrink $\Leftrightarrow$ TCE using for $\Rightarrow$ the following

LEMMA 4.2 (Denker, Przytycki, Urbański, [4]).

$$\sum_{j=0}^{n}{}' - \log |f^j(x) - c| \leq Qn$$

for a constant $Q > 0$ every $c \in \mathrm{Crit}(f)$, every $x \in K$ and every integer $n > 0$. Σ' means that we omit in the sum an index j of smallest distance $|f^j(x) - c|$.

Assuming UHP one proves CE2(z_0) for *safe* and *hyperbolic* z_0 by "shadowing", see Fig. 3.
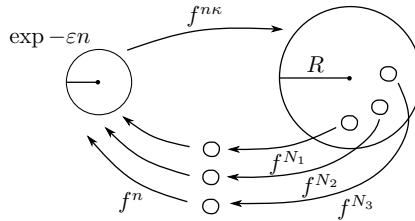

Figure 3. Shadowing

DEFINITION 4.3 (Safe point). We call $z \in K$ *safe* if $z \notin \bigcup_{j=1}^{\infty}(f^j(\mathrm{Crit}(f)))$ and for every $\varepsilon > 0$ and all n large enough

$$B(z, \exp(-\varepsilon n)) \cap \bigcup_{j=1}^{n}(f^j(\mathrm{Crit}(f))) = \varnothing.$$

Notice that this definition implies that all points except at most a set of Hausdorff dimension 0, are safe. Hyperbolic points (see below) are e.g. all points in invariant hyperbolic (expanding) subsets of K. Such sets are abundant.

DEFINITION 4.4 (Hyperbolic point). We call $z \in K$ *hyperbolic* if there exist $\lambda > 1, r > 0, C > 0$ such that for all $n \in \mathbb{N}$ the map f^n is injective on $\mathrm{Comp}_x(f^{-n}(B(f^n(x), r)))$ and $|(f^n)'(x)| \geq C\lambda^n$.

5. Geometric variational pressure and equilibrium states

For $\phi = \phi_t := -t \log |f'|$, the variational definition of pressure, here

$$P(t) := P_{\mathrm{var}}(f, \phi_t) = \sup_{\mu \in \mathcal{M}(f)} \left(h_\mu(f) - t \int_K \log |f'| \, d\mu \right)$$

still makes sense by the integrability of $\log |f'|$, [13]. Moreover $\int_K \log |f'| \, d\mu = \chi(\mu) \geq 0$ for all ergodic μ even in presence of critical points where $\phi = \pm\infty$. $t \mapsto P(t)$ is convex, monotone decreasing. We usually assume $t > 0$ later on.
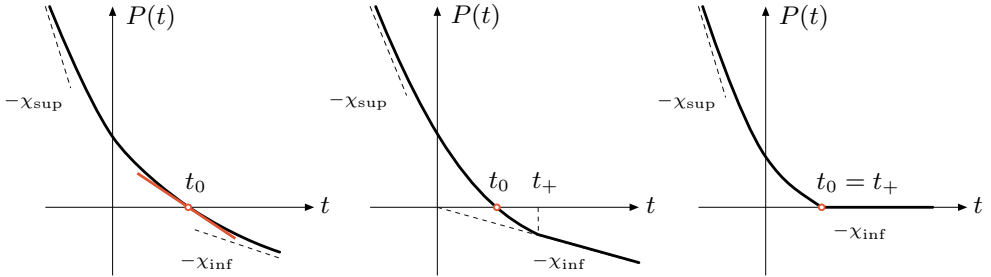

Figure 4. The geometric pressure: LyapHyp with $t_+ = \infty$, LyapHyp with $t_+ < \infty$, and non-LyapHyp

Here t_+ is the phase transition "freezing" parameter, where $t \mapsto P(t)$ is not analytic. $P(t)$ is equal to several other quantities, in the complex case see [15] and [22], in real [20]. E.g.

DEFINITION 5.1 (Hyperbolic pressure).

$$P_{\mathrm{hyp}}(t) := \sup_{X \in \mathcal{H}(f,K)} P(f|_X, -t \log |f'|),$$

where $\mathcal{H}(f, K)$ is defined as the space of all compact forward invariant, i.e. $f(X) \subset X$, expanding subsets of K, repellers.

