The dichotomy for the boundary of a parabolic simply connected basin: either it is analytic or its Hausdorff dimension is bigger than 1.

to a paper by F. Przytycki, J. Skrzypczak, A. Volberg,...

(notes by F. Przytycki, May 9, 1995)

Notation: Let $R : D \to U$ be a Riemann map from the unit disc to a simply-connected domain $U \in C$. Holomorphic f maps a neighb of ∂U to a neighb of ∂U, preserving ∂U. Denote by F its pull-back by R to a neighb of ∂D (extended by Symmetry Principle beyond ∂D). Let $P \in \partial D$ be an F-fixed parabolic point, weekly expanding along ∂D, with two petals (in D and in $C \setminus \mathrm{cl}D$). Assume F expansive on ∂D. (Cf [U1] M. Urbański Studia Math. 97 (3) (1991), 167- 188.)

Proposition 1. If U is a simply connected immediate parabolic basin for a rational map f such that $f(U) = U$, then the above conditions hold.

Proof. F is a Blaschke product. It does not have a fixed point in D because f does not have a fixed point in U, so $F^n|_D$ converge to a parabolic point $P \in \partial D$. To prove the expansiveness of F on ∂D, i.e. that ∂D is Julia set for F, it is sufficient to prove:

Claim. For almost every $z \in \partial D$ (for which R has a nontangential limit) for its Stolz angle S_z there is $x \in R(S_z)$ arbitrarily close to $R(z)$ such that $f^n(x)$ is well inside U for some $n > 0$.

Indeed then $F^n(R^{-1}(x))$ is well inside D hence z is not in a basin of attraction to P. Otherwise z together with its neighb in the basin would have forward trajectory close to ∂D.

Proof of Claim. Let a be an external component of the boundary of a small petal in U. Let n_0 be large enough that a component b of $f^{-n_0}(a)$ bounds all critical values of $f|_U$ (i.e. all these critical values are on the same side of b as the original petal). Choose points $q_1, q_2 \in b$ close to P, on different sides of P, take their pre-images q_1', q_2' by the branch g of f^{-1} fixing P, join q_i and q_i' with lines l_i in U so that

$\gamma :=$ (the long part of b between q_1 and q_2) $\cup \bigcup_{n\geq 0, i=1,2} g^n(l_i) \cup P$ is a Jordan curve. Then except for $\zeta \in \bigcup_{n\geq 0} f^{-n}(\{P\})$ every curve accessing ζ from U intersects $\Gamma := \bigcup_{n\geq 0} f^{-n}(\gamma)$ at a sequence of points converging to ζ. Observe finally that by the construction for every $x \in \Gamma$ there exists $n > 0$ so that $f^n(x)$ is within at least a fixed distance from ∂U. Claim is proved.

Remark. This proves (after an adequate correction for a non-simply connected case) that pre-images of every point in a parabolic basin converge densily to the boundary. So for example image of the maximal measure from the shift to ∂U through a geometric coding tree has support $= \partial U$, see [PZ] Przytycki, Zdunik, Fund. Math. , in particular accessible periodic sources are dense in ∂U. This was proved in [PZ] but the aboveproved density was commented in [PZ] only by the words "it is not hard..." p.76[1].

Notation. Let p denote the parabolic point in ∂U attracting U and $B = B(p, r_0)$ where r_0 is small enough that the twice larger disc $D(p, 2r_0)$ does not contain any critical point for f. Let $w := \partial U \setminus B$ and $W := R^{-1}(w)$ wherever it is defined ,(the radial limit R is defined only a.e.). Let λ, Λ be harmonic measures on ∂U, resp. ∂D (from a point in U, resp. D). Let f^*, F^* be first return maps for iterations of $f|_{\partial U}, F|_{\partial D}$ to w, W resp. . Let λ^*, Λ^* be f^* resp. F^*-invariant measures on w, W equivalent to λ, Λ. (The existence follows from Schweiger [?], the densities $d\lambda^*/d\lambda$, $d\lambda/d\lambda^*$, $d\Lambda^*/d\Lambda$, $d\Lambda/d\Lambda^*$ are bounded.

Lemma 1. $\log|(f^*)'|$ is λ (or λ^*)-integrable.

Proof. Let m be an f-invariant measure on ∂U, the "propagation" of λ^*, namely $m(E) := \lambda^*\{z \in w :$ there exists $n_z \geq 0$, such that $f^{n_z}(z) \in E, f^n(z) \notin w$ for every $0 < n \leq n_z\}$.

