Higher Lagrangians and strings

Janusz Grabowski

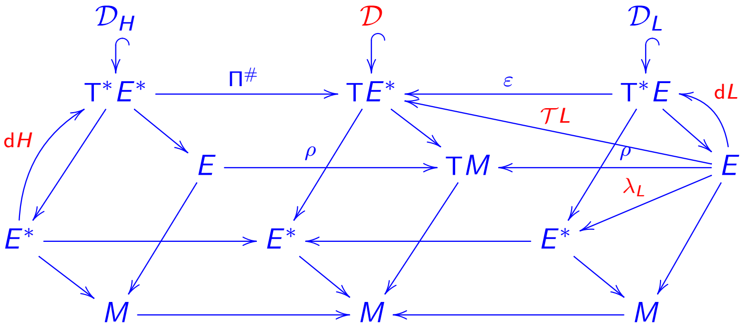
(Polish Academy of Sciences)

May 29, 2021

Plan of the talk

- Tulczyjew triples for algebroids
- Vakonomic constraints
- Higher-order Lagrangians
- Higher-order Euler-Lagrange equations
- Example: Higher Euler equations
- Higher order Lagrangian mechanics on Lie algebroids
- Dynamics of strings
- Tulczyjew triples for strings
- Lagrangian and Hamiltonian formalism for strings
- Example: Plateau problem
- References

Algebroid setting



$H : E^* \longrightarrow \mathbb{R}$ $\qquad$ $\mathcal{D} = \mathcal{T}L(E)$ $\qquad$ $L : E \longrightarrow \mathbb{R}$

$\mathcal{D}_H \subset \mathsf{T}^*E^*$ $\qquad$ $\mathcal{D} = \Pi^\#(\mathrm{d}H(E^*))$ $\qquad$ $\mathcal{D}_L \subset \mathsf{T}^*E$

The Euler-Lagrange equations read $\mathcal{T}L \circ \gamma = \mathrm{t}(\lambda_L \circ \gamma)$.

Euler-Lagrange equations for algebroids

If (q^a) are local coordinates in M,
(y^i) i (ξ_i) are linear coordinates in fibers of, respectively, E and E^*, and

$$\Pi = c_{ij}^k(q)\xi_k\partial_{\xi_i}\otimes\partial_{\xi_j} + \rho_i^b(q)\partial_{\xi_i}\otimes\partial_{q^b} - \sigma_j^a(q)\partial_{q^a}\otimes\partial_{\xi_j}\,,$$

then the Euler-Lagrange equations read

$$\begin{aligned} (1) \qquad \frac{\mathrm{d}q^a}{\mathrm{d}t} &= \rho_k^a(q)y^k\,, \\ (2) \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^j}\right)(q,y) &= c_{ij}^k(q)y^i\frac{\partial L}{\partial y^k}(q,y) + \sigma_j^a(q)\frac{\partial L}{\partial q^a}(q,y)\,. \end{aligned}$$

They are first-order differential equations (!) but for admissible curves in E, i.e. for curves satisfying (1). For $E = \mathsf{T}M$, they are exactly the tangent prolongations of curves in M, for which the equation is second-order.

Euler-Poincaré equations

A particular example of the equation (2) is not only the classical Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^a}(q,\dot{q}) = \frac{\partial L}{\partial q^a}(q,\dot{q})\,.$$

but also the Lagrange-Poincaré equation for G-invariant Lagrangians on principal G-bundle

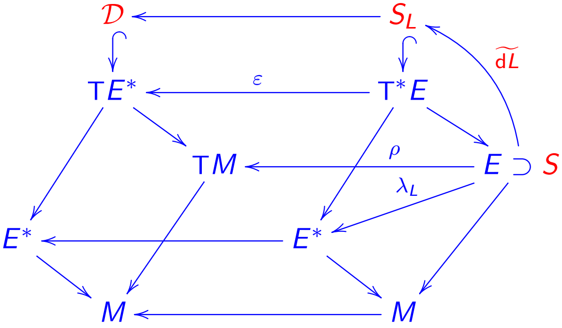
$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a}\right)(q,\dot{q},v) - \left(B^k_{ba}(q)\dot{q}^b + D^k_{ia}(q)v^i\right)\frac{\partial L}{\partial v^k}(q,\dot{q},v) = 0\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^j}(q,\dot{q},v) - \left(D^k_{aj}(q)\dot{q}^a + C^k_{ij}v^i\right)\frac{\partial L}{\partial v^k}(q,\dot{q},v) = 0\,,$$

and the Euler-Poincaré equations, for instance the rigid body equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^j}(v) - C^k_{ij}v^i\frac{\partial L}{\partial v^k}(v) = 0\,.$$