DEFINITION 5.2 (Hyperbolic dimension).

$$\mathrm{HD}_{\mathrm{hyp}}(K) := \sup_{X \in \mathcal{H}(f,K)} \mathrm{HD}(X).$$

Recall that for expanding $f \colon X \to X$, $t_0(X) = \mathrm{HD}(X)$, see (2.1). Passing to sup we obtain:

PROPOSITION 5.3 (Generalized Bowen's formula). *The first zero t_0 of $t \mapsto P_{\mathrm{hyp}}(K, t)$ is equal to $\mathrm{HD}_{\mathrm{hyp}}(K)$.*

It may happen $\mathrm{HD}_{\mathrm{hyp}}(J(f)) < \mathrm{HD}(J(f)) = 2$ for f quadratic polynomials, Avila & Lyubich.

THEOREM 5.4 (Przytycki, Rivera-Letelier, the real case, [20]). *Let $(f, K) \in \mathscr{A}_+^3$, f-periodic orbits in K be hyperbolic repelling. Then*
- $t \mapsto P(t)$ *is real analytic on an open interval (t_-, t_+) with $-\infty \le t_- < 0 < t_+ \le \infty$ defined by $P(t) > \sup_{\nu \in \mathcal{M}(f)} -t \int \log |f'| \, d\nu$. For $t \ge t_0$ $P(t)$ is affine.*
- *For each t in this interval there is a unique invariant equilibrium state μ_{ϕ_t}. It is ergodic and absolutely continuous with respect to an adequate conformal measure m_{ϕ_t} with $d\mu_{\phi_t}/dm_{\phi_t} \ge$ Const > 0 a.e.*
- *If furthermore f is topologically exact on K (that is for every V an open subset of K there exists $n \ge 0$ such that $f^n(V) = K$), then this measure is* mixing*, has* exponential decay of correrations *and satisfies CLT for Lipschitz observables.*

This generalizes results by Bruin, Iommi, Pesin, Senti, Todd.

THEOREM 5.5 (Przytycki, Rivera-Letelier, the complex case, [19]). *The assertion is the same. One assumes a very weak expansion:* the existence of arbitrarily small nice, or pleasant, couples *and* hyperbolicity away from critical points.

REMARK. For real f satisfying LyapHyp and $K = \hat{I}$, we have the unique zero of pressure $t_0 = 1$ and for $-\log |f'|$ we conclude that a unique equilibrium state exists which is absolutely continuous with respect to Lebesgue measure (probability), acip. In general for $K = I$ it holds assumed only e.g. $|(f^n)'(f(c))| \to \infty$ for all $c \in \mathrm{Crit}(f)$, see [2]. For $t > t_+$ for f LyapHyp, equilibria do not exist, see [9].

PROOFS use inducing (and Lai-Sang Young towers), compare Theorem 3.1 though here we find nice sets (pairs) geometrically, independently of equilibria. For a different proof, the real case, see a recent [5].

Lasota–Yorke Theorem

Sometimes to find an absolutely continuous invariant measure (probability) it is sufficient to find a function $u \colon I \to \mathbb{R}$ invariant for the transfer operator (Perron–Frobenius) directly for f rather than for a return map via

inducing as above. Then $u \cdot \mathrm{Leb}$ will be acip. This is so in the classical Lasota–Yorke Theorem.

THEOREM 5.6 (Lasota, Yorke, [10]). *Let $f \colon [0,1] \to \mathbb{R}$ be a piecewise continuous and piecewise C^2 (with finitely many pieces) and $\inf |f'| > 1$. Then there exists a measure absolutely continuous with respect to Lebesgue (acip).*

PROOF. We find $u := \lim \frac{1}{n} \sum_{k=0}^{n-1} P^k(\phi)$, where ϕ is an arbitrary function of class C^1, may be $\mathbb{1}$. The convergence follows from the conditional weak compactness of the family $P^{Nk}(\phi)$, where the Perron–Frobenius operator P is defined by $P(\phi)(x) := \sum_{f(y)=x} \phi(y)/|f'(y)|$.