Remark that m is infinite ([Schweiger], [Aaronson, Denker, Urbanski] ?).

Clearly $\int \log|f'|dm = \int \log|(f^*)'|d\lambda^*$. So we are done if we prove

$$\int |\log|f'||dm < \infty.$$

The latter follows from

1. The integrability far from p, where $dm/d\lambda$ is bounded;

2. $m(Q_n) \approx n^\alpha$ for a constant $\alpha > 0$, where $Q_n := f_\nu^{-n+1}(B(p, r_0)) \setminus f_\nu^{-n}(B(p, r_0))$ and the backward branch f_ν^{-1} keeping p invariant;

[1. holds because $\log|f'(R(z))|$ for $z \in D$ is comparable with Constdist$(R(z), c)$ close to each f-critical $c\partial U$ and $R(z) - c$ as univalent is in Nevanlinna class. Hence $f' \circ R$ is in Nevanlinna class, compare [Przytycki, Inv. Math.].

2. holds because $m(Q_n) \approx \sum_\tau \lambda(f_\tau^{-1} f_\nu^{-n}(B(p, r_0)))$, summing over all branches $f\tau^{-1}$ not mapping p to p. Each summand is bounded from above by Const$n^{-\beta/2}$ (where $\beta = 1$ if p is not a critical value and p has only one petal) by the geometry of harmonic measure ($\lambda(B(z, \rho) < \text{Const}\sqrt{\rho}$, for every z, ρ, [Beurling ?]).]

3. $\log|f'| \approx n-1$ on Q_n.

Remark. The estimate $\int \log|(f^*)'|d\lambda^* > -\infty$ is much simpler than Lemma 1: the integrability and $\int < \infty$, because close to p we have $|f'| \geq 1$ in ∂U. It seems that later on we use only this easier estimate.

Lemma 2. $\tilde{\lambda^*}$-almost every backward $\tilde{f}^*$-trajectory, i.e. sequence $\zeta_n \in \partial U, n = 0, 1, ...$ such that $f(\zeta_n) = \zeta_{n-1}$ and $\zeta_{n_k} \in w$ for a subsequence n_k, in other words an a.e. element ζ_{n_k} in the natural extension $(\tilde{w}, \tilde{f}^*, \tilde{\lambda^*})$ of (w, f^*, m), has the property that for every $\delta > 0$ there exists $C > 0$ such that for every k and every $n_k \leq n \leq n_{k+1}$

$$\text{dist}(\zeta_n, \text{Crit}(f)) > C \exp -\delta k \qquad (2).$$

In other words the trajectory approaches Crit at most subexponentially.

Proof. The assertion (2) for $n = n_k$ follows from Borel-Cantelli and the $\tilde{\lambda^*}$-integrability of $\log|f'(z)|$ on w (Nevanlinna, see 1. in Proof of Lemma 1).

[Explanation: for $\pi : \tilde{w} \to w$ the projection to the zero's coordinate, the integrability gives

$\sum_k \tilde{\lambda^*}(\tilde{f^*}^k \pi^{-1} B(\mathrm{Crit}(f), \theta^k)) = \sum_k \lambda^*(B(\mathrm{Crit}(f), \theta^k)) =$
$\sum_k k\lambda^*(B(\mathrm{Crit}, \theta^k) \setminus B(\mathrm{Crit}, \theta^{k+1}) \leq -\mathrm{Const} \int \log \mathrm{dist}(z, \mathrm{Crit}) d\lambda^*(z) < \infty$ for every $0 < \theta < 1$. Hence by Borel-Cantelli $\tilde{\lambda^*}$-a.e.trajectory in $\tilde{w}$ belongs only to a finite number of $\tilde{f^*}^k \pi^{-1} B(\mathrm{Crit}(f), \theta^k)$'s i.e. only to finite number of ζ_{n_k}'s belong to $B(\mathrm{Crit}(f), \theta^k$.]

For n strictly between n_k and n_{k+1} we have $\zeta_n \in B(p, r_0)$ hence by the definition of r_0 at the distance at least r_0 from Crit.]

Lemma 3. Characteristic Lyapunov exponent $\chi(f^*, \lambda^*) = \int \log |(f^*)'| d\lambda^*$ is positive. Moreover $\chi(f^*, \lambda^*) = \chi(F^*, \Lambda^*)$.

Proof. A.e.trajectory $(\zeta_{n_k}) \in \tilde{w}$ has the property that

$$\mathrm{diamComp} f^{-n_k}(B(\zeta_0, \exp -Ck)) \leq \exp -(C/2)k$$

for C large enough and all k (the components containing ζ_{n_k} resp.).