Algebroid setting with vakonomic constraints



where S_L is the lagrangian submanifold in T^*E induced by the Lagrangian on the constraint S, and $\widetilde{\mathrm{d}L} : S \to \mathsf{T}^*E$ is the corresponding relation,

$$S_L = \{\alpha_e \in \mathsf{T}^*_e E : e \in S \text{ and } \langle \alpha_e, v_e \rangle = \mathrm{d}L(v_e) \text{ for every } v_e \in \mathsf{T}_e S\}.$$

The vakonomically constrained phase dynamics is just $\mathcal{D} = \varepsilon(S_L) \subset \mathsf{T}E^*$.

Vakonomic E–L equations in coordinates

- Suppose that the vakonomic constraint S is defined as the zero-set of functions Φ^k.
- Then, for a Lagrangian $L(x,y)$ on E, we have

$$S_L = \left\{ \left(x, y, \frac{\partial L}{\partial x}(x,y), \frac{\partial L}{\partial y}(x,y) - \mu_k(x,y)\frac{\partial \Phi^k}{\partial y}(x,y) \right) \mid \Phi^k(x,y) = 0 \right\}$$

where $\mu_k \in C^\infty(S)$ are 'Lagrange multipliers'.

- Looking for curves in S_L which are mapped by $\varepsilon : \mathsf{T}^*E \to \mathsf{T}E^*$,

$$\varepsilon(x^a, y^i, p_b, \xi_j) = (x^a, \xi_i, \rho^b_k(x)y^k, c^k_{ij}(x)y^i\xi_k + \sigma^a_j(x)p_a)\,,$$

into admissible curves, we get the vakonomic E-L equations

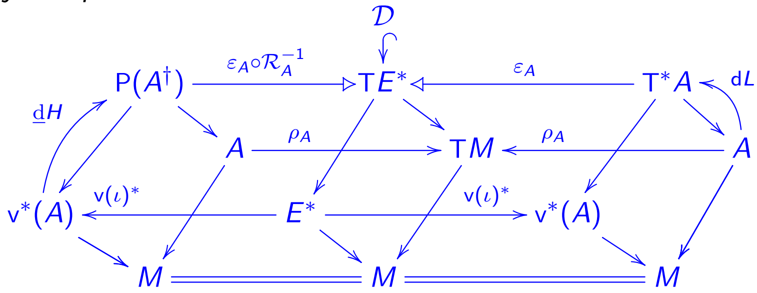
$$\Phi^k(x,y) = 0\,, \quad \frac{\mathrm{d}x^a}{\mathrm{d}t} = \rho^a_k(x)y^k\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial y^j}(x,y,t) - c^l_{ij}(x)y^i\frac{\partial L}{\partial y^l}(x,y,t) - \sigma^a_j(x)\frac{\partial L}{\partial x^a}(x,y,t) =$$

$$\dot{\mu}_k(t)\frac{\partial \Phi^k}{\partial y^j}(x,y) + \mu_k(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \Phi^k}{\partial y^j}(x,y) - c^l_{ij}(x)y^i\frac{\partial \Phi^k}{\partial y^l}(x,y) - \sigma^a_j(x)\frac{\partial \Phi^k}{\partial x^a}(x,y)\right)$$

Affine vakonomic constraints

In the case when $S = A$ is an affine subbundle of an algebroid E (assume for simplicity that A is supported on the whole M), we get the *reduced Tulczyjew triple* for an affine vakonomic constraint:



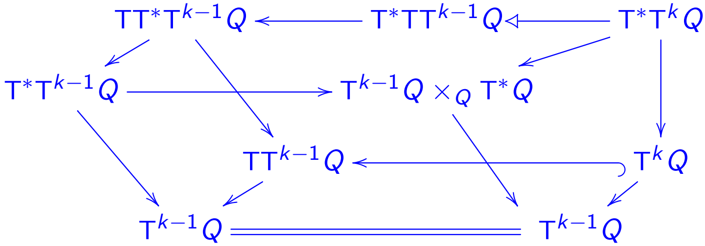
Here, $A^\dagger$ is the affine dual bundle, i.e. the bundle of affine functions on fibers of A, and Hamiltonians are sections of the so called affine phase bundle $\mathsf{P}(A^\dagger)$ over $\mathsf{v}^*(A)$ – the dual of the linear model $\mathsf{v}(A)$ of A.

