On Hausdorff dimension of some Cantor attractors

G. Levin
Dept. of Math., Hebrew Univ.
Jerusalem 91904, Israel
levin@math.huji.ac.il

F. Przytycki
Inst. of Math. of PAN
Warsaw 00-950, Poland
F.Przytycki@impan.gov.pl

Abstract

We study what happens with the dimension of Feigenbaum-like attractors of smooth unimodal maps as the order of the critical point grows

1 Introduction

Let f be a smooth unimodal map of an interval. We assume that f is infinitely-renormalizable with stationary combinatorics. Then f has an attractor $C(f)$ both in metric and topological senses, which is a Cantor set and which is the ω-limit set of the critical point of f. In this note we consider the following question motivated by [1], [15], and [8]: what happens with the Hausdorff dimension of $C(f)$ as the order ℓ of the critical point grows to infinity? We show that it must grow to at least 2/3. In the orientation reversing case (which includes the classical Feigenbaum's one) we also prove that the Hausdorff dimension has a limit as ℓ tends to infinity, this limit is less than 1, and it is equal to the Hausdorff dimension of an attractor of some limit unimodal dynamics defined in [8].

Denote by $HD(E)$ the Hausdorff dimension of a set E in $\mathbf{R}^n$.

It is well-known [9] (and follows from convergence of renormalizations), that the Hausdorff dimension $HD(C(f))$ of the attractor $C(f)$ of f depends actually only on the stationary combinatorics $\aleph$ of the map f and the criticality order ℓ of its critical point provided that ℓ is an even integer. It allows us to write $D(\aleph, \ell) = HD(C(f))$ for all smooth f with fixed $\aleph$ and ℓ.

(Note here that once the convergence of renormalizations is established for all real big enough criticalities ℓ all results and proofs of the paper hold true for such ℓ.)

We have a priori:

$$0 < HD(\aleph, \ell) < 1. \tag{1}$$

Comment 1 *(1) If $\ell = 2$, then the upper bound in (1) can be strengthened [5]: there is a number $\sigma < 1$, such that $HD(\aleph, 2) \le \sigma$ for all combinatorics $\aleph$.*

(2) Feigenbaum's case $|\aleph| = 2$ with the quadratic critical point ($\ell = 2$) has been studied intensively, see [16], [7], particularly in the framework of Feigenbaum's universality [3], [4]. Numerically, $D(\aleph, 2) = 0.538...$, see [6].

(3) Although $HD(\aleph, \ell)$ is always positive, it is not difficult to construct a sequence of stationary combinatorics $\aleph_n$, such that, for every ℓ, $HD(\aleph_n, \ell) \to 0$ as $n \to \infty$. For instance, $\aleph_n$ can be defined by the following first $n-1$ itineraries of the critical value: $n-2$ times "plus" and one time "minus". Then bounds (real or complex) imply that if $f_n(z) = z^\ell + c_n$ is infinitely-renormalizable with the stationary combinatorics $\aleph_n$, then $HD(C(f_n)) \to 0$ as $n \to \infty$.

Note that the number $D(\aleph, 2)$ ($|\aleph| = 2$) as well as the numbers $HD(\aleph_n, \ell)$ (with fixed ℓ and big n) are less than $2/3$.

Theorem 1 *For every $\aleph$,*

$$\liminf D(\aleph, \ell) > \frac{2}{3} \tag{2}$$

as ℓ tends to infinity along the even integers.

To state our result about the upper bound, we need to introduce some notions.

Non-symmetry. For a unimodal map f with a single critical point at c, denote by I_f the involution map defined in a neighborhood of c by $I_f : x \mapsto \hat{x}$, where $I_f(c) = c$, and otherwise $I_f(x)$ is the unique $\hat{x} \ne x$, such that $f(x) = f(\hat{x})$. If f is of the form $|E(x)|^\ell$, where $\ell > 1$ and E is a C^2-diffeomorphism, then I_f is also C^2, and $I'_f(c) = -1$. The *non-symmetry* $N(f)$ of f is said to be the number $N(f) = |I''_f(c)/2|$. It is easy to check that $N(f) = |E''(c)/E'(c)|$.

Orientation reversing combinatorics of an infinitely-renormalizable unimodal map f is such stationary combinatorics $\aleph$, that the rescaling factor of the renormalization is negative. In other words, the maps f and $f^{|\aleph|}$ have at the critical point of f different type of extrema (maximum and minimum). Examples: $|\aleph| = 2, 3$; more generally, $\aleph_n$ ($n \ge 1$) defined in Comment 1(3).

