Equivalence and topological invariance of conditions for non-uniform hyperbolicity in the iteration of rational maps

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Abstract. We show equivalence of several standard conditions for non-uniform hyperbolicity of complex rational functions (including the Topological Collet-Eckmann condition (TCE), Uniform Hyperbolicity on Periodic orbits, Exponential Shrinking of components, etc.). The condition TCE is stated in purely topological terms, so we conclude that all these conditions are invariant under topological conjugacy.

We show furthermore that for rational maps with one critical point in the Julia set all the conditions above are equivalent to the usual Collet-Eckmann and backward Collet-Eckmann conditions; thus the latter ones are invariant by topological conjugacy in the unicritical setting. We also prove that neither part of this stronger statement (for the unicritical case) is valid in general.

Introduction.

We establish the equivalence of several standard and widely used (and also some technical) conditions of non-uniform hyperbolicity of rational maps. Since one of them, Topological Collet-Eckmann (TCE), is formulated in purely topological terms this shows their invariance under topological conjugacy. The paper provides a comprehensive insight into the exponential features for the maps satisfying TCE; surprisingly, there is no need to involve any Markov type structure.

The condition TCE is natural because it can be formulated in several terms: differential, topological, measure-theoretic, geometric. The class of maps satisfying TCE is likely to be generic in the complement of the class of uniformly hyperbolic maps in measure-theoretic sense (see [B] and compare with [S] and [GSw1]). (This is true for quadratic maps of the interval, see [AM] and the earlier [Lyu].) Our methods contribute to the development of the theory of unimodal and multimodal maps of the interval (see [GSw2] and Swiatek's address at the ICM, Berlin 1998). Understanding non-uniformly hyperbolic dynamics, originated by J. Sinai in the sixties with billiards theory, became a challenge in various areas of dynamical systems in the past 20 years. We present a contribution in the complex 1-dimensional situation.

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1^o *The main results.*

Given a rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Riemann sphere $\overline{\mathbb{C}}$ we denote by $J(f)$ the Julia set of f, which is the closure of the repelling periodic points of f. Denote by Crit the set of critical points of f, which are the points where the derivative of f is equal to zero. Derivatives are taken with respect to the spherical metric. We will assume that all rational maps under consideration do not have parabolic cycles. (In fact all conditions below except CE yield this assumption. For elements of an analogous theory in the presence of parabolic cycles see [PU2].)

The conditions we discuss include the following ones.

• CE. *Collet-Eckmann condition.* There exist $\lambda_{CE} > 1$ and $C > 0$ such that for every critical point c in $J(f)$, whose forward orbit does not meet other critical points, and for every $n \geq 0$ we have

$$|(f^n)'(f(c))| \geq C\lambda_{CE}^n. \qquad (CE \text{ at } c)$$

• CE2(z_0). *Backward or second Collet-Eckmann condition at* $z_0 \in \overline{\mathbb{C}}$. There exist $\lambda_{CE2} > 1$ and $C > 0$ such that for every $n \geq 1$ and every $w \in f^{-n}(z_0)$,

$$|(f^n)'(w)| \geq C\lambda_{CE2}^n.$$

(In this case z_0 is necessarily not in the forward orbit of a critical point.)

• CE2. *Backward* or *second Collet-Eckmann condition.* The condition $CE2(c)$ holds for all critical points $c \in J(f)$ not in the forward orbit of other critical points.

• UHP. *Uniform Hyperbolicity on Periodic orbits.* There exists $\lambda_{\rm Per} > 1$ such that every repelling periodic point $p \in J(f)$ of period $k \geq 1$ satisfies,

$$|(f^k)'(p)| \geq \lambda_{\rm Per}^k.$$

• TCE. *Finite criticality* or *Topological Collet-Eckmann condition.* There exist $M \geq 0, P \geq 1$ and $r > 0$ such that for every $x \in J(f)$ there exists a strictly increasing sequence of positive integers n_j, for $j = 1, 2, ...$ such that $n_j \leq P \cdot j$ and for each j

$$\#\{i : 0 \leq i < n_j, \mathrm{Comp}_{f^i(x)} f^{-(n_j - i)} B(f^{n_j}(x), r) \cap \mathrm{Crit} \neq \emptyset\} \leq M,$$

where Comp_x means the connected component containing x.

• Lyapunov. *Lyapunov exponents of invariant measures are bounded away from zero.* There is a constant $c > 0$ such that the Lyapunov exponent of any invariant probability measure μ supported on Julia set satisfies $\Lambda(\mu) := \int \log |f'| d\mu \geq c$.

• ExpShrink. *Exponential shrinking of components.* There exist $\lambda_{\text{Exp}} > 1$ and $r > 0$ such that for every $x, y \in J(f)$ and $n > 0$ satisfying $f^n(y) = x$,

$$\operatorname{diam}(\operatorname{Comp}_y f^{-n}(B(x, r))) \leq \lambda_{\text{Exp}}^{-n}.$$

The following condition is defined only for polynomials (later on it will be generalized to all rational maps).

• Hölder. *The attracting basin of infinity is a Hölder domain.* In the case that $J(f)$ is connected this is equivalent to the condition that a conformal uniformization $\varphi : \mathbb{D} \to B$ of the attracting basin B, can be extended in a Hölder way to $\partial\mathbb{D}$. In the general case it is defined with an inequality involving the quasihyperbolic metric; see [GS1] and [Po].

With the help of new technical ideas we complete the theory started in [GS1], [PR1], [PR2], and [P3] to obtain the following result.

Main Theorem. *For a rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ without parabolic periodic points the following conditions are equivalent.*
(a) *CE2(z_0) holds for some $z_0 \in \overline{\mathbb{C}}$.*
(b) *UHP.*
(c) *TCE.*
(d) *Lyapunov.*
(e) *ExpShrink.*
(f) *Hölder (provided f is a polynomial).*
Supremums over possible constants $\lambda_{\text{CE2}}(z_0)$ (supremum also over z_0), λ_{Per}, $\log c$ in Lyapunov, and λ_{Exp}, are all equal to each other.

Moreover each of the conditions
(g) *CE.*
(h) *CE2.*
implies all the previous ones, and there are counterexamples to any other implication not mentioned.

In the case that there is only one critical point in the Julia set all conditions above are equivalent. Furthermore, if all critical points in the Julia set have the same dynamical multiplicity, the last two conditions are equivalent.

The *dynamical* multiplicity takes into account the possibility that a critical point may be mapped to another critical point; see Preliminaries for a definition.

The condition TCE is stated in purely topological terms and hence is invariant under topological conjugacy. So we obtain the following immediate corollary.

Corollary. *All conditions (a)-(f) are invariant under topological conjugacy. So are conditions (g) and (h) in the case where there is only one critical point in the Julia set.*

One may think that condition (g)CE or (h)CE2 may be invariant by topological conjugacy, even in the case when there are several critical points in the Julia set. We provide examples of two real polynomials of degree 4 that are quasiconformally conjugated, one satisfying both conditions and the other satisfying neither.

Conditions CE and CE2 were first introduced by P. Collet and J.-P. Eckmann in [CE], providing a large class of S-unimodal maps of an interval having a finite absolutely continuous invariant measure. Further investigations by T. Nowicki, S. van Strien, and D. Sands proved the equivalence of those and several similar properties in the real S-unimodal setting, in which our investigations have originated; see [NS] and also [N].

Rational Collet-Eckmann maps were first studied in [P1]. The condition TCE was first introduced in [PR1] in the complex setting and next in [NP] in the real setting.

The existence of a unique nonatomic conformal measure and of a finite invariant measure absolutely continuous with respect to it for rational Collet-Eckmann maps was proved in [GS2], see also [P1]. For TCE maps with more than 1 critical point in $J(f)$ the answer is not known.

The situation in the complex setting is not completely analogous to the real one and requires some new techniques. Furthermore, we obtain results in the presence of several critical points, whereas the results in the real setting are restricted to the unimodal case (one critical point).

2^o *On the equivalence of (a)-(f).*

Several implications in the theorem were known before.

In [PR1] it was proven that (e)ExpShrink$\Leftrightarrow$(c)TCE. Linking (d)Lyapunov to the other properties relies on Katok-Pesin theory and on [P5]; we devote Appendix C to this. To prove (e)ExpShrink$\Rightarrow$(b)UHP is easy (It was first noted in [NP] and [P3], see also our Proposition 2.2).

Let us discuss now (b)UHP$\Rightarrow$(e)ExpShrink. We shall prove this *via* (a) and a version of the condition (f)Hölder for general rational maps, that we call the Hölder coding tree condition, see the definition below.

We provide two proofs of (b)UHP$\Rightarrow$(a)CE2(some z_0) (one in section 2, one in Appendix B). They involve a sort of Bowen's specification property (shadowing by periodic orbit). They are not hard but tricky.

The most unexpected implication, and also the most involved, is (a)CE2(some z_0)$\Rightarrow$ (e)ExpShrink. A natural approach is to join z_0 to $B(x,r)$ with a curve slowly approached by critical values of f^n and deduce that the diameters of the connected components of $f^{-n}(B(x,r))$ shrink exponentially fast using the condition CE2(z_0) and controlling distortion along the curve. However it is hard to succeed this way directly.[2] At this point the rational map version of (f)Hölder provides help.

[2] In an early version of the paper we succeeded, assuming CE2(z_0) for all critical points. Troubles with $\mathrm{diamComp}_y f^{-n}(B(x,r))$ arising after meeting a critical point c can be overcome directly using the assumption CE2(c).

Already in [GS1] it was proved that for polynomials the condition (f)Hölder was equivalent to CE2(z_0), for some z_0 in the attracting basin of infinity. In [PR3] it was proved that CE2(z_0) for z_0 in the basin was equivalent to (e)ExpShrink.

We generalize these considerations to rational maps, replacing the basin of attraction of infinity by a *geometric coding tree* (*gct* for short) in the sense of [PUZ].

More precisely we prove that condition (a)CE2(z_0) for some $z_0 \in \overline{\mathbb{C}}$ implies (and *a posteriori* is equivalent to) the following condition

• Hölder coding tree. *There is $w \in \overline{\mathbb{C}}$ and a geometric coding tree $\mathcal{T}(w)$ that is Hölder.* Namely for each $w_1 \in f^{-1}(w)$ there exist a path $\gamma_{w_1} \subset \overline{\mathbb{C}} \setminus \cup_{n\geq 1} f^n(\mathrm{Crit})$ and constants $\lambda > 1$ and $C > 0$ so that for any $n \geq 0$ and any connected component γ of $f^{-n}(\gamma_{w_1})$ we have $\mathrm{diam}(\gamma) \leq C\lambda^{-n}$.

(gct $\mathcal{T}(w)$ is the graph with vertices $f^{-n}(w)$ and edges $f^{-n}(\gamma_{w_1})$. For a theory concerning convergence of general gct's see [PS].)

Finally we prove that the Hölder coding tree condition implies (e)ExpShrink (Section 1). This is similar to what happens for polynomials in [P3]; we give the proof for completeness and because it provides Lemma 1.2 that helps to replace z_0 by a better point w.

We remark that in the real unimodal setting the condition CE2(some z_0) does not imply ExpShrink or TCE. For example, for the Feigenbaum quadratic polynomial the condition CE2(z_0) holds (in the real sense) for all z_0 not accumulated by the critical orbit.

3^o On (b)UHP and variants of the condition (d)Lyapunov.

A step in (b)UHP⇒(b)Lyapunov is that UHP implies the following variants of the condition (d)Lyapunov (Appendix C).

• Lyapunov2. *Lyapunov exponents of invariant measures with positive Lyapunov exponents are bounded away from zero.* The same as the condition (d)Lyapunov, but only for all invariant probability measures μ satisfying $\Lambda(\mu) > 0$.

• Lyapunov$_+$. *Lyapunov exponents of invariant measures with positive entropy are bounded away from zero.* The same as the condition (b)Lyapunov, but only for all invariant probability measures μ of positive entropy $I(\mu) := \int \log \frac{d\mu(f)}{d\mu} d\mu > 0$.

Note that Lyapunov2 implies Lyapunov$_+$ since by Ruelle's inequality $I(\mu) \leq 2\Lambda(\mu)$.

Also the proof of (b)UHP⇒(a)CE2(some z_0) in Appendix B yields that (b)UHP implies $\Lambda(\mu) \geq c > 0$ for a large class of measures.

Note finally that the definition of UHP imposes a condition only on *repelling* cycles; it corresponds to Lyapunov2. However *a posteriori*, *via* ExpShrink, we obtain absence of indifferent cycles (in particular the absence of Siegel discs, see Proposition 2.2.) So,

we obtain condition UHP on all cycles in the Julia set; this corresponds to the condition (d)Lyapunov.

4^o *On CE and CE2.*

We have explained all equivalences in the first part of the Main Theorem. It remains consider the conditions (g)CE and (h)CE2.

The implication (h)CE2⇒(a)CE2(some z_0) is obvious and (g)CE⇒(c)TCE was proven in [PR1] so conditions (g)CE and (h)CE2 are stronger than all others.

We give examples of real polynomials of degree 5 that satisfy (g)CE but not (h)CE2 and *vice-versa*. S. van Strien considered one of such example (though not polynomial) for maps of the interval [vS]. We also provide a real polynomial of degree 3 satisfying (c)TCE but neither (g)CE nor (h)CE2 (Section 4). A similar example (in degree 4) was considered in [PR2], but the measurable Riemann mapping theorem was used to justify it.

All these examples (and the above mentioned example of CE and CE2 not invariant under topological conjugacy) are semihyperbolic in the sense of [CJY], that is all critical points are non-recurrent, or equivalently TCE with $P = 1$; see [M] and [CJY].

The troubles with CE or CE2 arise from the forward trajectory of one critical point closely approaching another critical point.