Higher order Lagrangians

The mechanics with a higher order Lagrangian $L : \mathsf{T}^k Q \to \mathbb{R}$ is traditionally constructed as a vakonomic mechanics, thanks to the canonical embedding of of the higher tangent bundle $\mathsf{T}^k Q$ into the tangent bundle $\mathsf{T}\mathsf{T}^{k-1} Q$ as an affine subbundle of holonomic vectors:

$$\left(q, \dot{q}, \ddot{q}, \dots, \overset{(k-1)}{q}, \overset{(k)}{q}\right) \mapsto \left(q, \dot{q}, \ddot{q}, \dots, \overset{(k-1)}{q}, \dot{q}, \ddot{q}, \dots, \overset{(k-1)}{q}, \overset{(k)}{q}\right) .$$

Thus we work with the standard Tulczyjew triple for $\mathsf{T}M$, where $M = \mathsf{T}^{k-1} Q$, with the presence of vakonomic constraint $\mathsf{T}^k Q \subset \mathsf{T}\mathsf{T}^{k-1} Q$:



Higher order Euler-Lagrange equations

The Lagrangian function $L = L(q, \dot{q}, \dots, \overset{(k)}{q})$ generates the phase dynamics

$$\mathcal{D} = \left\{ (v, p, \dot{v}, \dot{p}) : \ \dot{v}_{i-1} = v_i, \ \ \dot{p}_i + p_{i-1} = \frac{\partial L}{\partial \overset{(i)}{q}}, \dot{p}_0 = \frac{\partial L}{\partial q}, p_{k-1} = \frac{\partial L}{\partial \overset{(k)}{q}} \right\}.$$

This leads to the higher Euler-Lagrange equations in the traditional form:

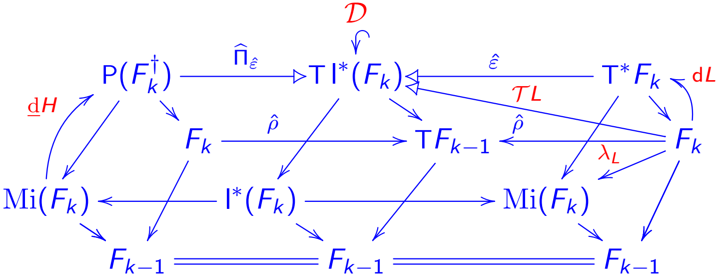
$$\overset{(i)}{q} = \frac{\mathrm{d}^i q}{\mathrm{d}t^i}, \ i = 1, \dots, k,$$

$$0 = \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \cdots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{\partial L}{\partial \overset{(k)}{q}}\right).$$

These equations can be viewed as a system of ordinary differential equations of order k on $\mathsf{T}^k Q$ or, which is the standard point of view, as an ordinary differential equation of order $2k$ on Q.

Lagrangian framework for graded bundles

A weighted Lie algebroid on $\mathsf{I}(F_k)$ gives the Tulczyjew triple



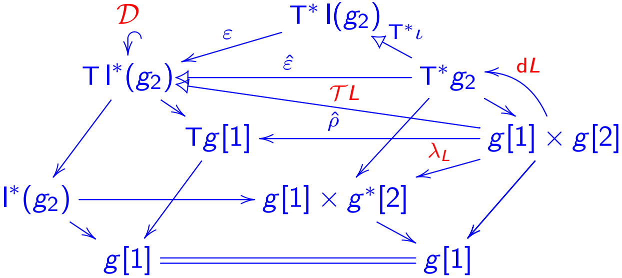
Here, the diagram consists of relations, $\hat\varepsilon : \mathsf{T}^*F_k \longrightarrow\!\!\triangleright\, \mathsf{T}^*\,\mathsf{I}(F_k) \to \mathsf{T}\,\mathsf{I}^*(F_k)$, and $\mathrm{Mi}(F_k) = F_{k-1} \times_M \bar{F}_k$ is the so called Mironian of F_k. In the classical case, $\mathrm{Mi}(\mathsf{T}^k M) = \mathsf{T}^{k-1}M \times_M \mathsf{T}^*M$.

$\mathcal{T}L$ is the Tulczyjew differential and λ_L the Legendre relation.

The fact that we obtain the Euler-Lagrange equations of higher order comes from the vakonomic constraint and the additional gradation.