For a combinatorial type $\aleph$ and an even integer ℓ, denote by $H_{\aleph,\ell}$ the unique universal unimodal map normalized so that $H_{\aleph,\ell} : [0, 1] \to [0, 1]$ and $H_{\aleph,\ell}(0) = 1$ (see next Section for complete definition). It is shown in [8], that the sequence $\{H_{\aleph,\ell}\}_\ell$ converges uniformly to a unimodal map $H_\aleph : [0, 1] \to [0, 1]$.

We prove in Lemma 4.3 that *if the combinatorial type $\aleph$ reverses orientation, then the sequence of non-symmetries $N(H_{\aleph,\ell})$, $\ell = 2, 4, ...$, is uniformly bounded.*

Theorem 2 *For a given combinatorial type $\aleph$, assume that the sequence of non-symmetries $N(H_{\aleph,\ell})$, $\ell = 2, 4, ...$, is uniformly bounded. Then the Hausdorff dimension of the attractor is continuous at $\ell = \infty$: there exists*

$$\lim_{\ell\to\infty} D(\aleph, \ell) = HD(C(H_\aleph)) < 1. \tag{3}$$

Consequently, (3) holds when $\aleph$ reverses orientation.

Comment 2 *It is not clear if the non-symmetry $N(H_{\aleph,\ell})$ is uniformly bounded in ℓ for any type $\aleph$.*

The proof of Theorems 1- 2 is based on recent results of [8]: see next Sect. where we reduce the statements to Theorem 4.

(Note however that in the proof of the lower 2/3-bound we use only a part of the main result of [8], namely, the compactness (Theorem 4 in [8]).)

In turn, to prove Theorem 4 we use some results of [10], [13], see Sect. 3.

From now on, we fix the type $\aleph$. Denote $p = |\aleph|$.

Acknowledgment. The first author thanks Benjamin Weiss for a helpful discussion. Second author acknowledges Polish KBN grant 2PO3A 034 25.

2 Reduction to fixed-point maps

2.1 Universal maps

For every real number $\ell > 1$, we consider a unimodal map $g_\ell : [-1,1] \to [-1,1]$ with the critical point at 0 of order ℓ. More precisely, g_ℓ is assumed to be in the following form: $g_\ell(x) = E_\ell(|x|^\ell)$, where $E_\ell : [0,1] \to \mathbb{R}$ is a C^2-diffeomorphism onto its image. The map $g = g_\ell$ is normalized so that $g_\ell(0) = 1$. It is further assumed to be infinitely renormalizable with the fixed combinatorial order type $\aleph$ and to satisfy the fixed point equation:

$$\alpha g^{|\aleph|}(x) = g(\alpha x) \ . \tag{4}$$

with $|\alpha| > 1$. By renormalization theory, see [14], a fixed point g_ℓ for any $\ell > 1$ can be represented as $E_\ell(|x^\ell|)$ with E_ℓ which is a diffeomorphism in Epstein class (i.e. a diffeomorphism E of a real interval T' onto another real interval T such that the inverse map $E^{-1} : T \to T'$ extends to a univalent map $E^{-1} : (\mathbb{C} \setminus \mathbb{R}) \cup T \to (\mathbb{C} \setminus \mathbb{R}) \cup T')$.

It will be useful to deal with another unimodal map H_ℓ, which is related to g_ℓ as follows: $H_\ell(x) = |g_\ell(x^{1/\ell})|^\ell = |E_\ell(x)|^\ell$, $0 \le x \le 1$. Then H_ℓ is a unimodal map of $[0,1]$ into itself, with a strict minimum attained at some $x_\ell \in (0,1)$. It also satisfies the equation:

$$\tau H^{|\aleph|}(x) = H(\tau x) \ . \tag{5}$$

with $\tau = |\alpha|^\ell$.

We denote by $C(g_\ell)$ and $C(H_\ell)$ the attracting Cantor sets of the maps $g_\ell : [-1,1] \to [-1,1]$ and $H_\ell : [0,1] \to [0,1]$ respectively. Clearly, $HD(C(g_\ell)) = HD(C(H_\ell))$. Indeed, the map $x \mapsto x^\ell$ maps $g_\ell^{2k+1}(0)$ to $H_\ell^{2k+2}(x_\ell)$, $k \ge 0$, hence, this map is a diffeomorphism of $C(g_\ell)\cap[g_\ell^3(0), g_\ell(0)]$ onto $C(H_\ell)\cap[H_\ell^4(x_\ell), H_\ell^2(x_\ell)])$. Therefore, $HD(C(g_\ell)) = HD(C(g_\ell)\cap[g_\ell^3(0), g_\ell(0)]) = HD(C(H_\ell)\cap[H_\ell^4(x_\ell), H_\ell^2(x_\ell)]) = HD(C(H_\ell))$.

Assume now that the order ℓ is an even integer. Then the equation (4) with the normalization as above does have a unique solution, for every fixed ℓ and $\aleph$, see [14], [11]. Consequently, $H_\ell = |g_\ell(x^{1/\ell})|^\ell$ is the unique solution of (5) with the normalization as above.