In [GS1] it is proved that (g)CE implies CE2(c) for critical points $c \in J(f)$ with maximal (dynamical) multiplicity. It is not difficult to prove that (h)CE2 together with (e)ExpShrink imply CE(c) at critical points $c \in J(f)$ with maximal (dynamical) multiplicity, see [GS1] and also [R-L] for a simpler case.

Hence, due to our new implication (a)CE2(some z_0)⇒(e)ExpShrink, we obtain the equivalence of the conditions (g)CE and (h)CE2 in the class of all rational functions for which all critical points in J have the same (dynamical) multiplicity (particularly when there is only one critical point in the Julia set).

It remains to note that for rational functions with only one critical point in the Julia set the implication (c)TCE⇒(g)CE was proven in [P3]. Let us mention that in [GP] J. Graczyk and I. Popovici proved directly the implication (b)UHP⇒CE for quadratic non infinitely renormalizable polynomials.

5^o *Pressure and dimension.*[3]

There are more conditions equivalent to TCE. For example it is not difficult to see that the condition (d)Lyapunov is equivalent to the following condition.

• Negative Pressure. *Pressure for large t is negative.* For large values of t the pressure function $P(t) \equiv P_f(-t \log |f'|) := \sup\{I(\mu) - t\Lambda(\mu) : \mu \text{ invariant probability}\}$ is negative.

In fact the condition $P(t_1) < 0$ implies, for every f-invariant μ on $J(f)$, that $\Lambda(\mu) \geq -P(t_1)/t_1 > 0$. Thus Negative Pressure⇒(d)Lyapunov.

[3] We plan to develop further this topic in relation with [P2] in a subsequent paper.

On the other hand, since entropies of the invariant measures are always non-negative and bounded above by the topological entropy of f (which is equal to $\log \deg f$), we conclude that $\lim_{t\to\infty} P(t)/t = -\inf\{\Lambda(\mu) : \mu\}$. Hence (d)Lyapunov⇒Negative Pressure.

In general $P(t)$ is decreasing since $\Lambda(\mu) \geq 0$, see [P5]. Here $\Lambda(\mu) \geq c > 0$ hence $P(t)$ is strictly decreasing so its zero is unique. We prove in Appendix C that this zero is equal to the box dimension of the Julia set. We conclude relying on [P2] that the box, hyperbolic and Haussdorff dimensions of the Julia set coincide. They are strictly less than 2 by mean porosity [PR1], provided $J(f) \neq \overline{\mathbb{C}}$.

6^o *Other conditions.*

In the proof of the Main Theorem we further consider the two following conditions.

• ExpShrinkCrit. *Exponential shrinking of components at critical points.* The same as ExpShrink, but only for $y \in \mathrm{Crit} \cap J(f)$.

• BExpShrink(z_0). *Backward exponential shrinking of components at* $z_0 \in \overline{\mathbb{C}}$. The same as ExpShrink, but only for $x = z_0$.

We state the following complement of the Main Theorem that includes the conditions stated after the Main Theorem.

Complement to the Main Theorem. *The condition TCE is equivalent to the following ones.*

- *CE2(z_0) holds for all $z_0 \in \overline{\mathbb{C}}$, except for an exceptional set E of Hausdorff dimension zero, with the same $\lambda_{CE2} > 1$ (and the same supremums of λ_{CE2}'s at each z_0) and possibly with different constant $C > 0$.*
- *Hölder coding tree.*
- *BExpShrink(z_0) holds for some $z_0 \in \overline{\mathbb{C}}$.*
- *ExpShrinkCrit.*
- *Lyapunov$_+$.*
- *Lyapunov2.*
- *Negative Pressure.*

For the relations (e)ExpShrink⇔ExpShrinkCrit⇔(c)TCE see [PR1] and [P3]. We discuss them briefly in Section 5.

The proofs of the implications (a)CE2(some z_0)⇒BExpShrink(z_0)⇒CE2(w) for a set of positive Lebesgue measure of $w \in \overline{\mathbb{C}}$ ⇒Hölder coding tree⇒(e)ExpShrink, are intermediate steps to prove (a)CE2(some z_0)⇒(e)ExpShrink (Section 1). This proof also yields that (e)ExpShrink⇒CE2(z_0) for most $z_0 \in \overline{\mathbb{C}}$, as above.

The rest of the implications were explained above.

We discuss further variants of the condition TCE. We are mainly interested in the following condition.

• TCECrit. *Finite criticality (or Topological Collet-Eckmann) at critical points.* The same as TCE, but only for $x \in \mathrm{Crit}$.

We say that TCE (resp. TCECrit) *has criticality* M, if (TCE at x) holds with the constant M for all x, (resp. all x such that $x \in f(\mathrm{Crit})$ and $\bigcup_{n=0}^{\infty} f^n(x) \cap \mathrm{Crit} = \emptyset$). Furthermore we consider the following auxiliary property.

• BLS. *Backward Lyapunov stability.* For every $\delta > 0$ there exists $r > 0$ such that for every $x \in J$ and any positive integer n, $\mathrm{diamComp}_x f^{-n} B(f^n(x), r)) \leq \delta$.

We prove that (TCECrit & BLS)⇒ ExpShrinkCrit. In fact it was already implicitly contained in [PR1], [P3], and [S]. Together with ExpShrinkCrit ⇔ TCE ⇔ ExpShrink ⇒ BLS, this gives a new topological condition equivalent to TCE, namely TCECrit & BLS.

We prove also that if an upper density in the set of all natural numbers of the set of integers n_j such that TCE(c) at a critical point $c \in J(f)$ holds with criticality M is positive, then it is also positive with $M = 0$ at a critical value $f(c')$ for a critical point c' in the closure of the forward orbit of c.

Organization of the Paper.

Section 1 is mainly devoted to the proof that CE2(some z_0)⇒ExpShrink with the intermediate steps mentioned above. In particular we replace CE2(z_0) by CE2(w) with a better point w for which we can construct a Hölder coding tree. The main tool to construct such a Hölder coding tree is a technical geometric lemma proved in Appendix A.

In *Section 2* we deal with periodic points. In particular we prove that UHP implies CE2(z_0) for most $z_0 \in \overline{\mathbb{C}}$ and that ExpShrink implies the absence of indifferent cycles.

In *Section 3* we provide counterexamples needed in the proof of the Main Theorem.

In *Section 4* we provide examples of two real polynomials of degree 4 that are quasi-conformally conjugate: one satisfying CE and CE2, the other satisfying neither of these conditions. In particular neither CE nor CE2 are invariant by topological conjugacy. All examples in Sections 3 and 4 are semi-hyperbolic.

In *Section 5* we discuss further variants of the condition TCE, as explained at the end of the introduction.

Appendix A contains a geometric technical lemma, which is essential to construct a Hölder coding tree, under the assumption that CE2(z_0) holds for some $z_0 \in \overline{\mathbb{C}}$ (Section 1).

We find curves from w to its preimages under f slowly approximable by critical values of f^n, that is, of slowly growing quasihyperbolic lengths in $\overline{\mathbb{C}} \setminus f^n(\mathrm{Crit})$.

An analogous geometric fact (however with cuves depending on n) was one of the main technical results in [P2], and was used independently in [HH] to study the distribution of zeros of the Riemann ζ-function (see Remark A.3).

Appendix B contains another proof that UHP implies CE2(z_0) for some (and most) z_0.

In *Appendix C* we deal with conditions involving Lyapunov exponents, pressure and dimensions.

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Preliminaries.

Distances, balls, diameters, and derivatives are considered with respect to the spherical metric on $\overline{\mathbb{C}}$. For $z \in \overline{\mathbb{C}}$ and $r > 0$, $B(z, r)$ denotes the ball centered at z with radius r. Const or C stands for various constants that can change even in one consideration. $a \sim b$ for numbers or functions a, b means $a/b < C$ and $b/a < C$

Consider a rational map f. We denote the Julia set $J(f)$ also by J when there is no danger of confusion. Recall that a parabolic cycle is a cycle whose multiplier is a root of unity. As mentioned in the introduction we assume that the rational maps under consideration do not have parabolic cycles.

In the case that a critical point in $J(f)$ is eventually mapped to another critical point, we treat whole blocks of critical points as just one critical point; that is if the critical points c_{i_1}, ..., $c_{i_k} \in J(f)$ are such that c_{i_s} is eventually mapped to $c_{i_{s+1}}$ by iteration, then we treat this block of critical points as one critical point with multiplicity equal to the product of the multiplicities of the c_i; this later is the *dynamical* multiplicity referred to in the introduction. So we can think that no critical point in $J(f)$ is mapped to a critical point.

Fix two periodic orbits O_1 and O_2 of period at least 2 and let $r_K > 0$ be such that for every $x \in \overline{\mathbb{C}}$ the disc $B(x, r_K)$ is disjoint from either O_1 or O_2. Then for every positive integer n and every component W of $f^{-n}(B(x, r_K))$, we have,

$$\mathrm{diam}(\overline{\mathbb{C}} \setminus W) \geq \min\{\mathrm{diam}(O_1), \mathrm{diam}(O_2)\} > 0.$$

Hence the following version of Koebe Distortion Theorem holds for the spherical metric for inverse branches of f^n (see also [P1, Lemma 1.2]).

Koebe Distortion Theorem. *For every $0 < \varepsilon < 1$ there exists a constant $C_K(\varepsilon) > 1$ such that for every disc $B(x,b)$ with $b \le r_K$, every n, and every component W of $f^{-n}(B(x,b))$ such that f^n is univalent on W, we have*

$$|(f^n)'(z_1)|/|(f^n)'(z_2)| \le C_K(\varepsilon),$$

for every $z_1, z_2 \in W$ such that $f^n(z_i) \in B(x, \varepsilon b)$ for $i = 1, 2$. Moreover $C_K(\varepsilon) \to 1$ as $\varepsilon \to 0$.

We call supremum of the left hand side ratios *the distortion* of f^n in $W(\varepsilon) = \mathrm{Comp}_W f^{-n}(B(x, \varepsilon b))$.

Section 1. Equivalence of ExpShrink and CE2(z_0).

In this section we prove that for a rational map $f \in \mathcal{C}(z)$, the condition ExpShrink is equivalent to CE2(z_0) for some $z_0 \in \overline{\mathcal{C}}$. We begin with the easier implication,

ExpShrink implies CE2(z_0) for a set of positive Lebesgue measure of $z_0 \in \overline{\mathcal{C}}$. Let $\lambda_{Exp} > 1$ and $r > 0$ be so that for every $w \in J(f)$ and every connected component W of $f^{-n}(B(w,r))$ we have $\mathrm{diam}(W) \le \lambda_{\mathrm{Exp}}^{-n}$. For $w \in J(f)$ fixed we have that,

$$\mathrm{Area}\Big(\bigcup_{n\ge 1} B(f^n(\mathrm{Crit}), C_2/n))\Big) \le C_2^2 \sum_{n\ge 1} (\#\mathrm{Crit})\pi/n^2,$$

which is smaller than the measure of $B(w,r)$, for C_2 small enough. So there is a set of positive measure of $z_0 \in B(w,r)$ not in $\bigcup_{n\ge 1} B(f^n(\mathrm{Crit}), C_2/n))$. Assume also that $C_2 \le \mathrm{dist}(z_0, \partial B(c,r))$, for all critical points $c \in \mathrm{Crit}$.

It follows from Koebe 1/4 Theorem applied to an arbitrary $y \in f^{-k}(z_0)$ and the branch g of f^{-k} on $B(z_0, C_2/k)$ mapping z_0 to y, that

$$C_3 \cdot (C_2/k) \cdot |(f^k)'(y)|^{-1} \le \mathrm{diamComp}_y f^{-k}(B(z_0, C_2/k)) \le$$

$$\mathrm{diamComp}_y f^{-k}(B(w,r)) \le \lambda_{\mathrm{Exp}}^{-k}.$$

The constant C_3 plays the role of 1/4; it takes into account that we consider the spherical metric, rather than the Euclidean one in $\mathcal{C}$, in which the Koebe 1/4 Theorem is usually considered. Hence for all $\lambda_2 \in (1, \lambda_{Exp})$ there is $C_4 = C_4(\lambda_2)$ such that for all $y \in f^{-n}(z_0)$,

$$|(f^k)'(y)| \ge C_4 \lambda_2^k. \qquad QED$$

Remark 1.1. The same proof yields that there is an exceptional set $E \subset B(w,r)$ of Hausdorff dimension zero such that CE2(z_0) holds for every $z_0 \in B(w,r) \setminus E$ with any constant $\lambda \in (1, \lambda_{\mathrm{CE2}})$. In fact E can be taken of zero Hausdorff measure with any gauge function ϕ (appearing in $\sum_j \phi(\mathrm{diam} B_j)$ in the definition of Hausdorff measure) satisfying $\sum_n \phi(\mu^n) < \infty$ for all $\mu < 1$.

Lemma 1.2. *Suppose that* CE2(z_0) *holds for f with some constant $\lambda_{\rm CE2} > 1$. Then* BExpShrink(z_0) *holds, for any $\lambda \in (1, \lambda_{\rm CE2})$; that is there is $r > 0$ such that for all $n \geq 0$ and every connected component W of $f^{-n}(B(z_0, r))$ we have that* $\mathrm{diam}(W) \leq \lambda^{-n}$.

The assumption CE2(z_0) can be replaced by the existence of a continium K containing z_0 and some other point, such that for every connected component K_0 of $f^{-n}(K)$ we have $\mathrm{diam}(K_0) \leq \lambda_{\rm Exp}^{-n}$.

This Lemma, stated for individual backward trajectories, was proved in [P4, Lemma 5, Remark 1], it was called there "telescope lemma." We provide here for completeness a simplified proof.