This is so by Lemma 1: $\int \log |(f^*)'| d\lambda^* > -\infty$, or by [Denker, Przytycki, Urbański].

By Lemma 3 there are no f^*-critical points in the abovementioned components. Moreover the respective branches of f^{-n_k} have uniformly bounded distortions (on $B(\zeta_0, \frac{1}{2} \exp -Ck)$).

For $R(\xi) = \zeta_0$ we have for the respective backward branches

$$|(F^{-n_k})'(t\xi)| = |(f^{-n_k})'(R(t\xi))| \frac{|R'(t\xi)|}{|R'(F^{-n_k}(t\xi)|} \qquad (3).$$

We apply it for $t = 1 - \exp -2Ck$. We obtain by $\lim_{t \to 1} \frac{\log R'(tz)}{\log |1-t|} = 0$ for a.e.z [Makarov] that

1. $R(t\xi) \in B(\zeta_0, \frac{1}{2} \exp -Ck)$
2. $|\log |R'(t\xi)|| \leq \kappa Ck$ for $\kappa > 0$ arbitrarily close to 0 and k large enough
3. $|\log |R'(F^{-n_k}(t\xi))|| \leq \kappa k(C + 2\chi(F^*, \Lambda^*))$

This is so for a subsequence of n_k to have uniform Makarov estimates, i.e. with the same κ

We conclude that taking $(1/k) \lim ...$ in (3) we can neglect R' and replace $t\xi$ by ξ.

Lemma 4. (Basic Lemma) For a.e. $(\zeta_{n_k}) \in \tilde{w}$ (Notation: ζ_n denotes a backward trajectory for f, n_k are integers for which $\zeta_{n_k} \in w$, more precisely $f^*(\zeta_{n_k}) = \zeta_{n_{k-1}}$, as before), there exists $\delta > 0$ and univalent maps g_n on $B(\zeta_0, \delta)$ being branches of f^{-n} mapping ζ_0 to ζ_n for all $n \geq 0$.

Proof. Using above Lemmas (also Birkhoff Ergodic Theorem and the ergodicity of f^*) we can assume that

1^o $\lim_{k \to \infty} \frac{1}{k} \log |g'_{n_k}(\zeta_0)| = \chi := \chi(f^*, \lambda^*)^{-1} < 0$,
in particular $|g'_{n_k}(\zeta_0)| < \exp -k\chi/2$ for k large enough. (Pay attention that we do not know yet how large are domains of g_{n_k}'s.)

2^o ζ_{n_k} approaches $\mathrm{Crit}(f)$ at most subexponentially.

For every $k = 1, 2, \ldots$ let $\delta_k = \delta_{k-1}(1 - \exp(-\frac{1}{4}\chi k))$, where $\delta_0 > 0$ will be specified later on. Denote for every $n_k \le n < n_{k+1}$

$$B_n := \mathrm{Comp}_{\zeta_n} f^{-n}(B(\zeta_0, \delta_k)),$$

the component containing ζ_n.

Suppose (by induction with respect to k)

1. the univalent branch g_{n_k} exists on $B(\zeta_0, \delta_k)$;
2. $\mathrm{diam} B_{n_k} \le \exp(-k\chi/4)$.

By "flower dynamics" close to p the branch G of $f^{-(n_{k+1}-n_k-1)}$ keeping p fixed exists on B_{n_k} because $\mathrm{diam}(B_{n_k})$ is much smaller than the distance from p (which is of order r_0). Moreover $|G'| < 1$. Hence for $B' := \mathrm{Comp} f^{-(n_{k+1}-n_k)}(B_{n_k})$ we have $\mathrm{diam} B' \le \mathrm{Const} \exp(-k\chi/2\nu)$ for ν the maximal multiplicity of f (at critical points). Hence by 2^o there are no f critical points in B'. So $g_{n_{k+1}}$ is univalent on $B(\zeta_0, \delta_k)$. So by Koebe distortion theorem $\mathrm{diam} B_{k_{n+1}} \le \exp(-(k+1)\chi/2)\exp(k\chi/4)$.

To begin the induction we take δ_0 small enough that 1.,2. hold for small k automatically hence we can assume in the induction that k is large.

Finally we set $\delta = \lim_{k\to\infty} \delta_k$.

Now we are in the position to prove

Theorem 1. If $\log |(f^*)' \circ R| - \log |F^*)'|$ is cohomologous to 0 in $L_1(W, \Lambda^*)$ then ∂U is real analytic.