In what follows, ℓ is an even integer, and H_ℓ denotes this unique solution of (5), with its own scaling constant $\tau_\ell > 1$. (Remind that the type $\aleph$ is fixed.)

2.2 Limit dynamics

The following result is proved in [8] (even for real ℓ), see Theorems 1-2 and Proposition 3 there:

Theorem 3 *The sequence of maps H_ℓ converges as $\ell \to \infty$, uniformly on $[0,1]$, to a unimodal function $H = H_\infty$, which satisfies the following properties:*

1. *$\lim_{\ell\to\infty} \tau_\ell = \tau > 1$ exists. and H, τ satisfy the fixed point equation $\tau H^p(x) = H(\tau x)$ for every $0 \le x \le \tau^{-1}$. Here (as always) $p = |\aleph|$.*
2. *H has analytic continuation to the union of two topological disks U_- and U_+ and this analytic continuation will also be denoted by H.*
3. *For some $R > 1$, H restricted to either U_+ or U_- is a covering (unbranched) of the punctured disk $V := D(0,R) \setminus \{0\}$ and $\overline{U_+ \cup U_-} \subset D(0,R)$.*
4. *$U_\pm$ are both symmetric with respect to the real axis and their closures intersect exactly at x_0; $[0, x_0) \subset U_-$, $(x_0, 1] \subset U_+$.*
5. *Each H_ℓ extends to complex-analytic map defined in $U_-\cup U_+$; this sequence of analytic extensions converges to H, as $\ell_m \to \infty$, uniformly on every compact subset of $U_- \cup U_+$.*
6. *For any two open intervals I, J of the real axis, if $0 \notin J$ and $H : I \to J$ is one-to-one, then the branch $H^{-1} : J \to I$ extends to a univalent map to the slit complex plane $(\mathbf{C} \setminus \mathbf{R}) \cup J$ (this follows from the same property for H_ℓ with ℓ finite)*
7. *The mapping $G_\infty(x) := H^{p-1}(\tau^{-1}x)$ fixes x_0 and G_∞^2 has the following power series expansion at x_0:*

$$G_\infty^2(x) = x - a(x - x_0)^3 + O(|x - x_0|^4)$$

   with $a > 0$.

8. *For each ℓ, the mapping $G_\ell := H_\ell^{p-1}(\tau_\ell^{-1}x)$ fixes the critical point x_ℓ of H_ℓ, $G_\ell'(x_\ell) = \pm 1/\tau_\ell^{1/\ell}$, and G_ℓ converge to G_∞ uniformly in a (complex) neighborhood of x_0.*

9. *The unimodal map map $H : [0,1] \to [0,1]$ has a unique attractor $C(H)$, which (as for finite ℓ) is the closure of iterates of the critical point.*

2.3 The reduction

Since we know already that $HD(C(f))$ depends merely on $\aleph$ and ℓ, Theorems 1- 2 are covered by the following statement

Theorem 4 *The following holds.*

(a)

$$\liminf_{\ell\to\infty} HD(C(H_\ell)) \geq HD(C(H_\infty)). \tag{6}$$

(b)

$$\frac{2}{3} < HD(C(H_\infty)) < 1; \tag{7}$$

(c) if the non-symmetries $N(H_\ell)$ are uniformly bounded as $\ell \to \infty$, then the Hausdorff dimension is continuous at infinity:

$$\lim_{\ell\to\infty} HD(C(H_\ell)) = HD(C(H_\infty)). \tag{8}$$

The rest of the paper is devoted to the proof of this statement.

3 Background in dynamics

We prove Theorem 4 by reducing it finally to known statements about infinite conformal iterated function systems (c.i.f.s.) [10] and asymptotics near parabolic maps [13], which are given here.

3.1 C.I.F.S.

We follow [10] restricting ourself to dimension one. Let X be a closed real interval, and σ be a positive continuous function on X, which defines a new metric $d\rho = \sigma dx$ on X. Let I be a countable index set, $|I| > 1$, and let $S = \{\phi_i : X \to X, i \in I\}$ be a collection of injective uniform contractions w.r.t. the metric ρ: there is $\lambda < 1$, such that $\rho(\phi_i(x), \phi_i(y)) \leq \lambda\rho(x, y)$ for all i and all x, y. For every finite word $w = w_1...w_n$, denote $\phi_w = \phi_{w_1} \circ ... \circ \phi_{w_n}$. (Note that the metric ρ can be replaced by the Euclidean one by replacing ϕ_i by ϕ_w, where w runs over all finite words of some fixed length n, s.t. $\lambda^n||\sigma|| < 1$.) For any infinite word of symbols $w = w_1w_2...w_j...$, $w_j \in I$, denote $w|n = w_1w_2...w_n$. The limit set L of S is $L = \cup_{w\in I^\infty} \cap_{n=1}^{\infty} \phi_{w|n}(X)$. The system S is said to be conformal if:

(a) $\phi_i(Int(X)) \subset Int(X)$ and $\phi_i(Int(X)) \cap \phi_j(Int(X)) = \emptyset$ for all indexes $i \neq j$.