Before providing a proof of Lemma 1.2 let us make two observations about distortion. First, it follows from Koebe Distortion Theorem that for any $D > 1$ there is $0 < \varepsilon < 1$ small enough, so that if the disc W satisfies $\mathrm{diam}(W) \leq \varepsilon \mathrm{dist}(W, f(\mathrm{Crit}))$ (in particular $W \cap f(\mathrm{Crit}) = \emptyset$), then the distortion of f in any connected component W_1 of $f^{-1}(W)$ is bounded by D. Secondly, once a small $r_0 > 0$ is fixed, there is $C_2 \geq 1$ such that for every disc $W \subset \overline{\mathbb{C}}$ and for every component W_1 of $f^{-1}(W)$ the inequality $\mathrm{diam}(W_1) \leq C_2 \mathrm{diam}(W)|f'(z)|^{-1}$ holds for every $z \in W_1$. We shall use this estimate for W close to $f(\mathrm{Crit})$.

Proof of Lemma 1.2. Let $C_1 > 0$ be the constant C that stands in the property CE2(z_0). Fix $D \in (1, \lambda_{\rm CE2}/\lambda)$ and consider the respective $\varepsilon > 0$ as above. Let $C_2 > 0$ be as above.

Let $l \geq 1$ be such that $C_2^{1+j/l} C_1^{-1} D^j \lambda_{\rm CE2}^{-j} \leq \frac{1}{2}\lambda^{-j}$ for $j \geq l$. Also let $r_1 > 0$ be so that for all $c \in \mathrm{Crit}$, $0 \leq k \leq l$, and every connected component W of $f^{-k}(B(c, 2r_1))$ we have $\mathrm{diam}(W) \leq \varepsilon \mathrm{dist}(W, \mathrm{Crit})$. Recalling convention that critical points are not hit by trajectories of other critical points and simple distortion argument allows to find such r_1.

Finally let $r > 0$ be so that for any connected component W of $f^{-k}(B(z_0, r))$ for $0 \leq k \leq l$ we have $\mathrm{diam}(W) \leq \varepsilon r_1$ (in particular $r \leq \varepsilon r_1$).

Choose any sequence of pull-backs: $W_0 = B(z_0, r)$, $W_1 = \mathrm{Comp} f^{-1}(W_0), ..., W_{k+1} = \mathrm{Comp} f^{-1}(W_k), ...$ and choose $z_k \in W_k$ so that $f(z_{k+1}) = z_k$, for $k \geq 0$.

Let $k_0 < k_1 < ...$ be all the integers k such that $\mathrm{diam}(W_k) > \varepsilon \mathrm{dist}(W_k, \mathrm{Crit})$.

So we have for all $0 \leq j < k_0$,

$$\mathrm{diam}(W_j) \leq D^j |(f^j)'(z_j)|^{-1} \mathrm{diam}(W_0). \tag{1.1}$$

Setting $j = k_0 - 1$ we obtain

$$\mathrm{diam}(W_{k_0}) \leq C_2 \mathrm{diam}(W_{k_0-1}) |f'(z_{k_0})|^{-1} \leq C_2 D^{k_0-1} |(f^{k_0})'(z_{k_0})|^{-1} \mathrm{diam}(W_0)$$

$$\leq C_2 C_1^{-1} D^{k_0} \lambda_{\rm CE2}^{-k_0} \mathrm{diam}(W_0) \leq \frac{1}{2}\lambda^{-k_0} \mathrm{diam}(W_0) \leq \varepsilon r_1 \tag{1.2},$$

the last line being true when $k_0 > l$. If $k_0 \leq l$ then directly $\mathrm{diam}(W_{k_0}) \leq \varepsilon r_1$.

We conclude that in both cases $\mathrm{dist}(W_{k_0}, \mathrm{Crit}) < \frac{1}{\varepsilon}\mathrm{diam}(W_{k_0}) \leq r_1$, and $W_{k_0} \subset B(\mathrm{Crit}, 2r_1)$.

Now by induction we prove that

$$\operatorname{diam}(W_{k_i}) \le C_2^{1+k_i/l} D^{k_i} |(f^{k_i})'(z_{k_i})|^{-1} \operatorname{diam}(W_0), \tag{1.3}$$

and the latter can be estimated by

$$\ldots \le C_2^{1+k_i/l} C_1^{-1} D^{k_i} \lambda_{\mathrm{CE2}}^{-k_i} \operatorname{diam}(W_0) \le \frac{1}{2} \lambda^{-k_i} \operatorname{diam}(W_0) \le \varepsilon r_1. \tag{1.4}$$

In fact, we already established (1.3) and (1.4) for $i = 0$. Assuming that (1.3) and (1.4) hold for $i < \ell$, we see that $\operatorname{dist}(W_{k_i}, \mathrm{Crit}) < \frac{1}{\varepsilon} \operatorname{diam}(W_{k_i}) \le r_1$ for $i < \ell$. Hence W_{k_i} is contained in a ball of radius $2r_1$ around some critical point and thus $k_{i+1} - k_i > l$ by the definition of r_1. Therefore for $i = \ell$ estimate (1.4) holds (since $k_\ell > l$) and to prove (1.3) we can write similarly to (1.2)

$$\operatorname{diam}(W_{k_\ell}) \le C_2 D^{k_\ell - k_{\ell-1}} |(f^{k_\ell - k_{\ell-1}})'(z_{k_\ell})|^{-1} \operatorname{diam}(W_{k_{\ell-1}})$$

$$\le C_2^{2+k_{\ell-1}/l} D^{k_\ell} |(f^{k_\ell})'(z_{k_\ell})|^{-1} \operatorname{diam}(W_0) \le C_2^{1+k_\ell/l} D^{k_\ell} |(f^{k_\ell})'(z_{k_\ell})|^{-1} \operatorname{diam}(W_0).$$

We have proved exponential decay of diameters of preimages of orders k_i. To deal with arbitrary order j, we deduce for $j \in [k_i, k_{i+1})$, similarly to (1.1), the desired estimate

$$\operatorname{diam}(W_j) \le D^{j-k_i} |(f^{j-k_i})'(z_j)|^{-1} \operatorname{diam}(W_{k_i})$$

$$\le C_2^{1+j/l} C_1^{-1} D^j |(f^j)'(z_j)|^{-1} \operatorname{diam}(W_0) \le \lambda^{-j},$$

when $j \ge k_1$, and by decreasing r it follows for all j.

If z_0 is replaced by a continuum K in the assumptions, then in the estimates we just replace $|(f^j)'(z_j)|^{-1}$ for $j < k_0$, by the ratio

$$\operatorname{diam}(\operatorname{Comp}_{z_j} f^{-j}(K \cap B(z_0, r)))/r,$$

replace $|f'(z_{k_\ell})|^{-1}$ by

$$\operatorname{diam}(\operatorname{Comp}_{z_{k_\ell}} f^{-(k_\ell)}(K \cap B(z_0, r)))/\operatorname{diam}(\operatorname{Comp}_{z_{k_\ell - 1}} f^{-(k_\ell - 1)}(K \cap B(z_0, r))),$$

and finally for $j \in [k_i, k_{i+1})$ replace $|(f^{j-k_i})'(z_j)|^{-1}$ by

$$\operatorname{diam}(\operatorname{Comp}_{z_j} f^{-j}(K \cap B(z_0, r)))/\operatorname{diam}(\operatorname{Comp}_{z_{k_i}} f^{-k_i}(K \cap B(z_0, r)).$$

QED

The following is the key lemma to prove ExpShrink; it is based on Lemma A of the Appendix A.

Lemma 1.3. *Suppose that* CE2(z_0) *holds for some* $z_0 \in \overline{\mathbb{C}}$ *with constant* $\lambda_{\mathrm{CE2}} > 1$ *and fix* $\lambda_0 \in (1, \lambda_{\mathrm{CE2}})$. *Then there is a set of positive Lebesgue measure of* $w \in \overline{\mathbb{C}}$ *for which*

we can construct a Hölder tree with root w. That is, we can join w to every preimage $w_1 \in f^{-1}(w)$ of w, by a path $\gamma_{w_1} \subset \overline{\mathbb{C}} \setminus \cup_{n\geq 0} f^n(\mathrm{Crit})$ such that for all $n \geq 0$ and all connected components γ of $f^{-n}(\gamma_{w_1})$,

$$\mathrm{diam}(\gamma) \leq C_0 \lambda_0^{-n},$$

where $C_0 = C_0(\lambda_0, w)$ is independent of n.

Proof. Fix $\lambda \in (\lambda_0, \lambda_{\mathrm{CE2}})$, then by Lemma 1.2 and by Remark 1.1 there is a set of positive Lebesgue measure of points w for which CE2(w) holds with the constant $\lambda > 1$ and $C > 0$. Thus one may choose such a point w also satisfying the conclusions of Lemma A of Appendix A. That is, for a fixed $\alpha \in (\frac{1}{2}, 1)$, we can join w to any $w_1 \in f^{-1}(w)$ with a path γ_{w_1} so that the distortion of f^n, $n \gg 1$ in any curve being a pull-back γ_n of γ_{w_1} by f^{-n} is bounded by D^{n^α} for some definite constant $D > 1$.

Since CE2(w) holds we have $\mathrm{diam}(\gamma_n) = \mathcal{O}(D^{n^\alpha} \lambda^{-n}) = \mathcal{O}(\lambda_0^{-n})$, for any $n \geq 1$. QED

Proof of CE2(z_0) at some $z_0 \in \overline{\mathbb{C}}$ implies ExpShrink. We proceed like in the proof of Hölder implies ExpShrink in [P3], as follows.

Let w be as in Lemma 1.3 and for $w_n \in f^{-n}(w)$ let γ_{w_n} be the pull-back of the path γ_{w_1}, where $w_1 = f^{n-1}(w_n)$, with w_n as one endpoint. Let $\sigma(w_n) \in f^{-(n-1)}(w)$ denote the other endpoint of γ_{w_n}.

Fix an arbitrary $z \in J(f)$ and let w_{n_j} be a sequence of n_j-preimages of w converging to z as $j \to \infty$. Taking a subsequence if necessary we may suppose that for all $n \geq 1$, $z_n = \sigma^{n_j - n}(w_{n_j})$ does not depend on j for big $j > \mathrm{Const}(n)$. By Lemma 1.3 it follows that,

$$\mathrm{dist}(z, z_n) = \lim_{j\to\infty} \mathrm{dist}(z, \sigma^{n_j - n}(w_{n_j})) \leq C_1 \lambda_0^{-n},$$

where $C_1 = C_1(\lambda_0, w) = \frac{C_0(\lambda_0, w)}{\lambda_0 - 1}$ does not depend on n. Moreover $\gamma_{z_n} \subset B(z, C_1 \lambda_0^{1-n})$. Thus the γ_{z_n}'s form a *ray* δ from w landing at z.

We can assume that none of γ_{w_1} is constant, so that δ contains more than one point.

For a small $r > 0$ (independent of z) and $W_0 = B(z, r)$ let δ_0 be the piece of ray inside W_0 that joins z to ∂W_0. So $\delta_0 \subset \cup_{n\geq m(r)} \gamma_{z_n}$ where $m(r) \to \infty$ as $r \to 0$. So for every component δ_n of $f^{-n}(\delta_0)$ we get $\mathrm{diam}(\delta_n) \leq C_1 \lambda_0^{-(n+m(r))}$. So $K = \delta_0$ satisfies the assumptions of Lemma 1.2, with λ_{CE2} replaced by λ_0, for r small enough. This proves ExpShrink with λ_{Exp} arbitrarily close to λ_0, and hence to λ_{CE2}. QED

Section 2. Uniform Hyperbolicity at Periodic orbits (UHP).

In this section we prove that UHP implies CE2(z_0) for some $z_0 \in \overline{\mathbb{C}}$ and that ExpShrink implies UHP plus the non-existence of indifferent cycles.

Let $f \in \mathcal{C}(z)$ be a rational map. Consider a compact set K in which f is uniformly expanding and so that the Hausdorff dimension, $HD(K)$, of K is positive. Such set can

be constructed given any two (or even one) repelling cycles (see Appendix C, construction in Proof of Lemma C.1.) Choose $\alpha > HD(K)^{-1}$ and let $h > \alpha^{-1}$. Since

$$\frac{n_0}{n_0^{\alpha h}} + \sum_{n \geq n_0} \frac{1}{n^{\alpha h}} \to 0 \text{ as } n_0 \to \infty,$$

it follows that for all z_0 in K, except for a set of Hausdorff dimension at most $\alpha^{-1} < HD(K)$, there is $n_0 = n_0(z_0)$ such that $dist(z_0, f^n(\text{Crit}))$ is at least $n_0^{-\alpha}$ if $0 \leq n \leq n_0$ and at least $n^{-\alpha}$ otherwise. Thus all pull-backs of $B(z_0, n^{-\alpha})$ by f^n, for $n \geq n_0$, are univalent.

Fix $z_0 \in K$ as above. Since f is uniformly expanding in K there is $w \in K$ and $\delta > 0$ such that for all small $r > 0$ there is a univalent pull-back $V = f^{-l}(B(w, 2\delta)) \subset B(z_0, r)$. Then $|(f^l)'| \sim r^{-1}$ in V so $l \leq C_1 \ln \frac{1}{r}$.

Lemma 2.1. *Let $f \in \mathcal{C}(z)$ be a rational map such that every repelling cycle of period n has eigenvalue of modulus at least $\lambda_n > 0$. Then there are constants C, C_0 and $C_1 > 0$ such that if z_0 is fixed as above, for any n-th preimage z_n of z_0 with $n \geq n_0(z_0)$, $|(f^n)'(z_n)| > C\lambda_{n+k} n^{-\alpha}$ for some $0 \leq k \leq C_0 + C_1 \ln n$.*

Proof. Let K, z_0, w, δ, ... be as above. By the eventually onto property of Julia sets there is m such that $J(f) \subset \cup_{0 \leq i \leq m} f^i(B(w, \delta))$.