Example

Let g be a Lie algebra and put $F_2 = g_2 = g[1] \times g[2]$, with coordinates (x^i, z^j) on g_2 and coordinates (x^i, y^j, z^k) on $\mathsf{l}(g_2) = g[1] \times g[1] \times g[2]$. The vector bundle projection is $\tau(x, y, z) = x$ and the corresponding diagram looks like



The embedding $\iota : g_2 \hookrightarrow \mathsf{l}(g_2)$ takes the form $\iota(x, z) = (x, x, z)$. In coordinates $(x, y, z, \alpha, \beta, \gamma)$ on $\mathsf{T}^*\,\mathsf{l}(g_2)$, the phase relation $\mathsf{T}^*\iota : \mathsf{T}^*g_2 \longrightarrow\!\!\!\!\rhd \mathsf{T}^*\,\mathsf{l}(g_2)$ relates $(x, z, \alpha + \beta, \gamma)$ with $(x, x, z, \alpha, \beta, \gamma)$.

Example continued

The Lie algebroid structure $\varepsilon : \mathsf{T}^* \mathsf{l}(g_2) \longrightarrow\!\!\triangleright \mathsf{T}\, \mathsf{l}^*(g_2)$ reads

$$(x, y, z, \alpha, \beta, \gamma) \mapsto (x, \beta, \gamma, z, \mathrm{ad}^*_y \beta, \alpha)\,,$$

so $\hat{\varepsilon}$ relates $(x, z, \alpha + \beta, \gamma)$ with $(x, \beta, \gamma, z, \mathrm{ad}^*_x \beta, \alpha)$.
Given a Lagrangian $L : g_2 \to \mathbb{R}$, the Tulczyjew differential relation $\mathcal{T}L : g_2 \to \mathsf{T}\, \mathsf{l}^*(g_2)$ therefore reads

$$\mathcal{T}L(x, z) = \left\{ \left(x, \beta, \frac{\partial L}{\partial z}(x, z), z, \mathrm{ad}^*_x \beta, \alpha \right) : \alpha + \beta = \frac{\partial L}{\partial x}(x, z) \right\}.$$

Hence, for the phase dynamics,

$$z = \dot{x}\,, \quad \mathrm{ad}^*_x \beta = \dot{\beta}\,, \quad \alpha = \frac{\mathsf{d}}{\mathsf{d}t} \left(\frac{\partial L}{\partial z}(x, z) \right),$$

and

$$\beta = \frac{\partial L}{\partial x}(x, z) - \frac{\mathsf{d}}{\mathsf{d}t} \left(\frac{\partial L}{\partial z}(x, z) \right).$$

Higher Euler equations

This leads to the Euler-Lagrange equations on g_2:

$$\begin{aligned} \dot{x} &= z\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,z)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right) &= \mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,z)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right)\,. \end{aligned}$$

These equations are second order and induce the Euler-Lagrange equations on g which are of order 3:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,\dot{x})-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right)=\mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,\dot{x})-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right)\,.$$

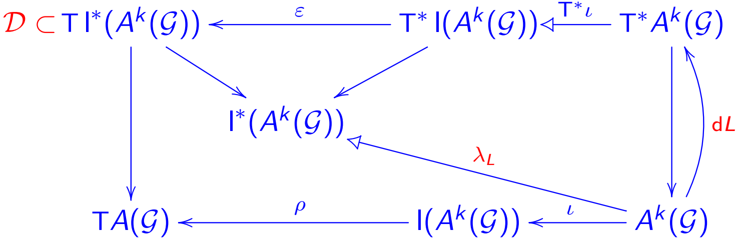
For instance, the 'free' Lagrangian $L(x,z)=\frac{1}{2}\sum_i I_i(z^i)^2$ induces the equations on g (c^k_{ij} are structure constants, no summation convention):

$$I_j\dddot{x}^j=\sum_{i,k}c^k_{ij}I_k x^i\ddot{x}^k\,.$$

The latter can be viewed as 'higher Euler equations'.

Higher order Lagrangian mechanics on Lie algebroids

Let us consider a general Lie groupoid $\mathcal{G}$ and a Lagrangian $L : A^k \to \mathbb{R}$ on $A^k = A^k(\mathcal{G})$. We will refer to such systems as a k-th order Lagrangian system on the Lie algebroid $A(\mathcal{G})$. The relevant diagram here is



Here, $\mathsf{I}(A^k(\mathcal{G}))$ is the corresponding Lie algebroid prolongation, $\mathcal{D} = \varepsilon \circ r \circ \mathrm{d}L(A^k(\mathcal{G}))$, and λ_L is the Legendre relation.