(b) There is an open set $Y \supset X$, such that all maps ϕ_i extend to $C^{1+\epsilon}$ diffeomorphisms of V into V.

(c) There is $K \geq 1$, such that $|D\phi_w(y)| \leq K|D\phi_w(x)|$ for every finite word w and all $x, y \in Y$, where $D\phi_w(x)$ means the derivative w.r.t. the metric ρ

The main object of our interest is the Hausdorff dimension of the limit set L. Note that it is the same w.r.t. the metric ρ as w.r.t. the standard Euclidean metric.

For every integer $n \geq 1$ and every $t \geq 0$ define $p_n(t) = \sum_w ||D\phi_w||^t$ where w runs over all words of length n, and $||.||$ means the sup-norm. Consequently, $P(t) = \lim_{n\to\infty} \frac{1}{n} \log p_n(t)$ is called the pressure of S at t. The parameter $\theta = \theta_S$ of the system is defined as $\inf\{t : p_1(t) < \infty\}$.

Theorem 5 *1. (see [10]. Prop. 3.3) $P(t)$ is non-increasing on $[0, \infty)$, strictly decreasing, continuous and convex on $[\theta, \infty)$.*

2. (see [10], Thm. 3.15) $HD(L) = \sup\{HD(L_F) : F \subset I \ \ is \ \ finite\} = \inf\{t : P(t) \leq 0\}$; if $P(t) = 0$ then $t = HD(L)$.

3. If the series $p_1(\theta)$ diverges, then $P(HD(L)) = 0$ and $\theta < HD(L)$.

(Note that 3 follows directly from 1-2.)

The system with $P(t) = 0$ is called *regular*. The system is regular if and only if there is a *t-conformal measure*, i.e. a probability measure m such that $m(L) = 1$ and for every Borel set $A \subset X$ and every $i \in I$ $m(\phi_i(A)) = \int_A |D\phi_i|^t dm$ and $m(\phi_i(X) \cap \phi_j(X)) = 0$ for all $i \neq j$ from I.

3.2 Dominant convergence and forward Poincaré series

Here we follow [13] adapting the statements sligthly for our applications.

Let $f_n : U \to \mathbf{C}$ be a sequence of holomorphic maps which converges uniformly in a topological disk U of the plane to a holomorphic map $f : U \to \mathbf{C}$. Assume that $c_n \to c \in U$, and the following expansions hold: $f_n(z) = c_n + \lambda_n(z - c_n) + b_n(z - c_n)^2 - a_n(z - c_n)^3 + ...$, where $0 < \lambda_n < 1$, $b_n, a_n \in \mathbf{R}$, and $f(z) = z - a(z - c)^3 + ...$, where $a > 0$, i.e. f is parabolic with two ("real") attracting petals at c. (In particular, $b_n \to 0$ and $a_n \to a$.) Then f_n is said to converge to f *dominantly*, if there is $M > 0$ such that $|b_n| \leq M|\lambda_n - 1|$ for all n.

For every $g = f_n$ and $t > 0$ define the (forward) Poincaré series $P_t(g, x) = \sum_{i\geq 0} |(g^i)'(x)|^t$, and, for any open set $V \subset U$, define $P_t(g, V, x) = \sum_{g^i(x)\in V} |(g^i)'(x)|^t$. We say the Poincare series for (f_n, t_n) converge uniformly, if, for any compact set K ($c \notin K$) in an attracting petal of f, and any $\epsilon > 0$ there exists a neighborhood V of c, such that $P_{t_n}(f_n, V, x) < \epsilon$ for all n large enough and all $x \in K$. We will need

Theorem 6 *Let f_n, f be as above, and $t_n \to t > 2/3$. If $f_n \to f$ dominantly, then the Poincare series for (f_n, t_n) converge uniformly.*

This is a particular case of Theorem 10.2 proven in [13]. For completeness, we give a short proof of Theorem 6, see Appendix.

4 Proof of Theorem 4

4.1 Presentation system for the Cantor attractor

We repeat (with modifications) a construction from [8] (cf. [7], [2]), which is crucial for our proof. Let H be either one of H_ℓ or the limit map H_∞. Consequently, let G be either the corresponding G_ℓ or G_∞. We construct the presentation system for the attractor $C(H)$, which is an infinite iterated function system Π on an interval I so that $C(H) \cap I$ is (up to a countable set) the limit set of Π. Moreover, this picture converges, as $\ell \to \infty$, to the corresponding picture of the limit map.