The weak compactness follows for ϕ with bounded variation, for $\alpha > 0$, $\beta < 1, k = 0, 1, ...,$ and some N, from

$$\mathrm{Var}\, P^{N(k+1)}(\phi) \le \alpha ||\phi||_1 + \beta \,\mathrm{Var}(P^{Nk}(\phi)). \qquad \square$$

This estimate with the use of two (semi)norms allows even to prove an exponential convergence to u (Ionescu–Tulcea, Marinescu).

Dimension spectrum

$P_{\mathrm{var}}(t)$ allows the study *dimension spectrum for Lyapunov exponent* via the Legendre transformation, proving in particular for $\alpha > 0$

$$\mathrm{HD}(\{x \in K : \chi(x) = \alpha\}) = \frac{1}{\alpha} \inf_{t \in \mathbb{R}} (P(t) + \alpha t).$$

Proof of $\ge$: Given α consider t where inf is attained. The tangent to $P(t)$ at t is parallel to $-\alpha t$ and for μ_t the equilibrium, it is $h_{\mu_t}(f) - t\chi(\mu_t)$. So the infimum is $h_{\mu_t}(f)$, see Fig. (By the variational definition, $P(t)$ and h_μ are mutual Legendre type transforms.) Dividing by α gives $\ge$ using Mañé's equality

(5.1)
$$\mathrm{HD}(\mu) = h_\mu(f)/\chi(\mu).$$

(Notice that this equality is related to (2.1).)

The proof of $\le$ uses conformal measures.

Using of the Legendre transform of $P(t)$, see Fig. 5 allows us to also give formulae for Hausdorff dimension of (irregular) sets of points with given lower and upper Lyapunov exponent

$$\mathrm{HD}(\{\underline{\chi}(x) = \alpha, \overline{\chi}(x) = \beta\})$$
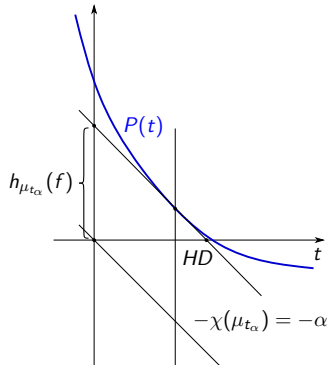
for $\beta > 0$, see [6] and [7].

Figure 5. Legendre transform

More on Lyapunov exponents

In analogy to $\chi(\mu) \geq 0$ one has:

THEOREM 5.7 (Levin, Przytycki, Shen, [11]). *If for a rational function* $f \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *there is only one critical point* c *in* $J(f)$ *and no parabolic periodic orbits, then* $\underline{\chi}(f(c)) \geq 0$.

For S-unimodal maps of interval this was proved much earlier by T. Nowicki and D. Sands.

6. Other definitions of geometric pressure

DEFINITION 6.1 (Tree pressure). For every $z \in K$ and $t \in \mathbb{R}$ define

$$(6.1) \qquad P_{\text{tree}}(z,t) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{f^n(x)=z,\, x \in K} |(f^n)'(x)|^{-t}.$$

THEOREM 6.2. $P_{\text{tree}}(z,t)$ *does not depend on* z *for* z *safe.*

- In the complex case to prove $P_{\text{tree}}(z_1, t) = P_{\text{tree}}(z_2, t)$ one joins z_1 to z_2 with a curve not fast accumulated by critical trajectories, see [15] and [22].
- In the real case there is no room for such curves. Instead, one relies on the *topological transitivity*. See [16] and [20].