Note that we deal with reductions: in the case $\mathcal{G}$ is a Lie group,

$$A^k(\mathcal{G}) = \mathsf{T}^k(\mathcal{G})/\mathcal{G} \quad \text{and} \quad \mathsf{I}(A^k(\mathcal{G})) = \mathsf{T}\mathsf{T}^{k-1}(\mathcal{G})/\mathcal{G}\,.$$

Higher order Lagrangian mechanics on Lie algebroids

For instance, using x^A as base coordinates, and y^a_i as fibre coordinates of degree $i = 1, \ldots, k$ in A^k, extended by the appropriate momenta π^j_b of degree $j = 1, \ldots, k$ in $\mathsf{I}^*(A^k)$, we get the equations for the Legendre relation in the form (no Lie algebroid structure appears!):

$$
\begin{aligned}
&k\pi^1_a = \frac{\partial L}{\partial y^a_k},\\
&(k-1)\pi^2_b = \frac{\partial L}{\partial y^b_{k-1}} - \frac{1}{k}\frac{d}{dt}\left(\frac{\partial L}{\partial y^b_k}\right),\\
&\vdots\\
&\pi^k_d = \frac{\partial L}{\partial y^d_1} - \frac{1}{2!}\frac{d}{dt}\left(\frac{\partial L}{\partial y^d_2}\right) + \frac{1}{3!}\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial y^d_3}\right) - \cdots\\
&+(-1)^k\frac{1}{(k-1)!}\frac{d^{k-2}}{dt^{k-2}}\left(\frac{\partial L}{\partial y^d_{k-1}}\right) - (-1)^k\frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial y^d_k}\right),
\end{aligned}
$$

which we recognize as the Jacobi–Ostrogradski momenta.

Higher order Lagrangian mechanics on Lie algebroids

The remaining equation for the dynamics is

$$\frac{d}{dt}\pi^k_a = \rho^A_a(x)\frac{\partial L}{\partial x^A} + y^b_1 C^c_{ba}(x)\pi^k_c\,,$$

where ρ^A_a and C^c_{ba} are structure functions of the Lie algebroid $A = A(\mathcal{G})$. The above equation can then be rewritten as

$$\rho^A_a(x)\tfrac{\partial L}{\partial x^A} = \left(\delta^c_a \tfrac{d}{dt} - y^b_1 C^c_{ba}(x)\right)\left(\tfrac{\partial L}{\partial y^c_1} - \tfrac{1}{2!}\tfrac{d}{dt}\left(\tfrac{\partial L}{\partial y^c_2}\right)\cdots - (-1)^k \tfrac{1}{k!}\tfrac{d^{k-1}}{dt^{k-1}}\left(\tfrac{\partial L}{\partial y^c_k}\right)\right)$$

which we define to be the k-th order Euler–Lagrange equations on $A(\mathcal{G})$.

The above higher order algebroid Euler-Lagrange equations are in complete agrement with the ones obtained by Jóźwikowski & Rotkiewicz, Colombo & de Diego, as well as Martínez. We clearly recover the standard higher Euler–Lagrange equations on $\mathsf{T}^k M$ as a particular example.

The tip of a javelin

For instance, let L be the Lagrangian, governing the motion of the tip of a javelin defined on $\mathsf{T}^2\mathbb{R}^3 \simeq \mathbb{R}^3 \times \mathbb{R}^3[1] \times \mathbb{R}^3[2]$,

$$L(x, y, z) = \frac{1}{2}\left(\sum_{i=1}^{3}(y^i)^2 - (z^i)^2\right).$$

We can understand $G = \mathbb{R}^3$ here as a commutative Lie group, and since L is G-invariant, we get immediately the reduction to the graded bundle $\mathbb{R}^3[1] \times \mathbb{R}^3[2]$. The Euler-Lagrange equations on $\mathsf{T}^2\mathbb{R}^3$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^i} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z^i}\right)\right) = 0\,,$$

give in this case

$$\frac{\mathrm{d}y^i}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}^2 z^i}{\mathrm{d}t^2}\,,$$

so the Euler-Lagrange equation on $\mathbb{R}^3$ ($y = \dot{x}$, $z = \ddot{x}$) reads

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \frac{1}{2}\frac{\mathrm{d}^4 x^i}{\mathrm{d}t^4}\,.$$

Dynamics of strings

- An evolution of strings is represented by surfaces in M. Passing to infinitesimal parts we will view a Lagrangian L as a function
$$L : \wedge^2 \mathrm{T}M \to \mathbb{R}\,.$$
If L is positive homogeneous, the action functional does not depend on the parametrization of the submanifold and the corresponding Hamiltonian (if it exists) is a function on the dual vector bundle $\wedge^2\mathrm{T}^*M$ (the phase space).
- The dynamics should be an equation (in general, implicit) for 2-dimensional submanifolds in the phase space, i.e.
$$\mathcal{D} \subset \wedge^2\mathrm{T}\wedge^2\mathrm{T}^*M\,.$$
- A submanifold S in the phase space $\wedge^2\mathrm{T}^*M$ is a solution of $\mathcal{D}$ if and only if its tangent space $\mathrm{T}_\alpha S$ at $\alpha \in \wedge^2\mathrm{T}^*M$ is represented by a bivector from $\mathcal{D}_\alpha$.
If we use a parametrization, then the tangent bivectors associated with this parametrization must belong to $\mathcal{D}$.