Take $n \gg n_0(z_0)$ and fix $z_n \in f^{-n}(z_0)$. Then there is $z_{n+q} \in B(w, \delta)$ such that $f^q(z_{n+q}) = z_n$ with $0 \leq q \leq m$. By the above the pull-back W' of $B(z_0, (n+m)^{-\alpha})$ by $f^{-(n+q)}$ containing z_{n+q} is univalent.

Hence we may choose $\varepsilon > 0$ independent of n such that the pull-back $W \subset W'$ of $B(z_0, \varepsilon(n+m)^{-\alpha})$ by $f^{-(n+m)}$ satisfies $\text{diam}(W) \leq \delta$. Thus $W \subset B(w, 2\delta)$ and therefore there is a univalent pull-back W' of W by f^{-l} such that $W' \subset B(z_0, \frac{1}{2}\varepsilon(n+m)^{-\alpha})$ and $|(f^l)'| \sim n^\alpha$ in W' thus $l \leq C_2 \ln n$.

Hence $f^{n+q+l} : W' \longrightarrow B(z_0, \varepsilon(n+m)^{-\alpha})$ is a bijection and therefore there is a repelling fixed point p of f^{n+q+l} in W'. Note that $k = q + l \leq C_0 + C_1 \ln n$ for definite constants C_0, $C_1 > 0$. Moreover,

$$\lambda_{n+k} \leq |(f^{n+k})'(p)| \leq \text{Const}\, n^\alpha |(f^n)'(f^k(p))| \leq \text{Const}\, n^\alpha |(f^n)'(z_n)|,$$

where the last inequality follows by Koebe Distortion Lemma. Therefore $|(f^n)'(z_n)| \geq C\lambda_{n+k} n^{-\alpha}$ as claimed. QED

Proof of UHP implies CE2(z_0) for some $z_0 \in \overline{\mathcal{C}}$. Suppose that UHP holds with constant $\lambda_{\text{Per}} > 1$. Letting $\lambda_n = \lambda_{\text{Exp}}^n$ in Lemma 2.1 we obtain that for every $\lambda \in (1, \lambda_{\text{Per}})$ there is $C > 0$ such that CE2(z_0) holds with constants $C > 0$ and $\lambda > 1$. QED

Proposition 2.2. *If the rational map $f \in \mathcal{C}(z)$ satisfies* ExpShrink *with a constant $\lambda_{\text{Exp}} > 1$, then for every non attracting periodic point p of period k we have $|(f^k)'(p)| \geq \lambda_{\text{Exp}}^k$. In particular* ExpShrink *implies* UHP *with $\lambda_{\text{Per}} \geq \lambda_{\text{Exp}}$.*

Note that the definition of UHP imposes a condition only for repelling cycles. However, *via* UHP⇒CE2(z_0) ⇒ExpShrink and by Proposition 2.2 we obtain that UHP implies the non existence of indifferent cycles, even outside Julia set.

Proof of Proposition 2.2. Let $r > 0$ be the radius constant involved in ExpShrink. We first prove that f does not have Siegel discs. Suppose that D is a Siegel disc for f, that we may suppose to be fixed by f. Denote by ϕ a conformal uniformization mapping from the unit disc $\mathbb{D}$ to D. Denote by p the indifferent fixed point in D and suppose $\phi(0) = p$. Take an arbitrary $x \in \partial D$ accessible from D. More precisely, take $\zeta \in \partial\mathbb{D}$ so that $\phi(a\zeta) \to x$ as $a \nearrow 1$. (By a theorem of Beurling all $\zeta \in \partial\mathbb{D}$ outside an exceptional set of capacity zero satisfy this property.)

Let $W_0 = B(x, r)$ and fix $a_0 < 1$ so that for $\gamma = \{a\zeta : a_0 \leq a < 1\}$ we have $\phi(\gamma) \subset W_0$. Consider $W_n = \mathrm{Comp}_{\phi(S^{-n}(\gamma))}(W_0)$, where S is the rotation of $\mathbb{D}$ by the angle $\mathrm{Arg} f'(p)$. This makes sense since $\phi S = f\phi$. By ExpShrink we have $\mathrm{diam} W_n \leq \lambda_{\mathrm{Exp}}^{-n} \to 0$. On the other hand for an arbitrary $a_1 : a_0 < a_1 < 1$ and $\hat{\gamma} = \{a\zeta : a_0 \leq a < a_1\}$, for all n we have $S^{-n}(\hat{\gamma}) \subset \mathrm{cl}B(0, a_1)$. Since ϕ^{-1} is defined and uniformly continuous on $D' = \phi(\mathrm{cl}B(0, a_1))$ (by compactness) we obtain $a_1 - a_0 = \mathrm{diam}\ \phi^{-1}(\phi(S^{-n}(\hat{\gamma})) \leq \mathrm{diam}\ \phi^{-1}(W_n \cap D') \to 0$, a contradiction. Therefore f has no Siegel discs, so every non-attracting periodic point belongs to $J(f)$.

Let $p \in J(f)$ be a periodic point of period $k \geq 1$ and consider the pull-back $W_0 = B_r(p)$, W_1, ... along the cycle of p. Since $\mathrm{diam}(W_i)$ decreases exponentially with i, it follows, by reducing $r > 0$ if necessary, that the W_i are disjoint from Crit; so the pull-back is univalent. Applying Schwarz lemma to the inverse of $f^{kn} : W_{kn} \to W_0$ we obtain that,

$$|(f^k)'(p)|^n = |(f^{kn})'(p)| \geq C\lambda_{\mathrm{Exp}}^{kn},$$

for some $C > 0$ depending in p. Hence $|(f^k)'(p)| \geq \lambda_{\mathrm{Exp}}^k$, as wanted. QED

Section 3. Counter-examples.

A class of rational maps of a large interest are the semi-hyperbolic ones.

• We call a rational function f *semi-hyperbolic* if all critical points in $J(f)$ are non-recurrent.

Semi-hyperbolicity does not exclude that one critical point belongs to the forward limit set of another. It was proved in [CJY], with the use of [M], that in absence of parabolic periodic points semi-hyperbolicity is equivalent to TCE with $P = 1$, namely n_j is the sequence of all natural numbers. Hence semi-hyperbolicity implies ExpShrink by [P3], this was proved also in [CJY].

All counterexamples constructed in this section and in the next section are semi-hyperbolic.

Consider a holomorphic one-parameter family of polynomials f_σ with the following properties: f_σ is real for σ real, 0 is a repelling fixed point with multiplier $f'(0) = \lambda_\sigma$ and such that there are critical points c_1 and c_2 with the criticalities μ_1 and μ_2 respectively such that $f_\sigma^2(c_2) \equiv 0$. Assume furthermore that for some real σ_0, $f_{\sigma_0}^k(c_1) = 0$ for a natural number k, but $f_{\sigma_0}^k(c_1) \not\equiv 0$, or stronger: $\partial_\sigma f_\sigma^k(c_1) \neq 0$ at $\sigma = \sigma_0$.

Moreover suppose that there is an open set $W \subset \mathbb{C}$ such that the restriction of f_σ to W is polynomial-like with c_2 as unique critical point. Then there is an invariant interval $I \subset W \cap \mathbb{R}$ and an holomorphic motion i of I compatible with dynamics such that i_{σ_0} is the identity. Note that

$|i_\sigma(x)| \sim |x|^{\alpha_\sigma}$ for x close to 0, where α_σ is defined by $|\lambda_\sigma| = |\lambda_{\sigma_0}|^{\alpha_\sigma}$.

Fix $\eta > 0$ close to 1 and consider a point $x \in I$ with lower Lyapunov exponent $\liminf_{n\to\infty} \frac{1}{n} \log |(f_{\sigma_0}^n)'(x)| = \log \eta$ for f_{σ_0} with the following properties. The orbit of x accumulates at c_2 and the closest return times of x to c_2 are $n_1 < n_2 < ...$ which are also the only times that $f_{\sigma_0}^n(x)$ is close to c_2.

Such x can be found with the use of induction. Suppose x^i satisfies $|(f_{\sigma_0}^{n_j+1})'(x^i)| \sim \eta^{n_j+1}$ for all $j \leq i$ and the closest return times to c_2 are are $n_1, ..., n_i$. Put $\rho_i = |f_{\sigma_0}^{n_i+1}(x^i) - f_{\sigma_0}(c_2)|$. Consider the first time $m > n_i + 1$ so that $|f_{\sigma_0}^m(x^i)| \sim 1$. By bounded distortion $|(f_{\sigma_0}^{m-n_i-1})'(f_{\sigma_0}^{n_i+1}(x^i))| \sim \rho_i^{-1}$ and therefore $m - n_i \sim \ln(\rho_i^{-1})$.

Choose y close to c_2, such that $|(f_{\sigma_0}^m)'(x^i)| \cdot |y - c_2|^{\mu_2 - 1} \sim \eta^{m+1}$. Set $n_{i+1} := m$ and define x^{i+1} as the preimage of y under the branch of $f_{\sigma_0}^{-m}$ mapping $f_{\sigma_0}^m(x^i)$ to x^i defined on an *a priori* fixed disc D disjoint from the set $\{f_{\sigma_0}(c_2), 0\}$ and such that $f_{\sigma_0}^m(x^i), y \in D'$ for another *a priori* fixed disc D' with closure in D. Hence by bounded distortion $|(f_{\sigma_0}^{n_j+1})'(x^{i+1})| \sim |(f_{\sigma_0}^{n_j+1})'(x^j)|$ for all $j = 1, ..., i$ and $|(f_{\sigma_0}^{n_{i+1}})'(x^{i+1})| \cdot |y - c_2|^{\mu_2 - 1} \sim |(f_{\sigma_0}^{n_{i+1}+1})'(x^{i+1})| \eta^{n_{i+1}+1}$. For $x := \lim_{i\to\infty} x^i$ for each j we obtain $|(f_{\sigma_0}^{n_j+1})'(x)| \sim \eta^{n_j+1}$

Note that

$$|f_{\sigma_0}^{n_{i+1}}(x) - c_2|^{\mu_2 - 1} \sim \rho_i \eta^{n_{i+1} - n_i} \ll \rho_i^{\frac{\mu_2 - 1}{\mu_2}} \sim |f_{\sigma_0}^{n_i}(x) - c_2|^{\mu_2 - 1},$$

if η is close enough to one, since $n_{i+1} - n_i \sim \ln(\rho_i^{-1})$. Thus n_{i+1} is a closer return than n_i, as we wanted, provided ρ_1 is small enough (to define it, let us set $n_0 = 0$), that is x close to 0. Moreover $n_{i+1} - n_i \to \infty$ exponentially fast; this assures the convergence of x^i.

Hence

$$|(f_\sigma^{n_i+1})'(i_\sigma(x))| \sim \prod_{j=1}^{i-1} |(f_\sigma^{n_{j+1}-n_j-1})'(f_\sigma^{n_j+1}(i_\sigma(x)))| \cdot |f_{\sigma_0}^{n_{j+1}}(i_\sigma(x)) - c_2|^{\mu_2 - 1}$$

$$\sim \prod_{j=1}^{i-1} \rho_j^{-\alpha_\sigma} |f_{\sigma_0}^{n_{j+1}}(i_\sigma(x)) - c_2|^{\alpha_\sigma(\mu_2 - 1)} \sim \eta^{\alpha_\sigma n_i}.$$

We conclude that the lower Lyapunov exponent of f_σ at the point $i_\sigma(x)$ is $\alpha_\sigma \log \eta$.

Semi-hyperbolicity does not imply neither CE nor CE2. An example, a degree 4 polynomial, was constructed already in [P4], with the use of Douady-Hubbard straightening of polynomial-like maps. Here we find directly degree 3 polynomials.

Just consider a family f_σ as above with an arbitrary $\eta < 1$. $f_{a,b}(z) = az(1+z)(1-z)-b$ in appropriate affine coordinates on $\mathbb{C}$ satisfies our assumptions. We have critical points $c_1 = -\sqrt{\frac{1}{3}}$ and $c_2 = \sqrt{\frac{1}{3}}$.

Parameterize the family by s, with $a(\sigma)$ and $b(\sigma)$, such that $a(\sigma_0) = \frac{3\sqrt{3}}{2}$ and $b(\sigma_0) = 0$; hence $f_{\sigma_0}(z) = \frac{3\sqrt{3}}{2}z(1+z)(1-z)$. We get $f^2_{\sigma_0}(c_i) = 0$ for $i = 1, 2$.

There exists a curve $(a(\sigma), b(\sigma))$ with $\partial_\sigma a\ , \partial_\sigma b > 0$ at $\sigma = \sigma_0$ and $f_\sigma(p) = p = f^2_\sigma(c_2)$ for $p = p(\sigma)$. One can see this by computing the gradients of the functions $f_{a,b}(p) - p$ and $f^2_{a,b}(c_2) - p$ in the coordinates $(a, b, p) \in \mathbb{R}^3$ at $a = \frac{3\sqrt{3}}{2}, b = p = 0$. One can see that $f^2_\sigma(c_1) - p(\sigma) > 0$ for $\sigma > \sigma_0$ by computing for example that $\partial_a f^2_{a,b}(c_1)$ and $\partial_b f^2_{a,b}(c_1)$ are large at $a = \frac{3\sqrt{3}}{2}, b = 0$.

Therefore there exists $\sigma > \sigma_0$ (arbitrarily close to σ_0) and $k \geq 2$ such that $f^k_\sigma(c_1) = i_\sigma(x)$. Hence the lower Lyapunov exponent at the critical value $f_\sigma(c_1)$ is $\log(\eta^{\alpha_\sigma}) < 0$. Thus f_σ is not CE. Since $\mu_1 = \mu_2 (= 2)$ here, f_σ does not satisfy CE2 neither, see introduction and [GS1].