- For $\phi = -t \log |f'|$ pressure via separated sets does not make sense. Indeed, in presence of critical points for f, for $t > 0$, it is equal to $+\infty$. So it is replaced by P_{tree}.
- One can consider however *spanning geometric pressure* $P_{\text{span}}(t)$ using (n, ε)-spanning sets (in place of separated) and infimum. Assumed *weak backward Lyapunov stability, wbls* (see the definition below) it is indeed equal to $P(t)$ in the complex case, see [16]. This is however not so in the real case, where wbls always holds if all periodic orbits are hyperbolic repelling. It happens that $P_{\text{span}}(t) = \infty$ for $t > 0$, if some x with big $|(f^n)'(x)|^{-1}$ is well isolated in the metrics ρ_n in Definition 1.3. See Fig. 6.
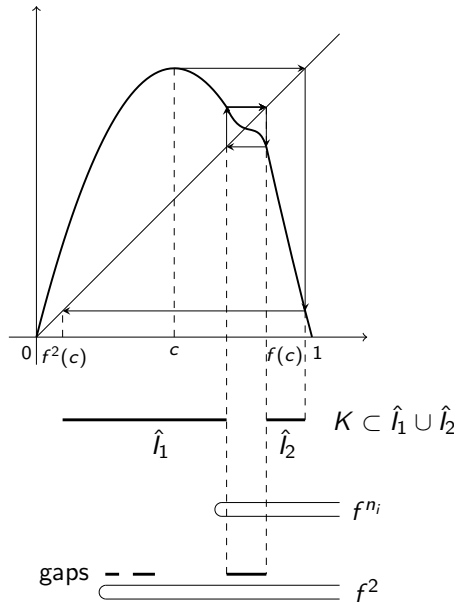

Figure 6. $P_{\text{span}}(t) = \infty$. The fold of f^{n_i} on $(-\varepsilon_{n_i}, \varepsilon_{n_i})$ is in the gap between $\hat{I}_1$ and $\hat{I}_2$ except a tiny neighbourhood of its tip

DEFINITION 6.3 (Weak backward Lyapunov stability, wbls). f is *weakly backward Lyapunov stable* if for every $\delta > 0$ and $\varepsilon > 0$ for all n large enough and every disc $B = B(x, \exp -\delta n)$ centered at $x \in K$, for every $0 \le j \le n$ and every component V of $f^{-j}(B)$ intersecting K, it holds that $\operatorname{diam} V \le \varepsilon$.

QUESTION. Does wbls hold for all rational maps?

7. Boundary dichotomy

Let $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map with $\deg(f) \geq 2$ and let $\Omega = \Omega_p(f)$ be a simply connected immediate basin of attraction to a fixed point p (that is the component of the (open) set attracted to p containing it), see Fig. 7. Let $R\colon \mathbb{D} \to \Omega$ be a Riemann map $R(0) = p$ and $g\colon \mathbb{D} \to \mathbb{D}$ defined by $g := R^{-1} \circ f \circ R$, extended conformally beyond the boundary $\partial\mathbb{D}$ (Schwarz symmetry), thus expanding on $\partial\mathbb{D}$.
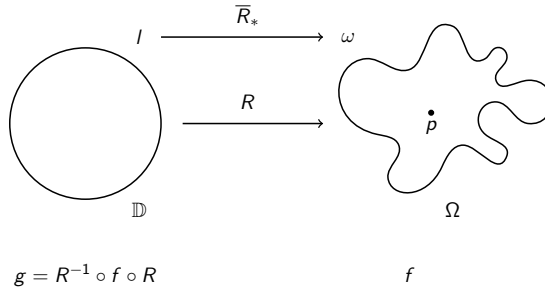

Figure 7. A Riemann map and its radial limit

Consider *harmonic measure* $\omega = \overline{R}_*(l)$, where l is normalized length measure on $\partial\mathbb{D}$ and $\overline{R}$ is radial limit, defined l-a.e. Since 0 is a fixed point for g, l is g-invariant, hence ω is f-invariant. Denote by H_1 Hausdorff measure in dimension 1.

THEOREM 7.1 (Przytycki, Urbański, Zdunik: 1985 – 2006). *For f, Ω as above,* $\mathrm{HD}(\omega) = 1$. *One of two cases holds:*
1) $\omega \perp H_1$, *which implies* $\mathrm{HD}_{\mathrm{hyp}}(\mathrm{Fr}\,\Omega) > 1$;
2) $\omega \ll H_1$ *and f is a finite Blaschke product or a two-to-one holomorphic factor of a Blaschke product with $\mathrm{Fr}\,\Omega$ being an interval, in some holomorphic coordinates on $\overline{\mathbb{C}}$.*

Consider $\psi := \log|g'| - \log|f'| \circ \overline{R}$. Notice that $\int_{\partial\mathbb{D}} \psi\, dl = 0$, hence $\mathrm{HD}(\omega) = 1$ as $\overline{R}_*$ does not change entropy [12] and using (5.1).