The Hamiltonian side for multivector bundles

Recall that $\wedge^2\mathrm{T}\wedge^2\mathrm{T}^*M$ is a double graded bundle (actually a GrL-bundle)

$$\begin{array}{ccc} & \wedge^2\mathrm{T}\wedge^2\mathrm{T}^*M & \\ \swarrow & & \searrow \\ \wedge^2\mathrm{T}^*M & & \wedge^2\mathrm{T}M \\ \searrow & & \swarrow \\ & M & \end{array} \quad .$$

We have:

- the canonical Liouville 2-form on $\wedge^2\mathrm{T}^*M$:

$$\theta^2_M = \frac{1}{2}p_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\,,\ \ p_{\mu\nu} = -p_{\nu\mu}\,;$$

- the canonical multisymplectic form

$$\omega^2_M = \mathrm{d}\theta^2_M = \frac{1}{2}\,\mathrm{d}p_{\mu\nu}\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\,;$$

- the vector bundle morphism

$$\beta^2_M\colon\ \wedge^2\mathrm{T}\wedge^2\mathrm{T}^*M\to\mathrm{T}^*\wedge^2\mathrm{T}^*M\,,\quad :u\mapsto i_u\omega^2_M\,.$$

The Lagrangian side for multivector bundles

In local coordinates,

$$\beta^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma,\delta,\epsilon,\zeta}) = (x^\mu, p_{\lambda\kappa}, -y^\eta_{\eta\rho}, \dot{x}^{\nu\sigma})\,.$$

Using the canonical isomorphism of double vector bundles

$$\mathcal{R} : \mathsf{T}^* \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M\,,$$

we can define $\alpha^2_M = \mathcal{R} \circ \beta^2_M$, which is another double graded bundle morphism,

$$\alpha^2_M\colon\ \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M\,,$$

(of double graded bundles over $\wedge^2\mathsf{T}M$ and $\wedge^2\mathsf{T}^*M$).

In local coordinates,

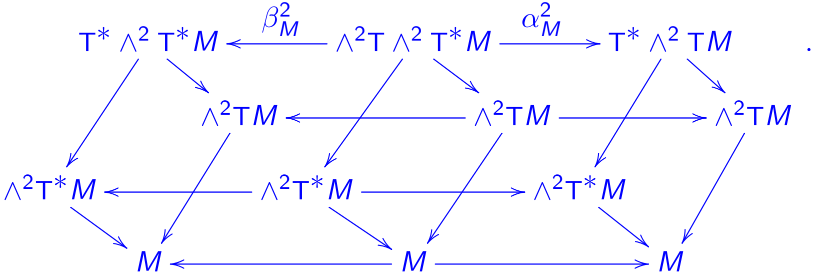
$$\alpha^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) = (x^\mu, \dot{x}^{\nu\sigma}, y^\eta_{\eta\rho}, p_{\lambda\kappa})\,.$$

The map α^2_M can also be obtained as the dual of the canonical isomorphism

$$\kappa^2_M : \mathsf{T} \wedge^2 \mathsf{T} M \to \wedge^2\mathsf{T}\mathsf{T} M\,.$$

The Tulczyjew triple for strings

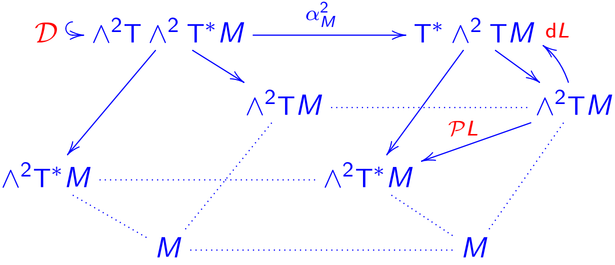
Combining the maps β^2_M and α^2_M, we get the following Tulczyjew triple for multivector bundles, consisting of double graded bundle morphisms:



The way of obtaining the implicit phase dynamics D, as a submanifold of $\wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M$, from a Lagrangian $L:\wedge^2\mathsf{T}M\to\mathbb{R}$ or from a Hamiltonian $H:\wedge^2\mathsf{T}^*M\to\mathbb{R}$ is now standard.