Proof (a sketch).

1. The cohomology equation $\log |(F^*)'| - \log |(f^*)' \circ R| = u^* \circ F^* - u^*$ extends from W to ∂D to

$$\log |F'| - \log |f' \circ R| = u \circ F - u, \tag{4}$$

where $u(z) := u^*(z') + \log |F'(z')| - \log |f' \circ R(z')|$ for $z' \in F^{-1}(z) \cap W$. As u^* is Λ^*-integrable (equiv. Λ integrable) and $\log |F'| - \log |f' \circ R|$ is Λ-integrable, so is u.

2. Let us extend u to a harmonic function h on D. Then (4) holds up to a harmonic function φ which is 0 on ∂D (this error function can be $\not\equiv 0$ if f does not extends holomorphically to U). Then

$$\log |R'(z)| - \log |R'(F^n(z))| = \log |(F^n)'(z)| - \log |(f^n)'(R(z))|$$

$$= h(F^n(z)) - h(z) + \sum_{j=0}^{n-1} \varphi(F^j(z))$$

hence denoting $\Phi(z) = \log |R'(z)| + h(z)$

$$\Phi(z) - \Phi(F^n(z)) = \sum_{j=0}^{n-1} \varphi(F^j(z)).$$

Let Γ be the union of curves in D as in Proof of Claim (page 1), but for F rather than for f. Assume $z \in \Gamma$. Then $|\sum_{j=0}^{n-1} \varphi(F^j(z))| <$ Const.

[Explanation: For $z = t\zeta$ for $|\zeta| = 1, 0 \le t < 1, t \approx 1$ let j_t be consecutive times for which $\mathrm{dist}(F^{j_t}(\zeta), P) > \varepsilon > 0$ for a small constant ε. Then $|\sum_{j=j_t}^{j_{t+1}-1} \varphi(F^j(z))| < (\sum_{n=1}^{\infty} n^{-\alpha})\mathrm{dist}(F^{j_t}(z), \partial D)$ for a constant $\alpha > 1$. We have used bounded distortion for above iterates of F and Lipschitz continuity of φ at ∂D.

By the expanding property of the first return map under the iteration of F to $\partial D \setminus B(P, \varepsilon)$ we have $\mathrm{dist}(F^{j_t}(z), \partial D)$ shrinking exponentially for decreasing t. Thus the above series is bounded by a constant.]

As $|\Phi(F^n(z))| <$ Const we obtain $|\Phi(z)| <$ Const on Γ. So by maximum principle (Phragmen-Lindelöf's version) $\Phi \in \mathbf{h}^\infty$. Hence $\log|R'| \in \mathbf{h}^1$. So R' is in Nevanlinna class, hence it has a nontangential limit a.e..

Remark. This proof relies on the idea (and is a simplification) of the analogous proof in the basin of attraction to a sink case [Zdunik, TAMS].

3. We choose a typical F^*-backward trajectory $\xi_{n_k} \in \tilde{W}$. We write $R = (f^*)^k \circ R \circ (F^*)^{-k}$ in a Stolz cone at ξ_0. For such k's that $\lim R'$ exists uniformly in S_k, Stolz cones at ξ_k's, we have

$$R(z) = \lim_{k \to \infty} (f^*)^k \circ \mathrm{Aff}_k \circ (F^*)^{-k}$$

where Aff_k is the affine part of R' at ξ_{n_k} (i.e. its nontangential limit). The family under lim is bounded hence normal in a neighbourhood of ξ_0 (this uses Lemma 4). Hence R extends holomorphically to a neigh of ξ_0. (This is the same large-small-large scale consideration as in the proof of rigidity in the expanding repellers case.)

Remark. To prove that the unsolvability of the cohomology equation in Theorem 1 yields $\mathrm{HD}(\partial U) > 1$ one needs to prove first that $\Psi := \log|(F^*)'| - \log|(f^*)' \circ R| \in L^p$ and that correlations $\int \Psi \cdot (\Psi \circ (F^*)^k) d\Lambda^*$ decrease fast enough. So the respective σ^2 exists and is positive. This is the strategy similar to [PUZ, Ann. Math.]. Next using Central Limit Theorem (fluctuations from the expectation value of sums $\sum \Psi \circ (F^*)^k$) using good backward trajectories (Lemma 4) one would like to produce a thick Cantor set as in [Zdunik, Inv. Math.]. All this does not seem technically simple.