$\mathrm{HD}(\omega) = 1$ was proved in 1985 by Makarov without assuming existence of f.

Consider the asymptotic variance $\sigma^2 = \sigma_l^2(\psi) := \lim_{n\to\infty} \frac{1}{n} \int_{\partial\mathbb{D}} (S_n\psi)^2\, dl$. Then $\omega \perp H_1$ is equivalent to $\sigma^2 > 0$ and equivalent to ψ not being cohomologous to 0 (not of the form $u \circ f - u$).

THEOREM 7.2 (LIL-refined-HD for harmonic measure, Przytycki, Urbański, Zdunik, [26] and [27]). *For f, Ω with $\sigma^2 > 0$, there exists $c(\Omega) > 0$, such that for $\alpha_c(r) := r \exp(c\sqrt{\log 1/r \log\log\log 1/r})$*

i) *$\omega \perp H_{\alpha_c}$ for the gauge function α_c, for all $0 < c < c(\Omega)$;*

ii) *$\mu \ll H_{\alpha_c}$ for all $c > c(\Omega)$.*

This theorem applies also e.g. to snowflake-type Ω's.

PROOFS. To prove $\mathrm{HD}(\Omega_{\mathrm{hyp}}) > 1$ in Theorem 7.1, we can find X with $\mathrm{HD}(X) \geq \mathrm{HD}(\omega) - \epsilon$ by A. Katok's method and using $\mathrm{HD} = h/\chi$, see (5.1). This is not enough. However we can do better:

$\sigma^2 > 0$ yields by CLT large fluctuations of the sums $\sum_{j=0}^{n-1} \psi \circ \varsigma^j$ from 0, allowing to find expanding X with $\mathrm{HD}(X) > \mathrm{HD}(\omega)$. One builds an iterated function system, for which X is the limit set. A special care is needed to get $X \subset \mathrm{Fr}\,\Omega$.

Substituting in LIL $n \sim (\log 1/r_n)/\chi(\omega)$ for $r_n = |(f^n)'(x)|^{-n}$, comparing $\log |(g^n)'| - \log |(f^n)'| \circ \overline{R}$ with $\sqrt{2\sigma^2 n \log\log n}$ for a sequence of n's, we get

LEMMA 7.3 (Refined Volume Lemma). *For ω-a.e. x*

$$\limsup_{n\to\infty} \frac{\omega(B(x, r_n))}{\alpha_c(r_n)} = \begin{cases} \infty, & \text{for } 0 < c < c(\omega), \\ 0, & \text{for } c > c(\omega). \end{cases}$$

yielding Theorem 7.1. Using $R = f^{-n} \circ R \circ g^n$ one obtains

THEOREM 7.4 (Radial growth). *For Lebesgue a.e. $\zeta \in \partial\mathbb{D}$*

$$G^+(\zeta) := \limsup_{r \nearrow 1} \frac{\log |R'(r\zeta)|}{\sqrt{\log(1/1-r) \log\log\log(1/1-r)}} = c(\Omega).$$

Similarly

$$G^-(\zeta) := \liminf_{r \nearrow 1} \frac{\log |R'(r\zeta)|}{\sqrt{\log(1/1-r) \log\log\log(1/1-r)}} = -c(\Omega).$$

The above theorems hold for every connected, simply connected open $\Omega \subset \mathbb{C}$, different from $\mathbb{C}$, without existence of f. Of course one should add ess sup over $\zeta \in \partial\mathbb{D}$ and over $z \in \mathrm{Fr}\,\Omega$ in Refined Volume Lemma and reformulate the case i). There is a universal Makarov's upper bound $C_{\mathrm{M}} < \infty$ for all $c(\Omega)$, $C_{\mathrm{M}} \leq 1.2326$ (Hedenmalm, Kayumov, 2007, [8]). In 1989 I gave a weaker estimate.

Geometric coding trees, g.c.t.