The phase dynamics - Lagrangian side

$\wedge^2\mathsf{T}M$ - (kinematic) configurations, $L:\wedge^2\mathsf{T}M\to\mathbb{R}$ - Lagrangian



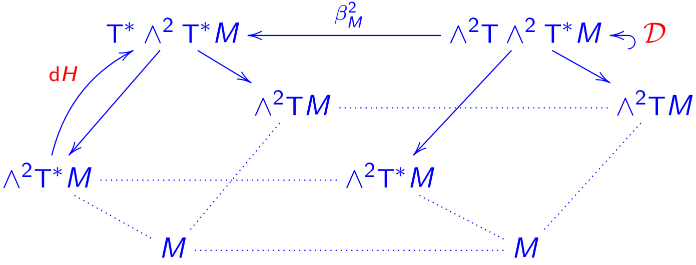
$$\mathcal{D}=(\alpha^2_M)^{-1}(\mathrm{d}L(\wedge^2\mathsf{T}M))$$

$$\mathcal{D}=\left\{(x^\mu,p_{\lambda\kappa},\dot{x}^{\nu\sigma},y^\eta_{\theta\rho},\dot{p}_{\gamma\delta\epsilon\zeta}):\quad y^\eta_{\eta\rho}=\frac{\partial L}{\partial x^\rho},\quad p_{\lambda\kappa}=\frac{\partial L}{\partial \dot{x}^{\lambda\kappa}}\right\}.$$

Thus we get Lagrange (phase) equations.

The phase dynamics - Hamiltonian side

$H : \wedge^2 \mathsf{T}^* M \to \mathbb{R}$



$$\mathcal{D} = (\beta^2_M)^{-1}(\mathrm{d}H(\wedge^2 \mathsf{T}^* M))$$

$$\mathcal{D} = \left\{ (x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) : \quad y^\eta_{\eta\rho} = -\frac{\partial H}{\partial x^\rho}, \quad \dot{x}^{\nu\sigma} = \frac{\partial H}{\partial p_{\nu\sigma}} \right\} .$$

Thus we get Hamilton equations.

The Euler-Lagrange and Hamilton equations

For a surface in $\wedge^2 \mathsf{T}M$,

$$(t,s) \mapsto \left(x^\sigma(t,s)\,,\dot{x}^{\mu\nu}(s,t)\right),$$

the Euler-Lagrange equations read

$$\begin{aligned}
\dot{x}^{\mu\nu} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}\,,\\
\frac{\partial L}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial}{\partial s}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right) - \frac{\partial x^\mu}{\partial s}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right).
\end{aligned}$$

As for the Hamilton equations, we have

$$\begin{aligned}
\frac{\partial H}{\partial p_{\mu\nu}} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}\,,\\
-\frac{\partial H}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial p_{\mu\sigma}}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial p_{\mu\sigma}}{\partial t}\,.
\end{aligned}$$

An example

In the relativistic dynamics of strings, the manifold of infinitesimal configurations is $\wedge^2 \mathsf{T}M$, where M is the space time with the Lorentz metric g. This metric induces a scalar product h in fibers of $\wedge^2 \mathsf{T}M$: for

$$w = \frac{1}{2}\dot{x}^{\mu\nu}\frac{\partial}{\partial x^\mu}\wedge\frac{\partial}{\partial x^\nu}, \quad u = \frac{1}{2}\dot{x}'^{\mu\nu}\frac{\partial}{\partial x^\mu}\wedge\frac{\partial}{\partial x^\nu}$$

we have

$$(u|w) = h_{\mu\nu\kappa\lambda}\dot{x}^{\mu\nu}\dot{x}'^{\kappa\lambda},$$

where

$$h_{\mu\nu\kappa\lambda} = g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}.$$

The Lagrangian is a function of the volume with respect to this metric, the so called Nambu-Goto Lagrangian,

$$L(w) = \sqrt{(w|w)} = \sqrt{h_{\mu\nu\kappa\lambda}\dot{x}^{\mu\nu}\dot{x}^{\kappa\lambda}},$$

which is defined on the open submanifold of positive bivectors.