Denote $c_j = H^{j-1}(0)$, $j \geq 0$, the j-iterate of the critical point c_0 of H (i.e., $c_0 = x_\ell$ for $H = H_\ell$ and $c_0 = x_0$ for $H = H_\infty$). Let $I = [c_p, c_{2p}]$. Then we define a sequence of maps $\psi_{k,m} : I \to I$, $k = 1, 2, ...,$ $m = 1, 2, ..., p-1$, as follows. Let $H^{-(p-m)} : [c_p, c_{2p}] \to [c_m, c_{p+m}]$ denote corresponding one-to-one branch of $H^{-(p-m)}$. Then set

$$\psi_{k,m} = G^k \circ H^{-(p-m)}. \tag{9}$$

Lemma 4.1 *(a)*

$$I_{k,m} := \psi_{k,m}(I) = [c_{p^k m}, c_{p^k(p+m)}] \subset I.$$

The intervals $I_{k,m}$ are pairwise disjoint.

(b) Let L be the limit set of the system $\{\psi_{k,m}\}$ (in other words, L is the set of non-escaping points of the inverse maps $\psi_{k,m}^{-1} : I_{k,m} \to I$). Then the closure $\overline{L} = L \cup P$, where P is a subset of pre-images of the critical point c_0, and

$$\overline{L} = C(H) \cap I.$$

Proof. From the functional equation for H, $G(c_j) = c_{pj}$, $j \in \mathbf{Z}$, where c_j, for $j < 0$ is an H^j-preimage of c_0. The rest follows.

$\square$

Denote by $\Pi_\ell = (\psi_{k,m}^{(\ell)})_{k,m}$, resp. $\Pi_\infty = (\psi_{k,m}^{(\infty)})_{k,m}$, the presentation system of H_ℓ, resp. H_∞.

The notation $B(E)$ stands for the round disk which is based on an interval $E \subset \mathbf{R}$ as a diameter.

Lemma 4.2 *Let $\Pi = \{\psi_{k,m} : I \to I_{k,m}\}_{k,m}$ be either Π_ℓ or Π_∞.*

(1) There exists a fixed open interval J, which contains I for all ℓ largh enough (including $\ell = \infty$), such that each $\psi_{k,m}$ extends to a univalent map $\psi_{k,m} : B(J) \to B(J_{k,m})$, where $J_{k,m} = \psi_{k,m}(J)$ are pairwise disjoint intervals properly contained in J.

Therefore, there is $\lambda < 1$ (dependent only on the type $\aleph$), such that $||D\psi_{k,m}||_\rho < \lambda$, for all k, m, and $\ell \leq \infty$ large enough, where $||D\psi_{k,m}||_\rho$ denotes the supremum on the interval I of the derivative of $\psi_{k,m}$ in the hyperbolic metric ρ of $B(J)$.

(2) Π (with the metric ρ restricted to the closed subinterval I of J) is an infinite conformal iterated function system, such that:

(a) $\theta_{\Pi_\ell} = 0$ for $\ell < \infty$;

(b) $\theta_{\Pi_\infty} = 2/3$, $P(\theta_{\Pi_\infty}) = \infty$;

(c) Π_ℓ, $\ell \leq \infty$, is regular.

Proof. (1) follows from Theorem 3 (7), and from another representation of the maps of the system: $\psi_{k,m} = H^{-1} \circ \tau^{-k} \circ H^{-(p-m-1)}$ which is a consequence of the eq. $H \circ G = \tau^{-1} \circ H$. (2a) is immediate because c_0 is the attracting fixed point of G for finite ℓ.

(2b)-(2c): since $G = G_\infty$ has a neutral fixed point with two attracting petals, and $\psi'_{k,m}(x) = (G^k)'(H^{-(p-m)}(x))(H^{-(p-m)})'(x)$, we obtain the following asymptotics, as $k \to \infty$, for the presentation system: $|\psi'_{k,m}(x)|/k^{-3/2} \to a_m(x)$ where, for fixed $m = 1, ..., p-1$, the function $a_m(x)$ is continuous and positive on I. It follows from here that the critical exponent θ of the system is $\theta = 2/3$. Thus, $p_1(\theta) = \infty$ for all $\ell \leq \infty$. Hence, by Theorem 5, the system $\{\psi_{k,m}\}$ is regular.

□

4.2 Hausdorff dimension for the limit map

As a corollary, we obtain Theorem 4, (a)-(b):

Corollary 4.1 *(1)* $2/3 < HD(C(H_\infty)) < 1$,

(2) $\liminf_{\ell\to\infty} HD(C(H_\ell)) \geq HD(C(H_\infty)) > \frac{2}{3}$.