Above theorems hold in an abstract setting of a *geometric coding tree* $\mathscr{T}$ in $f(U)$ for $f\colon U \to \overline{\mathbb{C}}$, $f(U) \supset U$ proper. We obtain a coding from the left shift space, see Introduction, $\pi\colon \Sigma^d \to \Lambda$ of the limit set Λ (in place of $\overline{R}\colon \partial\mathbb{D} \to \operatorname{Fr}\Omega$). If f extends holomorphically beyond $\operatorname{cl}\Lambda$ we call Λ a *quasi-repeller*.

More precisely, given $z \in f(U)$ and curves $\gamma^j\colon [0,1] \to f(U)$, $j = 1, \ldots, d$, joining z to $z^j \in f^{-1}(z)$, see Fig. 8, we define a graph $\mathscr{T}$ consisting of the set of vertices $f^{-n}(z)$ and edges $f^{-n}(\gamma^j)$, $n = 0, 1, \ldots$ and $j = 1, \ldots, d$, such that denoting the edges in $f^{-n}(\gamma^j)$ by $\gamma_n(\alpha)$ for all $\alpha \in \Sigma^d$ the following conditions hold

$$\gamma_0(\alpha) := \gamma^{\alpha_0}, \quad f \circ \gamma_n(\alpha) = \gamma_{n-1}(\varsigma(\alpha)), \quad \gamma_n(\alpha)(0) = \gamma_{n-1}(\alpha)(1).$$

The vertices are defined as the ends of $\gamma_n(\alpha)$, denoted then $z_n(\alpha)$ and $z_{n-1}(\alpha)$.
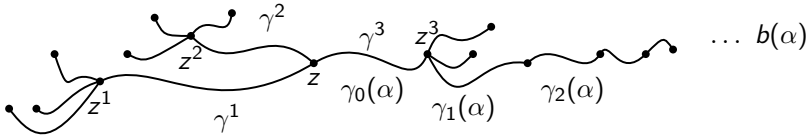

Figure 8. A geometric coding tree

For every $\alpha \in \Sigma^d$ the subgraph composed of $z, z_n(\alpha)$ and $\gamma_n(\alpha)$ for all $n \geq 0$ is called an *infinite geometric branch* and denoted by $b(\alpha)$. It is called *convergent* if the sequence $\gamma_n(\alpha)$ is convergent to a point in $\operatorname{cl} U$. This convergence holds for all a except a *thin set*, see [24]. Λ is defined as the set of limits of all convergent infinite branches.

For a Hölder potential $\phi\colon \Sigma^d \to \mathbb{R}$ (in place of $-\log|g'|$) and Gibbs measure μ_ϕ one gets dichotomies for $\mu := \pi_*(\mu_\phi)$ on Λ, analogous to the ones in Theorems 7.1 and 7.2.

For a constant potential, $\mu = \mu_{\max}$ is a measure of maximal entropy on Julia set $J(f)$ for $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ rational. Then

1) If $\sigma^2 > 0$ then $\operatorname{HD}_{\mathrm{hyp}}(J(f)) > \operatorname{HD}(\mu_{\max})$.
2) If $\sigma^2 = 0$ then for each $x, y \in J(f)$ not postcritical, if $z = f^n(x) = f^m(y)$ for some positive integers n, m, the orders of criticality of f^n at x and f^m at y coincide. In particular all critical points in $J(f)$ are pre-periodic, f is postcritically finite with parabolic orbifold, in particular z^d, Chebyshev or some Lattès maps, (Zdunik, 1990, [31]).

In the Ω version it is sufficient to assume f is defined only in a neighbourhood of $\partial\Omega$ repelling on the side of Ω, called *RB-domain*.

This applies to f polynomial and simply connected $\Omega = \Omega_\infty$ giving again the dichotomy on $\operatorname{Fr}\Omega = J(f)$.