CE2 does not imply CE. Consider now a family f_σ with $\mu_2 > \mu_1$. Start for example with $f_{a,b}(z) = az^4(1-z) + b$, write $c_1 = \frac{4}{5}$ and $c_2 = 0$, find an appropriate curve $a(\sigma), b(\sigma)$ and change coordinates so that the repelling fixed point is always at 0.

Put $\eta = 1$ and let σ close to σ_0 such that $f^k_\sigma(c_1) = i_\sigma(x)$ for some $k \geq 2$, there are such σ since $f_\sigma(c_1) \not\equiv 0$. By the above the lower Lyapunov exponent of f_σ is 0 so f_σ is not CE. Let us prove that f_σ is CE2.

Since c_1 is not an accumulation point of forward trajectories neither of c_1 nor of c_2, there is a neighborhood U_1 of c_1 such that all f^j on all components of $f^{-j}_\sigma(U_1)$ are univalent and, since f_σ as semi-hyperbolic is TCE (see [CJY]), the diameters shrink uniformly exponentially as $j \to \infty$. So c_1 satisfies CE2.

Similarly there is a neighborhood U_2 of c_2 such that the diameters of all pull-backs shrink uniformly exponentially. Note that a pull-back can encounter now the critical point c_1, but for any (x_j), a backward trajectory of c_2, this can happen at most for one component $\mathrm{Comp}_{x_j} f^{-j}(U_2)$ (since the diameters of the components shrink uniformly to 0 and c_1 is non-recurrent).

We can find a smaller neighborhood $\hat{U}_2 \subset U_2$ so that in all the components of $f^{-n}(\hat{U}_2)$, as long as f^n are univalent on the respective components of $f^{-n}(U_2)$, the derivatives of f^n are exponential at the respective preimages of c_2.

Consider now a pull-back $\hat{V}$ of $\hat{U}_2$ by f^{n+1}_σ so that $c_1 \in \hat{V}$. So f^n_σ is univalent in $f_\sigma(V)$, the respective pull-back of U_2 so the distortion of f^n_σ in $f_\sigma(\hat{V})$ is bounded by some definite constant, see Lemma 1.4 for, say, $U_2 = B(c_2, r)$ and $\hat{U}_2 = B(c_2, r/2)$, with $r \leq r_K$.

(If it happens that $c_1 \in V \setminus \hat{V}$ then we replace for all backward trajectories of c_2 through V the set U_2 by $\hat{U}_2$. As we cannot encounter c_1 again, only univalent cases happen.)

Note that $n = n_i + k - 1$ for some i, since the n_i are the only times that the orbit

of $i_\sigma(x)$ approaches c_2. By the previous considerations $|(f_\sigma^{n_i+k})'(f_\sigma(c_1))| > \text{Const } \theta^{i\alpha_\sigma}$, hence,

$$|f_\sigma^{n_i+k}(c_1) - c_2|^{\mu_2-1} \geq \text{Const} \cdot \theta^{i\alpha_\sigma} |(f_\sigma^{n_i+k-1})'(f_\sigma(c_1))|^{-1}.$$

Let ζ close to c_1 be such that $f_\sigma(\zeta)$ is the unique (n_i+k)-th preimage of c_2 in $\hat{V}$. So, by bounded distortion,

$$|f_\sigma(\zeta) - f_\sigma(c_1)| \sim |f_\sigma^{n_i+k}(c_1) - c_2||(f_\sigma^{n_i+k-1})'(f_\sigma(c_1))|^{-1}$$

$$\geq \text{Const } \theta^{i\frac{\alpha_\sigma}{\mu_2-1}} |(f_\sigma^{n_i+k-1})'(f_\sigma(c_1))|^{-\frac{\mu_2}{\mu_2-1}},$$

and considering that $|f'_\sigma(\zeta)| \sim |f_\sigma(\zeta) - f_\sigma(c_1)|^{\frac{\mu_1-1}{\mu_1}}$,

$$|(f_\sigma^{n_i+k})'(\zeta)| \geq \text{Const } \theta^{i\frac{\alpha_\sigma}{\mu_2-1}\frac{\mu_1-1}{\mu_1}} |(f_\sigma^{n_i+k-1})'(f_\sigma(\zeta))|^{1-\frac{\mu_1-1}{\mu_1}\frac{\mu_2}{\mu_2-1}}.$$

Since $\mu_2 > \mu_1$ it follows that $1 - \frac{\mu_1-1}{\mu_1}\frac{\mu_2}{\mu_2-1} > 0$ so the above estimate is exponentially large in n_i. Finally note that, by taking U_2 small we may assume that $\zeta \in U_1$ so it follows that the derivative at preimages of this ζ is exponential so c_2 also satisfies CE2.

CE does not imply CE2. Such an example was already considered by S. van Strien [vS]. Choose $\mu_1 > \mu_2$ this time and $\eta > 1$ and let f_σ as above.

Use for example again the family $f_{a,b}(z) = az^4(1-z) + b$ but change the roles of the critical points, $c_1 = 0$, $c_2 = \frac{4}{5}$, translate the repelling fixed point to 0 and find an appropriate curve $a(\sigma), b(\sigma)$.

Thus f_σ is CE and just as before we have an estimate,

$$|(f_\sigma^{n_i+k+1})'(\zeta)| \leq \text{Const}(\theta^{-i}\eta^{n_i})^{\frac{\mu_1-1}{\mu_1}\frac{\alpha_\sigma}{\mu_2-1}} |(f_\sigma^{n_i+k})'(f_\sigma(\zeta))|^{1-\frac{\mu_1-1}{\mu_1}\frac{\mu_2}{\mu_2-1}},$$

now $1 - \frac{\mu_1-1}{\mu_1}\frac{\mu_2}{\mu_2-1} < 0$ so the above estimate is exponentially small in n_i, if η is chosen close enough to 1. So c_2 does not satisfies CE2. Note that f_σ is not CE2 even in the real sense (for backward trajectories in $\mathbb{R}$).

Section 4. Neither CE nor CE2 are preserved by quasiconformal conjugacy.

It is proved in [P3] that, in the presence of only one critical point in the Julia set, CE is equivalent to TCE and therefore CE is invariant by topological conjugacy, see also [PR2].

In this section we prove the existence of two quasiconformally conjugated real polynomials of degree 4, so that one of them satisfies CE and CE2 and the other one does not satisfies neither one of these two properties. So it follows that neither CE nor CE2 are invariant by quasiconformal conjugacy.

Consider the family of polynomials

$$f_{\lambda,c_1}(z) = 2 + \lambda(z^2-4)(1-\frac{z}{c_1})^2, \text{ for } c_1 \neq 0.$$

For every such f_{λ,c_1}, 2 is a fixed point, c_1 is a critical point mapped to 2 and $f_{\lambda,c_1}(-2)=2$. We are mainly interested in $c_1\in[2,+\infty)$ big. If c_1 is big there is a unique critical point $c_2=c_2(c_1)$ of f_{λ,c_1} close to 0. In fact such c_2 satisfies $c_2^2=\frac{c_1}{2}c_2+2$ so $c_2=-4c_1^{-1}+\mathcal{O}(c_1^{-2})$. Then we will fix $\lambda=\lambda(c_1)$ close to one so that $f_{\lambda,c_1}(c_2)=-2$ and we denote f_{λ,c_1} just by f_{c_1}. Note that $\lambda(c_1)=1+\mathcal{O}(c_1^{-2})$.

Note also that f_{c_1} converges uniformly in compact sets to z^2-2 as $c_1\to\infty$ and moreover the restriction of f_{c_1} to some definite neighborhood W of $[-2,2]$ is polynomial like with c_2 as unique critical point and it is quasiconformally conjugated to z^2-2. Then we consider the fixed point $\alpha(c_1)$ of f_{c_1} in W, different from 2, so that $\alpha(c_1)=-1-2c_1^{-1}+\mathcal{O}(c_1^{-2})$. Moreover let $\lambda_1(c_1)$ and $\lambda_2(c_1)$ be the eigenvalues of 2 and $\alpha(c_1)$ respectively so that,

$$\lambda_1(c_1)=f'_{c_1}(2)=4(1-4c_1^{-1})+\mathcal{O}(c_1^{-2}) \text{ and}$$

$$\lambda_2(c_1)=f'_{c_1}(\alpha(c_1))=-2(1+c_1^{-1})+\mathcal{O}(c_1^{-2}).$$

Note that the function

$$s(c_1):=\frac{\ln|\lambda_1(c_2)|}{\ln|\lambda_2(c_1)|}=2(1-\frac{3}{\ln 2}c_1^{-1})+\mathcal{O}(c_1^{-2}),$$

is not constant. Moreover, for different values of c_1 the f_{c_1} are quasiconformally conjugated. We will find small perturbations $f_{c_1,\sigma(c_1)}$ of this family so that all elements are quasiconformally conjugated and so that for some values of c_1, $f_{c_1,\sigma(c_1)}$ is CE and CE2 but for other values of c_1 the polynomial $f_{c_1,\sigma(c_1)}$ does not satisfy neither CE nor CE2.

1.- Given η close to one we can choose $x_{c_1}\in[-2,2]$ with lower Lyapunov exponent $\ln\eta$ as x in Section 3 so that the orbit $x_n=f^n(x_{c_1})$ for $n\geq 0$ of x_{c_1}, is in W and accumulates c_2. Let $n_1<n_2<...$ be the for which x_{n_i} is closer to c_2 than the previous times. Then we may choose x_{c_1} so that these times are also the only times that the x_n are close to c_2 and moreover $|(f_{c_1}^{n_i+1})'(x_0)|\sim\eta^{n_i}$.

So if m_i is such that $|x_{m_i}-2|\sim 1$, then there is m'_i such that $|m'_i-m_i|<\text{Const}$ so $|x_{m'_i}-\alpha(c_1)|\sim|\lambda_2(c_1)|^{-2^{m_i}}$. Therefore, for some definite $k\geq 0$, we have that $|x_{m'_i+2^{m_i}+k}-\alpha(c_1)|\sim 1$. We further assume that n_{i+1} is such that $|n_{i+1}-(m'_i+2^{m_i})|<\text{Const}$. Note that $m_i-n_i\sim m'_i-n_i\sim\ln(n_i)\ll 2^{m_i}$,

$$|(f_{c_1}^{m'_i-n_i-1})'(x_{n_i+1})|\sim|\lambda_1(c_1)|^{m'_i-n_i} \text{ and } |(f_{c_1}^{2^{m_i}})'(x_{m'_i}))|\sim|\lambda_2(c_1)|^{2^{m_i}}, \text{ so,}$$

$$|f'_{c_1}(x_{n_{i+1}})|^{-1}\sim\eta^{-(n_{i+1}-n_i)}|\lambda_1(c_1)|^{m'_i-n_i}|\lambda_2(c_1)|^{2^{m_i}}.$$

2.- Consider the family,

$$f_{c_1,\sigma}(z)=f_{c_1}(z)-\sigma(z^2-4)(z-c_2(c_1))^2 \text{ for } \sigma\in\mathbb{C},$$

note that $f_{c_1,\sigma}$ is real when c_1 and σ are real. Moreover 2 is a fixed point for $f_{c_1,\sigma}$, $c_2=c_2(c_1)$ is a critical point mapped to 2 and $f_{c_1,\sigma}(-2)=2$. Fix c_1 so there is a

continuation $\tilde{c}_1 = \tilde{c}_1(\hat{c}_1, \hat{\sigma})$ of the critical point c_1 for $(\hat{c}_1, \hat{\sigma})$ close to $(c_1, 0)$ and note that $\tilde{c}_1$ is real when $\hat{c}_1$ and $\hat{\sigma}$ are real and $\tilde{c}_1(\hat{c}_1, 0) \equiv \hat{c}_1$.