Proof. Denote $H = H_\infty$. Since H is regular and $P(2/3) = \infty$, then $HD(C(H)) > 2/3$. On the other hand, the Lebesgue measure of $I \setminus \cup_{k,m} I_{k,m}$ is positive. Therefore ([10], Theorem 4.5), $HD(C(H)) = HD(C(H) \cap I) = HD(L) < 1$.

(2) follows from Theorem 5: for every $\delta > 0$, there is a finite subsystem F_∞ of Π_∞ with the Hausdorff dimension of its limit set at least $HD(C(H_\infty)) - \delta$. Since corresponding finite subsystem F_ℓ converges to F_∞ as $\ell \to \infty$, then the Hausdorff dimension of the limit set of F_ℓ is at least $HD(C(H_\infty)) - 2\delta$, for all ℓ large enough. The result follows.

□

4.3 Non-symmetry and dominant convergence

It remains to prove Theorem 4 (c).

Denote $\epsilon = 1$ or 2 depending on whether $G'_\infty(x_0) = 1$ or -1.

Lemma 4.3 *1. The sequence G_ℓ^ϵ converges to G_∞^ϵ dominantly if and only if the sequence of non-symmetries $N(H_\ell)$ is bounded.*

2. If the combinatorics reverses orientation, then G_ℓ^2 converges dominantly to G_∞^2, and the non-symmetries $N(H_\ell)$ are uniformly bounded.

Proof. Let $H = H_\ell$ and $G = G_\ell$, $\tau = \tau_\ell$, and $I = I_H$. We have: $H(G(I(x))) = \tau^{-1}H(I(x)) = \tau^{-1}H(x) = H(G(x))$, i.e. $I \circ G = G \circ I$. The latter equation gives us: $|(G^\epsilon)''(x_\ell)| = N(H)\lambda(1-\lambda)$, where $\lambda = \lambda_\ell = (G^\epsilon)'(x_\ell) \in (0,1)$. This implies 1.

To prove 2, notice that the combinatorics reverses orientation if and only if $G'_\infty(x_0) = -1$. Then we get the dominant convergence, because $|(G^2)''(x_\ell)| = |G''(x_\ell)||\lambda|(1-|\lambda|)$ and $G''(x_\ell) = G''_\ell(x_\ell)$ converges to the number $G''_\infty(x_0)$, as $\ell \to \infty$. (One can also refer formally to [13], Proposition 7.3.)

□

4.4 Conformal measures of the presentation systems

Remind that $\Pi_\ell = (\psi_{k,m}^{(\ell)} : I^\ell \to I_{k,m}^\ell)_{k,m}$, resp. $\Pi_\infty = (\psi_{k,m}^{(\infty)} : I^\infty \to I_{k,m}^\infty)_{k,m}$, the presentation system of H_ℓ, resp. H_∞. We know that Π_ℓ, Π_∞ are regular. Denote by μ_ℓ, resp. μ_∞, the unique probability h_ℓ-conformal, resp. h_∞-conformal, measure of Π_ℓ, resp. Π_∞, where $h_\ell = HD(C(H_\ell) \cap I^\ell) = HD(C(H_\ell))$, $h_\infty = HD(C(H_\infty) \cap I^\infty) = HD(C(H_\infty))$. (Notice that the measures have nothing to do with conformal measures of H_ℓ, H_∞, because the dynamics are completely different.) Since any regular system has a unique conformal measure, to prove that $h_\ell \to h_\infty$, it is enough to prove that a weak limit ν of a subsequence of μ_ℓ is a conformal measure of Π_∞. For this to be true, it is enough to check that the support of ν is contained in the limit set L_∞ of Π_∞. Note that by Lemma 4.1(b), the set $\overline{L}_\infty \setminus L_\infty$ is countable. Therefore, it is enough to prove that ν has no atoms. Thus Theorem 4(c) follows from

Lemma 4.4 *If the non-symmetries $N(H_\ell)$ are uniformly bounded, then the measure ν has no atoms.*

Proof. Let the point $a \in supp(\nu) = \overline{L}_\infty$, where L_∞ is the limit set of Π_∞, be an atom of ν. Then there is $\sigma > 0$ such that for all $r > 0$ small enough $\mu_\ell(B(a,r)) > \sigma$ along a subsequence of ℓ's. Since $\psi_{k,m}$ are uniform contractioncs and the measures are probabilities, one sees that $a \in \overline{L}_\infty \setminus L_\infty$, i.e., afterall, one can assume that $a = x_0$. Now $\mu_\ell(B(x_0,r)) \le \sum_{I_{k,m}^\ell \cap B(x_0,r) \neq \emptyset} \int_{I^\ell} |D\psi_{k,m}^{(\ell)}|^{h_\ell} d\mu_\ell \le C \sum |(G_\ell^k)'(y_{\ell,m})|^{h_\ell}$, for some fixed $C > 0$, some points $y_{\ell,m}$ from a fixed compact set K, $x_0 \notin K$ (if ℓ is big enough), and the latter sum runs over such k that $G_\ell^k(y_{\ell,m}) \in B(x_0, r')$, where $r' \to 0$ as $r \to 0$. Then a contradiction follows directly from Lemma 4.3 and Theorem 6 (note that $t > 2/3$ by Corollary 4.1(2)).