Integral mean spectrum

For a simply connected domain $\Omega \subset \mathbb{C}$ one considers the *integral means spectrum*:

$$\beta_\Omega(t) := \limsup_{r \nearrow 1} \frac{1}{|\log(1-r)|} \log \int_{\zeta \in \partial\mathbb{D}} |R'(r\zeta)|^t \, |d\zeta|.$$

This, in presence of f, e.g. for an RB-domain Ω and for $\phi = -\log|f'|$ on $\operatorname{Fr}\Omega$, for $g(z) = z^d$, e.g. Ω being a simply connected basin of ∞ for a polynomial of degree d, satisfies

$$\beta_\Omega(t) = t - 1 + \frac{P(t\phi)}{\log d}. \quad \text{(N. Makarov, F. Przytycki \& S. Rohde)}$$

One considers

$$\sigma^2(\log R') := \limsup_{r \nearrow 1} \frac{\int_{\partial\mathbb{D}} |\log R'(r\zeta)|^2 \, |d\zeta|}{-2\pi |\log(1-r)|}.$$

It holds $\sigma^2(\log R') = 2 \frac{d^2 \beta_\Omega(t))}{dt^2}|_{t=0}$ (O. Ivrii). It is related to the Weil–Petersson metric (McMullen).

Recall that $\sigma_\mu^2(t\phi) = \frac{d^2 P(f, t\phi)}{dt^2}$ for μ Gibbs in expanding case, see [28] and [25].

8. Accessibility

THEOREM 8.1 (Douady, Eremenko, Levin, Petersen on accessibility of periodic sources; Przytycki on accessibility of more points, [14]). *Let Λ be a limit set for a g.c.t. $\mathscr{T}$ for holomorphic $f \colon U \to \overline{\mathbb{C}}$. Assume* uniform shrinking, *that is* $\operatorname{diam}(\gamma_n(\alpha)) \to 0$, *as $n \to \infty$ uniformly with respect to $\alpha \in \Sigma^d$. Then every good $q \in \operatorname{cl}\Lambda$ is a limit of a convergent infinite branch $b(\alpha)$, i.e. $q \in \Lambda$. In particular, this holds for every q with $\underline{\chi}(q) > 0$ and satisfying a local backward invariance of U.*

COROLLARY 8.2 (Lifting of measure, [14] and [17]). *Consider a g.c.t. $\mathscr{T}$ as above, uniformly shrinking, with no self-intersections, and a non-atomic hyperbolic probability measure μ on $\mathrm{cl}\,\Lambda$, i.e. satisfying $\chi(\mu) > 0$. Assume μ-a.e. local backward invariance of U. Then μ is the π_* image of a probability ς-invariant measure ν on Σ^d.*

In particular a lift ν exists for every completely invariant RB-domain, e.g. for every hyperbolic μ on $\mathrm{Fr}\,\Omega_\infty$ for f polynomial.

References

[1] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lect. Notes in Math., 470, Springer-Verlag, Berlin-New York, 1975.

[2] H. Bruin, J. Rivera-Letelier, W. Shen, and S. van Strien, *Large derivatives, backward contraction and invariant densities for interval maps*, Invent. Math. **172** (2008), no.3, 509–533.

[3] W. de Melo and S. van Strien, *One-Dimensional Dynamics*, Springer-Verlag, Berlin, 1993.

[4] M. Denker, F. Przytycki, and M. Urbański, *On the transfer operator for rational functions on the Riemann sphere*, Ergodic Theory Dynam. Systems **16** (1996), no. 2, 255–266.

[5] N. Dobbs and M. Todd, *Free energy and equilibrium states for families of interval maps*, to appear in Mem. Amer. Math. Soc. Avaliable at arXiv:1512.09245.

[6] K. Gelfert, F. Przytycki, and M. Rams, *On the Lyapunov spectrum for rational maps*, Math. Ann. **348** (2010), no. 4, 965–1004.

[7] K. Gelfert, F. Przytycki, and M. Rams, *Lyapunov spectrum for multimodal maps*, Ergodic Theory Dynam. Systems **36** (2016), no. 5, 1441–1493.

[8] H. Hedenmalm and I. Kayumov, *On the Makarov law of the iterated logarithm*, Proc. Amer. Math. Soc. **135** (2007), no. 7, 2235–2248.

[9] I. Inoquio-Renteria and J. Rivera-Letelier, *A characterization of hyperbolic potentials of rational maps*, Bull. Braz. Math. Soc. (N.S.) **43** (2012), no. 1, 99–127.