Furthermore there is canonic holomorphic motion i of $[-2, 2]$ defined in a neighborhood of $(c_1, 0)$ which is compatible with dynamics and so that $i_{c_1,0}$ is the inclusion. Note that $i_{\hat{c}_1,\hat{\sigma}}(\alpha(c_1))$ is a fixed point of $f_{\hat{c}_1,\hat{\sigma}}$. Denote by $\lambda_i(\hat{c}_1, \hat{\sigma})$, $i = 1, 2$ the eigenvalues of 2 and $i_{\hat{c}_1,\hat{\sigma}}(\alpha(c_1))$ respectively. Thus,

$$|(f_{\hat{c}_1,\hat{\sigma}}^{2^{m_i}})'(i_{\hat{c}_1,\hat{\sigma}}(x_{m_i'}))| \sim |\lambda_2(\hat{c}_1, \hat{\sigma})|^{2^{m_i}},$$

and as in Section 3,

$$|(f_{\hat{c}_1,\hat{\sigma}})'(i_{\hat{c},\hat{\sigma}}(x_{n_i+1}))| \sim |f_{c_1}'(x_{n_i})|^{\frac{\ln|\lambda_1(\hat{c}_1,\hat{\sigma})|}{\ln|\lambda_1(c_1)|}}.$$

Considering that $m_i' - n_i$, $n_{i+1} - n_i - 2^{m_i} \ll 2^{m_i}$ it follows that the lower Lyapunov exponent of $f_{\hat{c}_1,\hat{\sigma}_n}$ at $i_{\hat{c}_1,\hat{\sigma}}(x_{c_1})$ is equal to

$$\ln|\lambda_2(\hat{c}_1, \hat{\sigma})| + \frac{\ln|\lambda_1(\hat{c}_1, \hat{\sigma})|}{\ln|\lambda_1(c_1)|}(\ln\eta - \ln|\lambda_2(c_1)|)$$

$$= \ln|\lambda_1(\hat{c}_1, \hat{\sigma})| \left(\frac{\ln|\lambda_2(\hat{c}_1, \hat{\sigma})|}{\ln|\lambda_1(\hat{c}_1, \hat{\sigma})|} - \frac{\ln|\lambda_2(c_1)|}{\ln|\lambda_1(c_1)|}\right) + \frac{\ln|\lambda_1(\hat{c}_1, \hat{\sigma})|}{\ln|\lambda_1(c_1)|}\ln\eta.$$

3.- For $\sigma > 0$ and c_1 real, $f_{c_1,\sigma}(\tilde{c}_1) < 2$ so we may choose a sequence $\{\sigma_n\}_{n\geq 1}$ such that $\sigma_n \to 0$ and such that $\tilde{c}_1$ is eventually mapped to $i_{c_1,\sigma_n}(x_{c_1})$ by f_{c_1,σ_n}. Hence, for c_1' in some definite neighborhood of c_1 there is $\sigma_n(c_1')$ so that

$$f_{c_1',\sigma_n(c_1')}(\tilde{c}_1) = i_{c_1',\sigma_n(c_1')} \circ i_{c_1,\sigma_n}^{-1}(f_{c_1,\sigma_n}(\tilde{c}_1)).$$

Therefore $f_{c_1',\sigma_n(c_1')}$ is quasiconformally conjugated to f_{c_1,σ_n} and moreover the lower Lyapunov exponent of $f_{c_1',\sigma_n(c_1')}$ at $f_{c_1',\sigma_n(c_1')}(\tilde{c_1})$ is equal to the lower Lyapunov exponent of $i_{c_1',\sigma_n(c_1')}(x_{c_1})$. By 2 these lower Lyapunov exponent converges to,

$$\ln|\lambda_1(c_1')|(s(c_1') - s(c_1)) + \frac{\ln|\lambda_1(c_1')|}{\ln|\lambda_1(c_1)|}\ln\eta$$

as $n \to \infty$, where the function s is as in 1. By 1 the function s is not constant and since it is analytic one may choose c_1' arbitrarily close to c_1 so that $s(c_1') \neq s(c_1)$. Once fixed such c_1' one may choose η so that,

$$\ln\eta \quad \text{and} \quad \ln|\lambda_1(c_1')|(s(c_1') - s(c_1)) + \frac{\ln|\lambda_1(c_1')|}{\ln|\lambda_1(c_1)|}\ln\eta,$$

have different signs. Hence, for n big one of the f_{c_1,σ_n} or $f_{c_1',\sigma_n(c_1')}$ satisfies CE and the other one does not. Since the polynomials considered have all critical points with the same

multiplicity, CE is equivalent to CE2. Hence, neither CE nor CE2 are preserved by quasiconformal conjugacy.

Section 5. The property TCE and critical points.

Theorem 5.1. (TCECrit & BLS) $\Rightarrow$ ExpShrinkCrit.

Proof. TCECrit implies *mean exponential shrinking of components at critical points*, compare [NP], that is for every $c \in J \cap \mathrm{Crit}$ there is $P > 0, \delta > 0, \lambda > 1$ and a strictly increasing sequence of positive integers $n_j, j = 1, 2, ...$ such that $n_j \leq Pj$ and for each j and $i : 0 \leq i \leq n_j$

$$\mathrm{diamComp}_{f^i(c)} f^{-(n_j - i)} B(f^{n_j}(c), \delta) \leq \lambda^{-(n_j - i)}.$$

The proof is a part of the proof of Proposition 3.1 in [PR] (rewritten in [NP, Sec.2] for unimodal maps of the interval). The idea is the *telescope* construction. We can assume that all $n_{j+1} - n_j$ larger than a constant so that

$$\mathrm{diamComp}_{f^{n_j}(c)} f^{-(n_{j+1} - n_j)} B(f^{n_{j+1}}(c), \delta) \leq \delta/2.$$

Hence

$$\mathrm{diamComp}_c f^{-n_j} B(f^{n_j}(c), r) \leq \xi^j \leq (\xi^{1/P})^{n_j}.$$

for a constant $\xi < 1$. See also [GSw2, p.279-280].

Next notice that for $n : n_j < n < n_{j+1}$, by BLS, we have

$$\mathrm{diamComp}_{f^{n_j}(c)} f^{-(n - n_j)} B(f^n(c), r) < \delta.$$

Hence $\mathrm{Comp}_{f^{n_j}(x)} f^{-(n-n_j)} B(f^n(x), r) \subset B(f^{n_j}(x), \delta)$ and finally $\mathrm{diam}_x f^{-n} B(f^n(x), r) \leq \xi^j \leq \xi^{-1} \xi^{j+1} \leq \xi^{-1} (\xi^{1/P})^n$. QED

Note that in [PR1, Proposition 3.1] we coped with $n : n_j < n < n_{j+1}$ by using CE, referring to [P1]. Here, instead, we refer to BLS.

Problem. Does TCECrit $\Rightarrow$ BLS ?

Corollary 5.2. (TCECrit & BLS) $\Leftrightarrow$ ExpShrinkCrit $\Leftrightarrow$ TCE $\Leftrightarrow$ ExpShrink.

Proof. The implication (TCECrit & BLS) $\Rightarrow$ ExpShrinkCrit is provided by Theorem 5.1. The proof of ExpShrinkCrit $\Rightarrow$ TCE is a repetition of [NP, Sec.1] or [PR1]. It is even simpler than in the latter paper, where to prove CE$\Rightarrow$ TCE, *shrinking neighbourhoods* technique was needed, not needed in [NP] and here.

TCE $\Rightarrow$ ExpShrink again goes via a telescope, first giving as in Proof of Theorem 5.1 *mean exponential shrinking of components at critical points.* However there is now no

problem with $n : n_j < n < n_{j+1}$, since the gaps (n_j, n_{j+1}) can be filled in by secondary telescopes at $f^{n_j}(x)$, secondary gaps filled in with the next generation telescopes, etc, yielding finally ExpShrink. See [P3, Lemma 4.3] for details.

Finally ExpShrink $\Rightarrow$ BLS is obvious and ExpShrink $\Rightarrow$ TCECrit is weaker than already proved ExpShrinkCrit $\Rightarrow$ TCE. QED

Notation. For any A, a subset of the set of natural numbers, we write

$$\underline{d}(A) := \liminf_{n\to\infty} \#([1,...,n] \cap A)/n$$

$$\overline{d}(A) = \limsup_{n\to\infty} \#([1,...,n] \cap A)/n.$$

For every $x \in \overline{\mathbb{C}}, \delta > 0$ and $n \geq k \geq 0$ denote the set $\mathrm{Comp}_{f^k(x)} f^{-(n-k)}(B(f^n(x), \delta))$, the connected component containing $f^k(x)$, by $W_{n,k,x,\delta}$.

Given $M \geq 0, \delta > 0, x \in \overline{\mathbb{C}}$ we say $n \geq 0$ is an (x, M, δ) good time, and write $n \in G(x, M, \delta)$ if

$$\sharp\{0 \leq k < n : W_{n,k,x,\delta} \cap \mathrm{Crit} \neq \emptyset\} \leq M.$$

We assume that δ is small enough that for all $n \in G(x, M, \delta)$ and $0 \leq k \leq n$, the sets $W_{n,k,x,\delta}$ are topological discs of diameters smaller than $\min \mathrm{dist}(c, c')$, the minimum taken over all pairs of distinct critical points, see [M]; therefore each $W_{n,k,x,\delta}$ contains at most one critical point.

Theorem 5.3. *$\overline{d}(G(c, M, \delta)) > 0$ at a critical point $c \in J$ implies $\overline{d}(G(f(c'), 0, \delta')) > 0$ for some $\delta' > 0$ and $c' \in \mathrm{Crit} \cap \mathrm{cl} \bigcup_{n\geq 0} f^n(c)$.*

Remark. *We do not know whether $\overline{d}$ can be replaced by $\underline{d}$ in this this Theorem, which would be in accordance to the* TCE *at c and c' properties.*

Proof. The proof goes by induction with respect to M. Fix c, M, δ and assume $\overline{d}(G(c, M, \delta)) > 0$, $M > 1$. For every $n \in G(c, M, \delta)$ let $\delta \geq \delta_n \geq 2^{-M}\delta$ be chosen so that for every $0 < k < n$, $(W_{n,k,x,\delta_n} \setminus W_{n,k,x,\delta_n/2}) \cap \mathrm{Crit} = \emptyset$. Denote $W_{n,k} := W_{n,k,c,\delta_n}$.

Shrinking δ and taking only every n_0-th n for n_0 large enough if necessary, we can assume the following *telescope property*: For every $n_1 \leq n_2 \leq n_3 \in G(c, M, \delta)$ it holds $W_{n_3,n_1} \subset W_{n_2,n_1}$.

For all $s = 0, ..., n, n \in G(c, M, \delta)$, for $\delta'(s) := \frac{\delta}{2^M \sup |f'|^s}$, we have

$$B(f^{n-s}(c), \delta'(s)) \subset W_{n,n-s}$$

hence, for all $k : 0 < k \leq n - s$,

$$W_{n-s,k,c,\delta'(s)} \subset W_{n,k}.$$

Let $t(n) : 0 \leq t(n) < n$ be the largest integer such that $W_{n,t(n)}$ contains a critical point.

Since $W_{n,t(n)}$ captures a critical point, there are at most $M-1$ integers $k : 0 \leq k < n - t(n)$ such that $W_{n,k}$ captures a critical point. Therefore for $s \geq n - t(n)$, $n - s \in G(c, M-1, \delta'(s))$.

For every integer $s > 0$ denote $G_s := \{n \in G(c, M, \delta) : n - t(n) \leq s\}$. Suppose there exists $s > 0$ such that

$$\overline{d}(G(c, M-1, \delta 2^{-M})) \cup G_s) > 0.$$

Suppose $n \in G_s$. Then $n - s \in G(c, M-1, \delta'(s))$. Of course if $n \in G(c, M-1, \delta 2^{-M})$ then $n - s \in G(c, M-1, \delta'(s))$. Therefore

$$\overline{d}(G(c, M-1, \delta'(s)) \geq \overline{d}(G(c, M-1, \delta 2^{-M}) \cup G_s) > 0.$$

Suppose now that s as above does not exist. Then there are sequences $n_j, s_j \to \infty$ such that for

$$G^j := \{n \in G(c, M, \delta) \setminus G(c, M-1, \delta 2^{-M}) : 0 < n \leq n_j, n - t(n) \geq s_j\}$$

it holds

$$\#G^j / n_j \geq \varepsilon_0 := \frac{1}{2}\overline{d}(G(c, M, \delta)) > 0.$$

Note that for an arbitrary j and all $n \in G^j$ and $t(n) < m < n, m \in G(c, M, \delta)$, we have $t(m) = t(n)$. Otherwise, if $t(m) > t(n)$, the set $W_{m,t(m)}$ contains a critical point. If $W_{n,k}$ for $k \leq t(n)$ contains a critical point then also $W_{m,k}$ contains it by the *telescope property*. Since $n \notin G(c, M-1, \delta 2^{-M})$ there are M of such k's. So $M+1$ number of $W_{m,k}$'s, for $k < m$ capture critical points, i.e. $m \notin G(c, M, \delta)$, a contradiction.

We obtain $\#\{t(n) : n \in G^j\}/n_j \leq 1/s_j \to 0$ as $j \to \infty$, since for each $t(m)$, the closest $n > t(m), n \in G^j$, satisfies $n \geq t(n) + s_j = t(m) + s_j$. Hence there exists a pair $n < n' \in G^j$ such that there is no integer of the form $t(m)$ between $t(n)$ and $t(n')$ and

$$\#(G^j \cap [t(n)+1, t(n')])/s_j \geq \varepsilon_0.$$

(We include to the set of $t(n')$'s the last in G_j integer n_j.)

For each $m \in G^j \cap [t(n)+1, t(n')]$ no set $W_{m,k}, k = t(n)+1, ..., m-1$ contains a critical point. Recall finally that a critical point $c' \in W_{m,k} = \mathrm{Comp} f^{-(m-k)}(B(f^m(c), \delta_m))$ belongs in fact to $\mathrm{Comp} f^{-(m-k)}(B(f^m(c), \delta_m/2))$ by the definition of δ_m. Hence $m \in G(c', 0, \delta_m/2) \subset G(c', 0, \delta 2^{-(M+1)})$.

Thus, by $s_j \to \infty$, choosing c' appearing infinitely many times for j's, we get

$$\overline{d}(G(f(c'), 0, \delta 2^{-(M+1)})) \geq \varepsilon_0 > 0.$$

By the construction, $c' \in B(\bigcup_{n \geq 0} f^n(c), \delta)$, so $c' \in \mathrm{cl} \bigcup_{n \geq 0} f^n(c)$ if we assumed that $\delta \leq \mathrm{dist}(\bigcup_{n \geq 0} f^n(c), \mathrm{Crit} \setminus \mathrm{cl} \bigcup_{n \geq 0} f^n(c))$. QED

Appendix A.

The purpose of this appendix is to prove the following lemma.