□

5 Appendix: proof of Theorem 6

1. If $h_n \to h$ is a sequence of injective holomorphic maps in a fixed neighborhood of c, which converges to an injective h uniformly, then the Poincaré series for (f_n, t_n) converge uniformly iff the Poincaré series for $(h_n \circ f_n \circ h_n^{-1}, t_n)$ converge uniformly. In particular, one can assume that $c_n = c = 0$.

2. (see Theorem 7.2 of [13]). Let $h_n(z) = z - B_n z^2$, where $B_n = b_n/(\lambda_n(\lambda_n - 1))$. Since $|b_n| \le M|\lambda_n - 1|$ for all n,there is a subsequence of h_n as in Step **1**. On the other hand, $h_n \circ f_n \circ h_n^{-1}(z) = \lambda_n z + O(z^3)$. It means one can assume that $f_n(z) = \lambda_n - a_n z^3 + ...$ where $a_n \to a > 0$, $0 < \lambda_n < 1$ and $\lambda_n \to 1$.

3. For f_n, make a change $z = \hat{h}_n(w) = d_n w^{-1/2}$, where $w \in F = \{w : Re(w) > R_0\}$ and $d_n = (\lambda_n^3/(2a_n))^{1/2}$. For $g_n = \hat{h}_n^{-1} \circ f_n \circ \hat{h}_n$, it holds $g_n(w) = \sigma_n w + 1 + \alpha_n(w)$, where $\sigma_n = \lambda_n^{-2} > 1$ and $\sigma_n \to 1$, α_n converge uniformly in F to the corresponding α for $g = \hat{h}^{-1} \circ f \circ \hat{h}$, $\hat{h} = \lim \hat{h}_n$, and $\alpha_n(w) = O(|w|^{-1/2})$, $\alpha(w) = O(|w|^{-1/2})$.

To deal with g_n^i, we prove the following simple Claim. This is weaker than Theorems 8.1-8.3 of [13], but still enough for our needs.

Claim 1: *For every $\delta > 0$ there is $R_\delta > R_0$ and, for every n, there is $1+\delta$-quasiconformal map ϕ_n of the plane that fixes $0, 1$, and ∞, such that $\phi_n^{-1} \circ g_n \circ \phi_n = T_n$, where $T_n(w) = \sigma_n w + 1$, for $Re(w) > R_\delta$. Passing to a subsequence, one can assume that $\phi_n \to \phi$, so that $\phi^{-1} \circ g \circ \phi = T$, $T(w) = w + 1$.*

Proof. Fix $\delta > 0$. Denote $\Pi(R_1, R_2) = \{w : R_1 < Re(w) < R_2\}$. Then $|\alpha_n(w)|$ and $|\alpha_n'(w)| \le \sup\{|\alpha_n(t)| : |t - w| < 1\}$ are uniformly arbitrary small as $w \in L := \{\Re(w) = R_\delta\}$ and $R_\delta \to \infty$. Therefore, all $\sigma_n w$ can be joined to $z(w) := \sigma_n w + 1 + \alpha_n(w)$ by disjoint intervals $I(w)$ in the strip between $\sigma_n L$ and $z(L)$. The mapping ϕ_n, which is affine on each interval $[\sigma_n w, \sigma_n w + 1]$ onto $I(w)$ together with the identity on $\Pi(R_\delta, \sigma_n R_\delta)$, is $1 + \delta$ quasi-conformal on $\Pi(R_\delta, \sigma_n R_\delta + 1)$. Then we extend ϕ_n to $Re(w) > \sigma_n R_\delta + 1$ by the (conformal) dynamics of g_n, T_n, and define it identity on the rest of the plane.