Nambu-Goto dynamics

The dynamics $\mathcal{D} \subset \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M$ is the inverse image by α^2_M of the image $\mathrm{d}L(\wedge^2 \mathsf{T} M)$ and it is described by the Lagrange (phase) equations

$$\begin{aligned} y^{\alpha}_{\alpha\nu} &= \frac{1}{2\rho} \frac{\partial h_{\mu\kappa\lambda\sigma}}{\partial x^{\nu}} \dot{x}^{\mu\kappa} \dot{x}^{\lambda\sigma}, \\ p_{\mu\nu} &= \frac{1}{\rho} h_{\mu\nu\lambda\kappa} \dot{x}^{\lambda\kappa}, \end{aligned}$$

where

$$\rho = \sqrt{h_{\mu\nu\lambda\kappa} \dot{x}^{\mu\nu} \dot{x}^{\lambda\kappa}}\,.$$

The dynamics $\mathcal{D}$ is also the inverse image by β^2_M of the lagrangian submanifold in $\mathsf{T}^* \wedge^2 \mathsf{T}^* M$, generated by the Morse family

$$\begin{aligned} H \quad &: \wedge^2 \mathsf{T}^* M \times \mathbb{R}_+ \to \mathbb{R}\,, \\ &: (p, r) \mapsto r\left(\sqrt{(p|p)} - 1\right). \end{aligned}$$

In the case of minimal surface, i.e. the Plateau problem, we replace the Lorentz metric with a positively defined one.

Plateau problem

In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the Lagrangian reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} (\dot{x}^{\kappa\lambda})^2}\,.$$

The Euler-Lagrange equation for surfaces, being graphs of maps $(x, y) \mapsto (x, y, z(x, y))$, provides the well-known equation for minimal surfaces, found already by Lagrange :

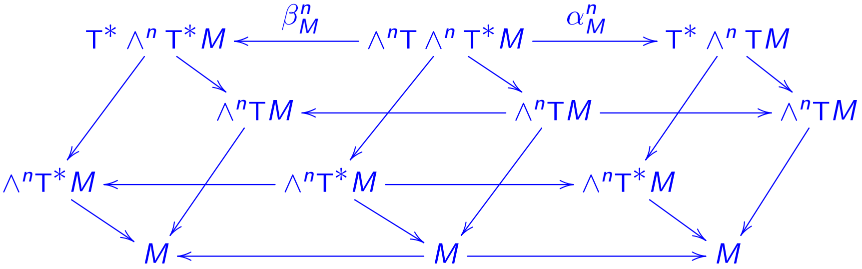
$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = 0\,.$$

In another form:

$$(1 + z_x^2) z_{yy} - 2 z_x z_y z_{xy} + (1 + z_y^2) z_{xx} = 0\,.$$

A generalization

We have a straightforward generalization for all integer $n \geq 1$ replacing 2:



The map

$$\beta^n_M \colon \wedge^n \mathrm{T} \wedge^n \mathrm{T}^* M \to \mathrm{T}^* \wedge^n \mathrm{T}^* M$$

comes from the canonical multisymplectic $(n+1)$-form ω^n_M on $\wedge^n \mathrm{T}^* M$, being the differential of the canonical Liouville n-form

$$\theta^n_M = p_{\mu_1 \mu_2 \dots \mu_n} \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \cdots \wedge \mathrm{d}x^n \, .$$

The map α^n_M is just the composition of β^n_M with the canonical isomorphism of double vector bundles $\mathrm{T}^* \wedge^n \mathrm{T}^* M$ and $\mathrm{T}^* \wedge^n \mathrm{T} M$.

References

- **A.J. Bruce, K. Grabowska & J. Grabowski,** Higher order mechanics on graded bundles, J. Phys. A **48** (2015), 205203 (32pp).
- **A.J. Bruce, K. Grabowska & J. Grabowski,** Graded bundles in the category of Lie groupoids, *SIGMA* **11** (2015), 090 (25pp).
- **A.J. Bruce, K. Grabowska & J. Grabowski,** Linear duals of graded bundles and higher analogues of (Lie) algebroid, J. Geom. Phys. **101** (2016), 71–99.
- **A.J. Bruce, J. Grabowski & M. Rotkiewicz,** Polarisation of graded bundles, *SIGMA* **12** (2016), 106 (30pp).
- **K. Grabowska, J. Grabowski & P. Urbański,** Geometry of Lagrangian and Hamiltonian formalisms in the dynamics of strings, *J. Geom. Mech.* **6** (2014), 503–526.
- **J. Grabowski& M. Rotkiewicz,** Higher vector bundles and multi-graded symplectic manifolds, *J. Geom. Phys.* **59** (2009), 1285–1305.
- **J. Grabowski & M. Rotkiewicz,** Graded bundles and homogeneity structures, *J. Geom. Phys.* **62** (2012), 21–36.

THANK YOU FOR YOUR ATTENTION!