Lemma A. *Fix $\alpha \in (\frac{1}{2}, 1)$ and Let $f \in \mathcal{C}(z)$ be a rational map. Then there is a set of full Lebesgue measure in $\overline{\mathbb{C}}$ of points w so that we can join w to any $w_1 \in f^{-1}(w)$ with a path $\gamma_{w_1} \subset \overline{\mathbb{C}} \setminus \cup_{n\geq 1} f^n(Crit)$ so that the distortion of f^n, $n \gg 1$ in any pull-back of γ_{w_1} by f^{-n} is bounded by D^{n^α}, where $D > 1$ is a definite constant independent of f.*

Compare with Hall and Hayman lemma in [HH] and independently [P2], see remark at the end. The proof of this lemma is at the end of this section. It is based in the following observation. For a given $n \geq 0$ suppose that there is a disc $U \subset W_n = \overline{\mathbb{C}} \setminus \cup_{1\leq k\leq n} f^n(\text{Crit})$ containing a path γ. Then there is an estimate of the distortion of f^n in any pull-back γ_n of γ by f^{-n}, given by Koebe Distortion Lemma. It turns out that this distortion is bounded by D^h, where h is the hyperbolic diameter of γ in U and $D > 1$ is a universal constant, see version of Koebe Distortion Lemma in [Po].

Hence to prove Lemma A is enough to find paths γ_{w_1} with an estimate of the infimum of the hyperbolic diameter of γ in U, over all such discs $U \subset W_{n_j}$, $j \geq 1$. If γ is a rectificable path, this latter quantity is comparable to,

$$\int_\gamma \frac{|dw|}{\text{dist}(w, \partial W)},$$

which is called the *quasihyperbolic* length of γ_{w_1} in W_n or qh length for short, see [Po].

Lemma A easily follows from Lemma A.2 below, but we begin with a technical lemma.

Lemma A.1. *Fix $R > 0$ and let S be the strip $(-1, R+1) \times (-1, 1) \subset \mathbb{R}^2$ and $l = [0, R] \times \{0\}$. Consider $r \in (0, \frac{1}{2})$ and a finite set $Z \subset S$ so that $dist(Z, l) > r$. Then the qh diameter of l in $S - Z$ is less than $R + 6\#Z \ln \frac{1}{r}$.*

Proof. Consider the vertical projection Z' of Z to $[-1, R+1] \times \{0\}$. Then,

$$\int_{l\cap B_r(Z')} \frac{1}{\text{dist}(l, Z \cup \partial S)} dx \leq 2\#Z.$$

Note that for $x \in l - B_1(Z')$ we have $\text{dist}(x, Z \cup \partial S) \geq 1$ so,

$$\int_{l - B_1(Z')} \frac{1}{\text{dist}(x, Z \cup \partial S)} dx \leq R.$$

Moreover, if $x \in l \cap (B_1(Z') \setminus B_r(Z'))$ then $\text{dist}(x, Z \cup \partial S) \geq \text{dist}(x, Z')$. Thus,

$$\int_{l\cap(B_1(Z')\setminus B_r(Z'))} \frac{1}{\text{dist}(x, Z \cup \partial S)} dx \leq \#Z \int_{(-1,1)-(-r,r)} \frac{1}{x} dx = 2\#Z \ln \frac{1}{r}.$$

This proves the desired estimate. QED

Let n_j be the integer part of e^j for $j \geq 0$ and fix $\alpha \in (\frac{1}{2}, 1)$ as in Lemma A.

Lemma A.2. *Let $Q = (-2,2) \times (-2,2)$ and let $\{x_n\}_{n\geq 0}$ be a sequence of points in Q. There is a set of full Lebesgue measure of x in $(-1,1)$, for which there is $j(x)$ such that for all $j \geq j(x)$ the qh length of $l_x = (-1,1) \times \{x\}$ in $Q - \{x_0, ..., x_{n_j}\}$ is $\mathcal{O}(n_j^\alpha)$.*

Proof. Since $\{\frac{n_j}{n_j^{2\alpha}}\}_{j\geq 0}$ is summable, it follows that for almost every $x \in (0,1)$ there is $j_0(x)$ such that for all $j \geq j_0(x)$ we have that $dist(l_x, x_n)$ is at least $n_j^{-2\alpha}$ if $0 \leq n \leq n_j$. The measure of those $x \in (-1,1)$ for which the strip,

$$S_x^j = (-1 - \frac{1}{n_j^\alpha}, 1 + \frac{1}{n_j^\alpha}) \times (x - \frac{1}{n_j^\alpha}, x + \frac{1}{n_j^\alpha})$$

contains at least $\frac{n_j^\alpha}{\ln n_j}$ points in $\{x_0, ...x_n\}$, is at most

$$n_j^{-\alpha} \frac{n_j}{n_j^\alpha / \ln n_j} = (\ln n_j) n_j^{(1-2\alpha)} \sim j e^{(1-2\alpha)j},$$

which is summable. Thus for almost every $x \in (-1,1)$ there is $j_1(x)$ such that for all $j \geq j_1(x)$ the strip S_x^j contains at most $\frac{n_j^\alpha}{\ln n_j}$ points in $\{x_0, ..., x_{n_j}\}$.

If $x \in (-1,1)$ belongs to the two described sets, then for every $j \geq \max(j_0(x), j_1(x))$ we have that $\text{dist}(l_x, \{x_0, ..., x_{n_j}\}) \geq \frac{1}{n_j^{2\alpha}}$ and the strip S_x^j contains at most $\frac{n_j^\alpha}{\ln n_j}$ points in $\{x_0, ..., x_{n_j}\}$. Applying Lemma A.1 to a scaled copy of S_x^j by a factor of $n_j^\alpha = R$ and with $r = n_j^{-\alpha}$, it follows that the qh length of l_x in $S_x^j - \{x_0, ..., x_{n_j}\}$ is less than $n_j^\alpha + 6\frac{n_j^\alpha}{\ln n_j} \ln n_j^\alpha \leq 7n_j^\alpha$. QED

Proof of Lemma A. By Lemma A.2, for a set of full measure of pairs $z, w \in Q$ the qh distance between z and w in $Q - \{x_0, ..., x_{n_j}\}$ is $\mathcal{O}(n_j^\alpha)$. So, if $f \in \overline{\mathcal{C}}(z)$ is a rational map, there is $D_1 > 1$ so that for a set of full Lebesgue measure of $w \in \overline{\mathcal{C}}$ we can join w to $w_1 \in f^{-1}(w)$ with a path γ_{w_1} so that for any $n_j \leq n \leq n_{j+1}$ the qh length of γ_{w_1} in $\overline{\mathcal{C}} \setminus \cup_{0\leq m\leq n} f^m(Crit)$ is $\mathcal{O}(n_{j+1}^\alpha) = \mathcal{O}(n_j^\alpha)$. Lemma A follows with $D > D_1^{3^\alpha}$. QED

Remark A.3. *With the same method it follows that, given any sequence $\{x_n\}_{n\geq 0} \subset \overline{\mathcal{C}}$ we can join any two points in $\overline{\mathcal{C}}$, outside an exceptional set of Lebesgue measure zero, by a path γ such that the qh length of γ in $\overline{\mathcal{C}} \setminus \{x_0, ..., x_n\}$ is $\mathcal{O}(n^{\frac{1}{2}} \ln n (\ln \ln n)^\beta)$ for any $\beta > \frac{1}{2}$. The mentioned lemma of Hall, Hayman and the first author yields a path* depending on n, *with qh length $\mathcal{O}(n^{\frac{1}{2}})$ in $\overline{\mathcal{C}} \setminus \{x_0, ..., x_n\}$. Furthermore, if we require the exceptional set of Hausdorff dimension at most $h > 0$, then we obtain the estimate $\mathcal{O}(n^{\frac{1}{2}} (\ln n)^{\frac{1}{2}+\frac{1}{h}})$ for the qh length.*

Appendix B.

In this Appendix we present an alternative proof that UHP implies CE2(some z_0).

Lemma B.1. *For any integer $L \geq 2$ there exists $R(L) > 0$ such that if for a ball $B(x,R)$, intersecting J and of radius $R \leq R(L)$, a component of $f^{-k}B(x,R)$ contains a critical point, and also $f^{-l}B(x_k,R)$ contains a critical point, then $k+l \geq L$. Here $x_k \in f^{-k}x$ is a preimage of x belonging to the component of $f^{-k}B(x,R)$ under consideration.*

Proof. Take δ to be so small that the return time of the set of critical points to its δ-neighborhood is greater than L (recall our convention that no critical point in J is in the forward orbit of another critical point). By compactness there is such R that for $k \leq L$ any component of connectivity of the pull-back $f^{-k}B$ of any ball of radius R has diameter less than δ. This R is the desired one. QED

Lemma B.2. *Fix ε and let R be sufficiently small. Then for sufficiently large n any $a \in J$ has a "good" preimage $b \in f^{-[\varepsilon n]}a$, meaning that for some $k < \varepsilon n$ the ball $B(f^k b, R)$ can be pulled univalently along any branch of $f^{-(n+k)}$ passing through a.*

Proof. Denote by d the degree of f. Take $R < R(L)$ with L large enough to satisfy $4\log(4dL)/L < \epsilon\log(d)$ with $\epsilon < \varepsilon/2$, and assume $n > L/\epsilon$. For every branch of $f^{-(n+[\epsilon n])}$ at a do the following procedure: take a ball $B(a,R)$ and pull it back until it hits some critical point after k_1 iterations, take a ball $B(f^{-k_1}a,R)$ and pull it back until it hits some critical point after k_2 iterations. We continue by induction until we obtain a ball $B(f^{-m}a,R)$, with $m = k_1 + \ldots + k_\ell$, which can be pulled back univalently till $f^{-(n+[\epsilon n])}a$.

For any ball B and positive integer i for at most $4d$ (more precisely: at most $2(2d-2)$) (counting multiplicities) branches of f^{-i} preimage of B hits critical points first time exactly after i iterates. Thus one specification $(k_1,\ldots,k_\ell)$ can correspond to at most $(4d)^\ell$ distinct branches of length m (depending on the branch). Also, as was shown above for any specification we have $k_i + k_{i+1} > L$, thus in any interval $[j, j+L)$, with $0 \leq j \leq [(n+[\epsilon n])/L]$, there are at most two numbers of the form $k_1 + \ldots + k_i$. Considering every second i each block $[tL,(t+1)L)$ contains no $\sum_{s=1}^{i} k_s$ or at most one with L possible positions. Therefore there are at most $(L+1)^{2+2(n+[\epsilon n])/L} < L^{4n/L}$ ways to choose a specification, and necessarily $\ell \leq 2+2(n+[\epsilon n])/L < 4n/L$. Hence the number of such "critical" branches is at most

$$(4d)^\ell L^{4n/L} < (4dL)^{\frac{4n}{L}} = \exp\left(4n\log(4dL)/L\right)) < d^{\epsilon n} \ .$$

But counting multiplicities there are $d^{[\varepsilon n]} > d^{\epsilon n}$ preimages $f^{-[\varepsilon n]}a$, thus there is a point $b \in f^{-[\varepsilon n]}a$, such that for all branches of $f^{-(n+[\varepsilon n])}a$ passing through it $m < [\varepsilon n]$. Clearly b is the desired point, and $k = [\varepsilon n] - m$ is the desired iterate. QED

Lemma B.3. *Let μ be the invariant measure of maximal entropy. Then for almost all z with respect to μ there is a positive integer $n(z)$ such that for $n > n(z)$*

1. *The ball $B(f^k z, R)$ can be pulled back univalently along any branch of $f^{-(n+k)}$ passing through z, where $k = k(n) < \epsilon n$.*
2. *The ball $B(f^{n+k}z, R)$ can be pulled back univalently along branch of $f^{-(n+k)}$ passing through z, where $k = k(n) < \epsilon n$.*

Proof. Choosing arbitrarily small ϵ in the Lemma B.2, we deduce that when n is large, except for a set of exponentially small measure μ, for all z in the Julia set the ball $B(f^k z, R)$ can be pulled univalently along any branch of $f^{-(n+k)}$ passing through z, where $k = k(n) < \epsilon n$. Here we use the fact that Jacobian of f with respect to μ is equal to d, hence exponentially small proportion of preimages supports only exponentially small proportion of μ. Letting $n \to \infty$ and using Borel-Cantelli lemma, we deduce the first claim.

Similarly, passing to preimages (and using invariance of the measure μ), we obtain the second claim. It also follows immediately from Pesin's theory. QED

Lemma B.4. *There is a constant $j = \mathrm{Const}(R)$ such that if a component U of $f^{-n}B(R)$ is a univalent pull back of some ball $B(R)$ of radius R for sufficiently large n, then component U' of $f^{-n}B(R/2)$ inside U contains a repelling periodic point of period $n + j$.*

Proof. Choose j to be large enough so that preimages of any point in J by f^j are $R/4$-dense in J. By compactness there is $\delta > 0$ such that any component of the pull-back $f^{-j}B(\delta)$ of any ball $B(\delta)$ of radius δ has diameter less than $R/4$. By a standard argument (dating back to Fatou) for sufficiently large n component U' would have diameter less than δ. Components of pull backs $f^{-j}U'$ are $R/4$-dense and have diameters less than $R/4$, hence one of them, say V, should be inside $B(R/2)$. Using an index argument for the map $f^{n+j} : V \to B(R/2)$, we find the desired periodic point, which is repelling by Schwarz lemma. QED

Now assume UHP. Take any $\lambda < \lambda_{\mathrm{Per}}$. Choose ε small enough, so that $\lambda_{\mathrm{Per}}/\lambda > M^{\varepsilon}$. Parameters R, L, j are given by Lemmas above. Then for n larger than some n' we have $(\lambda_{\mathrm{Per}}/\lambda)^n > 7M^{\varepsilon n + j}$

Take as above the measure μ of maximal entropy (balanced measure). Then by Lemma B.3 for μ-almost all points z_0 for $n > n(z_0)$ the ball $B(f^k z_0, R)$ can be pulled univalently along any branch of $f^{-(n+k)}$ passing through z, where $k = k(n) < \epsilon n$.