Claim 2. *For every real $p > 1$, there is M such that $\frac{|(T_n^i)'(w)|}{|T_n^i(w)|^p} \le M i^{-p}$ for all i, n, and all $w > 1$.*

Indeed, denote $C(i, n) = \sigma_n^i$. Consider any subsequence $(i_j, n_j), j \to \infty$. If $C(i, n)$ is bounded from above along this subsequence, then applying as in [12], Sect.6, the inequality between arithmetic and geometric means, we can write

$T_n^i(w) = \sigma_n^i w + (1 + \sigma_n + ... + \sigma_n^{i-1}) \geq (i+1) w^{1/(i+1)} \sigma_n^{i/2} \geq C(i,n)^{1/2} i$, so that $\frac{|(T_n^i)'(w)|}{|T_n^i(w)|^p} \leq C(i,n)^{1-p/2} i^{-p} = O(i^{-p})$ along the subsequence. If now $C(i,n) \to \infty$ along (i_j, n_j) (and $\sigma_{n_j} \to 1$), then $\frac{|(T_n^i)'(w)|}{|T_n^i(w)|^p} = \frac{|\sigma_n^i|}{|\sigma_n^i w + (\sigma_n^i - 1)/(\sigma_n - 1)|^p} \sim C(i,n)|\sigma_n - 1|^p / C(i,n)^p \sim (\log C(i,n))^p / C(i,n)^{p-1} i^{-p} = o(i^{-p})$.

4. From Steps 1-2, Claim 1, and Koebe distortion theorem, it follows that it is enough to prove the theorem assuming that the compact K is a point x, which moreover lies on an attracting direction of f, and small neighborhood V can be replaced by big indexes. We have: $|(f_n^i)'(x)| = K|(g_n^i)'(w)|/|g_n^i(w)|^{3/2}$, where $K > 0$ and $w > R$ depend only on $x > 0$. Thus we need to show that, if $t_n \to t > 2/3$, for a given $w > 0$ close enough to $+\infty$, for any $\epsilon > 0$ there exists an index i_0, such that $S(g_n, i_0, t_n) := \sum_{i \geq i_0} |(g_n^i)'(w)/g_n^i(w)^{3/2}|^{t_n} < \epsilon$ for all n large enough. Claim 2 (with $p = 3/2$) implies immediately that this is true for $g_n = T_n$.

To handle $S(g_n, i_0, t_n)$ in general, we compare it with $S(T_n, i_0, t_n)$ and proceed similar to [13], Sect.10. Due to Koebe distortion theorem, one can replace the derivative by the ratio of diameters. By Claim 1, the change of the diameters when passing from g_n to T_n is Hölder with the exponent arbitrary close to 1. Then we apply Claim 2 with p arbitrary close to 3/2.

References

[1] Bruin H., Keller, G., Nowicki, T. & Van Strien, S.: *Wild Cantor attractors exist*, Ann. Math. **143**, 97-130 (1996)

[2] Collet, P. & Eckmann, J.-P.: *Iterated maps on the interval as dynamical systems*, Progress in Physics, Birkhauser Boston (1980)

[3] Feigenbaum, M.: *Qualitative universality for a class of non-linear transformations*, J. Stat. Phys. **19** (1978), 25-52

[4] Feigenbaum, M.: *The universal metric properties of non-linear transformations*, J. Stat. Phys. **21** (1979), 669-706

[5] Graczyk, J. & Kozlovski, O. S.: *Global universality in smooth unimodal maps,* Warwick preprint 05/2002, February 2002

[6] Grassberger, P.: *On the Hausdorff dimension of fractal attractors*, J.Statist.Phys. 26 (1981), no.1, 173-179

[7] Ledrappier, F. & Misiurewicz, M.: *Dimension of invariant measures for maps with exponent zero*, Ergodic Theory Dynamical Systems, 5 (1985), 595-610

[8] Levin, G. & Świątek, G. *Dynamics and universality of unimodal mappings with infinite criticality*, matharxiv 0306033, 2003

[9] De Melo, W & van Strien, S.: *One-dimensional dynamics.* Springer-Verlag, New-York, 1993

[10] Mauldin, D. & Urbanski, M.: *Dimensions and measures in infinite iterated function systems*, Proc. London Math. Society, v.73 (1996), 105-154

[11] Mc Mullen, C.: *Renormalization and 3-manifolds which fiber over the circle*, Ann. of Math. Studies **142**, Princeton University Press (1998)

[12] Mc Mullen, C.: *Hausdorff dimension and conformal dynamics I: Kleinian groups and strong limits*, J. Diff. Geom. 51 (1999), 471-515

[13] Mc Mullen, C.: *Hausdorff dimension and conformal dynamics II: Geometrically finite rational maps*, Comment. Math. Helv. 75 (2000), 535-593

[14] Sullivan, D.: *Bounds, quadratic differentials and renormalization conjectures*, in: *Mathematics into the Twenty-First Century*, AMS Centennial Publications (1991)

[15] Van Strien, S. & Nowicki, T.: *Polynomial maps with a Julia set of positive Lebesgue measure: Fibonacci maps*, manuscript (1994)

[16] Vul, E. B., Sinai, Ya. G. & Khanin, K. M.: *Feigenbaum universality and thermodynamical formalism*, Russ.Math.Survey, 39 (1984), 3, 1-40