[10] A. Lasota and J.A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. **186** (1973), 481–488.

[11] G. Levin, F. Przytycki, and W. Shen, *The Lyapunov exponent of holomorphic maps*, Invent. Math. **205** (2016), no. 2, 363–382.

[12] F. Przytycki, *Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map*, Invent. Math. **80** (1985), no. 1, 161–179.

[13] F. Przytycki, *Lyapunov characteristic exponents are nonnegative*, Proc. Amer. Math. Soc. **119** (1993), no. 1, 309–317.

[14] F. Przytycki, *Accessibility of typical points for invariant measures of positive Lyapunov exponents for iterations of holomorphic maps*, Fund. Math. **144** (1994), no. 3, 259–278.

[15] F. Przytycki, *Conical limit set and Poincaré exponent for iterations of rational functions*, Trans. Amer. Math. Soc. **351** (1999), no. 5, 2081–2099.

[16] F. Przytycki, *Geometric pressure in real and complex 1-dimensional dynamics via trees of pre-images and via spanning sets*, Monatsh. Math. **185** (2018), no. 1, 133–158.

[17] F. Przytycki, *Thermodynamic formalism methods in one-dimensional real and complex dynamics*, in: B. Sirakov et al. (Eds.), *Proceedings of the International Congress of Mathematicians 2018, Rio de Janeiro, Vol. III*, World Scientific, Singapore, 2018, pp. 2105–2132.

[18] F. Przytycki, *Metody formalizmu termodynamicznego w jednowymiarowej dynamice rzeczywistej i zespolonej*, Wiad. Mat. **54** (2018), no. 1, 23–53.

[19] F. Przytycki and J. Rivera-Letelier, *Nice inducing schemes and the thermodynamics of rational maps*, Comm. Math. Phys. **301** (2011), no. 3, 661–707.

[20] F. Przytycki and J. Rivera-Letelier, *Geometric pressure for multimodal maps of the interval*, Mem. Amer. Math. Soc. **259** (2019), no. 1246, 81 pp.

[21] F. Przytycki, J. Rivera-Letelier, and S. Smirnov, *Equivalence and topological invariance of conditions for non-uniform hyperbolicity in the iteration of rational maps*, Invent. Math. **151** (2003), no. 1, 29–63.

[22] F. Przytycki, J. Rivera-Letelier, and S. Smirnov, *Equality of pressures for rational functions*, Ergodic Theory Dynam. Systems **24** (2004), no. 3, 891–914.

[23] F. Przytycki and S. Rohde, *Porosity of Collet-Eckmann Julia sets*, Fund. Math. **155** (1998), no. 2, 189–199.

[24] F. Przytycki and J. Skrzypczak, *Convergence and pre-images of limit points for coding trees for iterations of holomorphic maps*, Math. Ann. **290** (1991), no. 3, 425–440.

[25] F. Przytycki and M. Urbański, *Conformal Fractals: Ergodic Theory Methods*, London Mathematical Society Lecture Note Series, 371, Cambridge University Press, Cambridge, 2010.

[26] F. Przytycki, M. Urbański, and A. Zdunik, *Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps, I*, Ann. of Math. (2) **130** (1989), no. 1, 1–40.

[27] F. Przytycki, M. Urbański, and A. Zdunik, *Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps, II*, Studia Math. **97** (1991), no. 3, 189–225.

[28] D. Ruelle, *Thermodynamic Formalism*, Encyclopedia of Mathematics and its Applications, vol. 5, Addison-Wesley Publ. Co., London, 1978.

[29] Ya.G. Sinai, *Gibbs measures in ergodic theory*, (Russian) Uspehi Mat. Nauk **27** (1972), no. 4(166), 21–64. Russ. Math. Surv. **27** (1972), no. 4, 21–69 (English).

[30] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics, 79, Springer-Verlag, New York-Berlin, 1982.

[31] A. Zdunik, *Parabolic orbifolds and the dimension of the maximal measure for rational maps*, Invent. Math. **99** (1990), no. 3, 627–649.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
UL. ŚNIADECKICH 8
00-656 WARSZAWA
POLAND
e-mail: feliksp@impan.pl