Take one of such points z_0, some $n > n(z_0)$, and some $w \in f^{-n}(z_0)$. Let W be the component of $f^{-(n+k)}B(f^k z_0, R/2)$ containing w. By the Lemma B.4 there is a periodic point $p \in W$ with a period $n + k + j$. Denoting $M := \sup_J |f'|$ and using Koebe Distortion Theorem and UHP, we can write

$$|(f^n)'(w)| \geq |(f^n)'(p)|/7 \geq |(f^{n+k+j})'(p)|/(7M^{k+j}) \geq \lambda_{\mathrm{Per}}^n/(7M^{k+j}),$$

which is greater than λ^n whenever n is large enough.

So we have shown that UHP implies CE2(z_0) for almost all z_0 with respect to μ.

Using the second statement of Lemma B.3 we similarly note that for μ-almost all points $|(f^n)'(z)| > \lambda^n$ whenever n is large enough. Thus the lower Lyapunov exponent $\liminf_{n\to\infty} \frac{1}{n} \log |(f^n)'(z)|$ is at least $\log \lambda > 0$ for μ-almost all points z, and we infer that $\Lambda(\mu) \geq \log \lambda > 0$.

Remark B.5. Lemma B.3, hence the assertions above, hold for any invariant measure μ of Jacobian bounded away from 1 on a set of full measure μ. These are however not the best results. In Appendix C it is proved with the use of Pesin-Katok method that $\Lambda(\mu) \geq \log \lambda > 0$ for every μ of positive Lyapunov exponent (in fact even this assumption can be skipped) and every $\lambda < \lambda_{\mathrm{Per}}$. Furthermore $\mathrm{CE}(z_0)$ for just one z_0 implies $\mathrm{CE}(z_0)$ for almost all z_0 for all invariant ergodic measures of positive entropy (even for a larger class of invariant measures) by Remark 1.1.; any set of positive measure has positive Hausdorff dimension by the formula: dimension of measure = entropy / Lyapunov exponent. Nevertheless the direct method *via* elementary Lemma B.2 seems to be of an independent interest.

Appendix C.

Here we discuss Lyapunov exponents and pressure function $P(t)$, see Introduction. Denote $c_+ = \inf\{\Lambda(\mu) : I(\mu) > 0\}$ (that is infimum of Lyapunov exponents of invariant measures of positive entropy), $c2 = \inf\{\Lambda(\mu) : \Lambda(\mu) > 0\}$ and $\lambda_{\rm Per} = \inf |(f^{k(p)})'(p)|^{1/k(p)}$, infimum taken over all repelling periodic $p \in J$, where $k(p)$ denotes period of p.

Lemma C.1. $c_+ = c2 = \log \lambda_{\rm Per}$,

Proof. As we noted in Introduction $I(\mu) > 0$ implies $\Lambda(\mu) > 0$, so infimum in $c2$ is taken over a larger set of measures than in c_+. Therefore $c_+ \geq c2$.

The inequality $c2 \geq \log \lambda_{\rm Per}$, i.e. looking for periodic orbits with appropriately small exponents, belongs to so-called Pesin-Katok theory, see for example [PU1, Ch.9]. Here is the sketch of the proof. Let μ be any f-invariant measure on J of positive Lyapunov exponent (it is sufficient to consider ergodic measure) and let $\tilde{\mu}$ be its natural extension in Rohlin's sense (that is the lift of μ to the shift invariant measure on the inverse limit of the system $... \to_f J \to_f J$ and Borel σ-algebra, see for example [PU1, Ch.1.]). Then for $\tilde{\mu}$-a.e. backward trajectory $x_0, x_1, ...$ where $f(x_n) = x_{n-1}$, for every $\varepsilon > 0$ there exists $C > 0$ such that for every $n \geq 0$

$$C \exp(n(1-\varepsilon)\Lambda(\mu)) \leq |(f^n)'(x_n)| \leq C^{-1} \exp(n(1+\varepsilon)\Lambda(\mu)).$$

Moreover, there exists $\delta > 0$ such that for a positive measure $\tilde{\mu}$ set of backward trajectories $x_0, x_1, ...$ for every n the function f^n is univalent on $\mathrm{Comp}_{x_n} f^{-n}(B(x_0, \delta))$ with distortion bounded by a constant C_1 for all n. This follows from Pesin's theory, see [PUZ, Lemma 8].

Fix $x_n, n = 0, 1, ...$ satisfying the above properties. For every $n \geq 0$ we can extend x_0, x_1, ..., x_n to a backward trajectory x_0, x_1, ..., x_n, $x_{n+1}^{(n)}$, ..., $x_{n+m}^{(n)}$, $m \leq N$, for a constant N independent of n, so that $x_{n+m}^{(n)} \in B(x_0, \delta/3)$ (compare lemma B.4 or beginning of Proof of Lemma 2.1). Hence, for n large enough,

$$W = \mathrm{Comp}_{x_{n+m}^{(n)}} f^{-(n+m)}(B(x_0, \delta)) \subset B(x_0, \delta/2)$$

(With an additional effort we could find W such that f^m is univalent on it, but we do not need it in the sequel.)

Hence there exists a repelling periodic point $y \in W$, of period $n+m$, such that

$$|(f^{n+m})'(y)| \leq |(f^m)'(y)| \cdot |(f^n)'(f^m(y))| \leq C_1 C^{-1} \exp(n(1+\varepsilon)\Lambda(\mu)) \sup |f'|^N.$$

Taking arbitrarily small ε and increasing n yields $\lambda_{\rm Per} \leq \Lambda(\mu)$.

Finally let us prove that $\log \lambda_{\rm Per} \geq c_+$. For every periodic x write $\Lambda(x) := \frac{1}{n(x)} \log |(f^{n(x)})'(x)|$, where $n(x)$ is a period of x. Denote the periodic orbit of x by $O(x)$. Given p repelling periodic point of period $n(p)$ we want, for an arbitrary positive

ε, to find a measure μ of positive entropy such that $\Lambda(\mu) \leq \Lambda(p) + \varepsilon$. We shall construct an invariant hyperbolic set of positive entropy and demanded exponent using an idea of "building bridges" from $O(p)$ to itself, by E. Prado [Prado]. (This corresponds in higher dimension to finding a "horseshoe" close to a homoclinic point.) Let $x_0 = p, x_1, ...$ be a backward trajectory such that $x_1, x_2, ... \notin O(p)$ and $x_n \to O(p)$ as $n \to \infty$. There exists $\delta > 0, m > 0$ such that all $f^t(B(p,\delta))$ for $0 < t \leq n(p)$ are pairwise disjoint with f univalent on them, $f^{n(p)}(B(p,\delta)) \supset \mathrm{cl}B(p,\delta)$, $x_m \in f^{n(p)}(B(p,\delta)) \setminus \mathrm{cl}B(p,\delta)$, $x_n \notin f^{n(p)}(B(p,\delta))$ for all $0 < n < m$ and $x_n \in \bigcup_{0 \leq t < n(p)} f^t(B(p,\delta))$ for $n > m$.

Fix k large enough that for $V_0 = \mathrm{Comp}_p f^{-kn(p)}(B(p,\delta))$. Denoting also $V_n = \mathrm{Comp}_{x_n} f^{-n}(V_0)$ for $n = 1, 2, ...$, we get

$$V_m \subset f^{n(p)}(B(p,\delta)) \setminus \mathrm{cl}B(p,\delta).$$

and all sets $V_n, n = 0, ...m-1$ are disjoint from $f^{n(p)}(B(p,\delta))$.

Considering $f^t(V_0), 0 \leq t < n(p)$ and $V_n, 0 < n <\leq n(p) + m$, we obtain a family of topological discs U_j on which f is "generalized polynomial-like", that is $f(\bigcup_j \partial U_j) \cap \bigcup_j \partial U_j = \emptyset$. More precisely, we can guarantee this by making small changes of U_j to destroy $f(\partial V_{n+1}) = \partial V_n$ and $f^t(\partial V_0) = \partial f^{t+1}(V_0)$.

If x_n do not meet Crit then choosing δ small enough, denoting $\bigcup_j U_j$ by U, we guarantee $U \cap \mathrm{Crit} = \emptyset$, hence f on the related "Julia-like set" $X = \{z : (\forall n \geq 0) f^n(z) \in U\}$ is expanding (that is $(\forall x \in X)|(f^n)'(x)| > 1$).

In the case one cannot find such x_n, that is all backward trajectories from $O(p)$ meet Crit, we replace first $O(p)$ by $O(p')$ of minimal period $kn(p) + m$, with $p' \in V_0$ defined above, with large k. $O(p')$ follows $O(p)$ for time $kn(p)$. For every $\varepsilon > 0$ we find k large enough that $\Lambda(p') \leq \Lambda(p) + \varepsilon$ and backward trajectories from $O(p')$ do not meet Crit (use $\sharp\mathrm{Crit} < \infty$) so we can repeat the previous construction for p replaced by p'. (Note that $\Lambda(p')$ can be much smaller than $\Lambda(p)$ for every $p' \neq p$, consider for example $z \mapsto z^2 - 2$ and $p = 2$.)

We conclude with expanding X on which f is a topological Markov chain, topologically transitive and not just one periodic orbit. Hence the toplogical entropy of f on X is positive.

Note finally that for every $\varepsilon > 0$ one can find δ so that Lyapunov exponents of all points in X are smaller than $\Lambda(p) + \varepsilon$ because the smaller V_0 the longer time each trajectory follows $O(p)$ before a consecutive passing "the bridge" along $x_m, x_{m-1}, ..., x_0$. So every μ on X has Lyapunov exponent less than $\Lambda(p) + \varepsilon$, in particular the balanced measure, i.e. with maximal, hence positive, entropy. QED

Corollary. UHP, Lyapunov$_+$ and Lyapunov2 *are equivalent to each other.*

Lemma C.2. ExpShrink *implies* Lyapunov, *with all* $\Lambda(\mu) \geq \log \lambda_{\mathrm{Exp}}$.

Proof. Take any f-invariant probability measure μ on J. By [P5] $\Lambda(\mu) \geq 0$. Hence for every $\delta > 0$ and μ-a.e. $x \in J$ we have $\mathrm{dist}(f^n(x), \mathrm{Crit}(f)) \geq \exp(-\delta n)$ for all n large enough. Hence distortion of f^n on $\mathrm{Comp}_x f^{-n}(B(f^n(x), \frac{1}{2}\exp(-\delta n))$ is bounded by

a constant independent of n. Hence $|(f^n)'(x)| \geq C\exp(-\delta n)/\lambda_{\mathrm{Exp}}^{-n}$. Taking δ arbitrarily small, we get $\Lambda(\mu) \geq \log \lambda_{\mathrm{Exp}}$. QED

Since obviously the condition Lyapunov implies Lyapunov$_+$, we obtain the following conclution from the Main Theorem, see Introduction.

Conclusion. Lyapunov$_+$ $\Leftrightarrow$ Lyapunov2 $\Leftrightarrow$ Lyapunov $\Leftrightarrow$ a)-f) in the Main Theorem.

Recall again that a compact set $X \subset \overline{\mathbb{C}}$ is called *hyperbolic*, or *expanding*, if there exists $n > 0$ such that $|(f^n)'(x)| > 1$ for every $x \in X$. Then the *hyperbolic dimension* of $J(f)$, is the supremum of Hausdorff dimensions of f-invariant hyperbolic subsets of $J(f)$.

Corollary. Let f be a rational map satisfying the condition TCE. Then the pressure function $P(t)$ has a unique zero, and it coincides with the hyperbolic dimension of $J(f)$.

Proof. By the above the condition TCE is equivalent to the condition Lyapunov. The condition Lyapunov implies by the definition of $P(t)$ that at every t the left and right hand side derivatives are less than a negative constant, supremum achieved for $t \to \infty$ equal to $-\inf \Lambda(\mu)$ (see Introduction).

By [P2] for every rational function f the first zero of the pressure function is equal to the hyperbolic dimension of $J(f)$. QED

Proposition C.1. *If f is* Topological Collet-Eckmann (TCE) *then the unique zero of pressure is equal to the box dimension of $J(f)$, in particular hyperbolic, Hausdorff and box dimensions of $J(f)$ coincide.*

This assertion was proved under the Collet-Eckmann condition in [P1] and under a summability condition in [GS2].

Proof of Proposition C.1. As in [PR1] (the proof of mean porosity of $J(f)$), we see that for every $p_1 > 1, \xi > 0$ there exist $\delta > 0, M \geq 0$ such that for every $x \in J(f)$ there exist increasing sequences of integers n_j, k_j such that $n_j \leq p_1 j$, $W_j := \mathrm{Comp}_x f^{-k_j}(B(f^{k_j}(x), \delta))$ is contained in $B(x, 2^{-n_j})$ and contains $B(x, \xi 2^{-n_j})$ and for each j there is only finite, bounded by M, number of $i : 0 \leq i \leq k_j$ such that $f^i(W_j)$ contains a critical point (δ replaced by 2δ in the latter W_j). Let m be any probability α-conformal measure on J. Then there exists a constant C depending on M such that $m(W_j) \geq C(\mathrm{diam} W_j)^\alpha$, compare [P1]. Hence for every $r > 0$ and every $x \in J$ we obtain $m(B(x,r)) \geq C_1 r^{\alpha+\epsilon}$ where ϵ depends on p_1 and tends to 0 for $p_1 \searrow 1$ and $C_1 = C_1(\varepsilon) > 0$ is a constant not depending on x and r.

For every r one can find a bounded multiplicity cover of J by discs B_i of diameter r and $m(B_i) \geq C_1 r^{\alpha+\epsilon}$ (Besicovitch) hence $\sum_i (\mathrm{diam} B_i)^{\alpha+\epsilon} \leq C_1^{-1} \sum_i m(B_i) \leq Const.$ Hence upper box dimension $\mathrm{BD}(J(f)) \leq \alpha + \epsilon$. Taking α equal to hyperbolic dimension of J (possible by [DU] and [P5]) ends the proof. QED

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