On Local Induction and Collection Principles

Part I: Basic Notions and Applications to Reflection Principles

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Model Theory and Proof Theory of Arithmetic
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Introduction

General Goal: To find natural restrictions on an axiom scheme to obtain axiomatizations of its Γ–consequences.

Axiom Scheme	Γ	Restriction
$I\Sigma_n, B\Sigma_n$	Π_{n+1}	Inference rule version
$I\Sigma_n, B\Sigma_n$	Σ_{n+2}	Parameter free version
$I\Sigma_n, B\Sigma_n$	Σ_{n+1}	??*

- Kaye–Paris–Dimitracopoulos'88 and Beklemishev–Visser'05 gave axiomatizations of Σ_{n+1}–consequences of $I\Sigma_n$. But they don't correspond to a restriction of the induction scheme.
- Axiomatizations of Σ_{n+1}–consequences of $B\Sigma_n$ not known.

Outline

1. We introduce axiom schemes restricted up to definable elements and study their basic properties.

2. We show that these restrictions give axiomatizations of the Σ_{n+1}–consequences of $I\Sigma_n$ and $B\Sigma_n$.

3. Applications to Local Reflection Principles.

Local axiom schemes

- Induction

$$\forall v \in B\, [\varphi(0,v) \wedge \forall x\, (\varphi(x,v) \rightarrow \varphi(x+1,v)) \rightarrow \forall x \in A\, \varphi(x,v)]$$

- Collection

$$\forall v \in B\, [\forall x\, \exists y\, \varphi(x,y,v) \rightarrow \forall z \in A\, \exists u\, \forall x \leq z\, \exists y \leq u\, \varphi(x,y,v)]$$

Definition

1. $E(\Gamma, A, B)$ denotes the E–scheme up to elements in A restricted to Γ–formulas with parameters in B.

2. $E(\Gamma^-, A)$ denotes the E–scheme up to elements in A restricted to *parameter free* Γ–formulas.

Submodels of Definable elements

- a is Γ–definable in $\mathfrak{A}$ with parameters in X if there are $\varphi(x, v) \in \Gamma$ and $b \in X$ s.t. $\mathfrak{A} \models \varphi(a, b) \wedge \exists! x\, \varphi(x, b)$.

- $\mathcal{K}_n(\mathfrak{A}, X) = \Sigma_n$–def. elements of $\mathfrak{A}$ (parameters in X)

- $\mathcal{I}_n(\mathfrak{A}, X) =$ initial segment determined by $\mathcal{K}_n(\mathfrak{A}, X)$

$$\mathfrak{A} \quad [\!\text{———})_{\omega}\text{———})_{\mathcal{I}_n}\text{————————————})$$

Expressing "$\forall x \in \mathcal{K}_n$" in the language

- Put $Def_\delta(x) \equiv \delta(x) \wedge \forall x_1,x_2\,(\delta(x_1) \wedge \delta(x_2) \rightarrow x_1 = x_2)$.

$$\text{“}\forall x \in \mathcal{K}_n\,\varphi(x)\text{”}$$
$$\Updownarrow$$
$$\{\forall x\ [Def_\delta(x) \rightarrow \varphi(x)]\ :\ \delta \in \Sigma_n\}$$

$$\text{“}\forall x \in \mathcal{I}_n\,\varphi(x)\text{”}$$
$$\Updownarrow$$
$$\{\forall x, z\ [Def_\delta(x) \wedge z \leq x \rightarrow \varphi(z)]\ :\ \delta \in \Sigma_n\}$$

- Fragments of Arithmetic up to definable elements $\rightsquigarrow$ local schemes restricted to classes of definable elements.

What do fragments "up to" look like?

- Σ_n^-–induction up to Σ_m–definable elements, $I(\Sigma_n^-, \mathcal{K}_m)$:

$$\varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow$$
$$\forall x\, (Def_\delta(x) \rightarrow \varphi(x)\,)$$

  where $\varphi \in \Sigma_n$, $\delta \in \Sigma_m$.

- Σ_n^-–collection up to Σ_m–def. elements, $B(\Sigma_n^-, \mathcal{K}_m)$:

$$\forall x\, \exists y\, \varphi(x, y) \rightarrow$$
$$\forall z\, (Def_\delta(z) \rightarrow \exists u\, \forall x \leq z\, \exists y \leq u\, \varphi(x, y))$$

  where $\varphi \in \Sigma_n$, $\delta \in \Sigma_m$.

- and so on ...

An axiomatization of $Th_{\Sigma_{n+1}}(B\Sigma_n)$

Theorem ($n \geq 1$)

Over $I\Sigma_{n-1}^{-}$, $\quad Th_{\Sigma_{n+1}}(B\Sigma_n) \equiv B(\Sigma_n^{-}, \mathcal{K}_n)$.

An axiomatization of $Th_{\Sigma_{n+1}}(B\Sigma_n)$

Theorem ($n \geq 1$)

Over $I\Sigma^-_{n-1}$, $\quad Th_{\Sigma_{n+1}}(B\Sigma_n) \equiv B(\Sigma^-_n, \mathcal{K}_n)$.

Proof:

($\vdash$): $B(\Sigma^-_n, \mathcal{K}_n) \subseteq B\Sigma_n$ and Σ_{n+1}–axiomatizable.

($\dashv$): Assume $B(\Sigma^-_n, \mathcal{K}_n)$.

Case 1: $\mathcal{I}_n(\mathfrak{A}) = \mathfrak{A}$.

Then, $\mathfrak{A} \models B\Sigma^-_n$ and so $\mathfrak{A} \models Th_{\Sigma_{n+1}}(B\Sigma_n)$.

Case 2: $\mathcal{I}_n(\mathfrak{A}) \neq \mathfrak{A}$.

- $\mathcal{I}_n(\mathfrak{A}) \models B\Sigma^-_n$ (end–extension properties in $I\Sigma^-_{n-1}$)
- $\mathcal{I}_n(\mathfrak{A}) \models Th_{\Pi_{n+1}}(\mathfrak{A})$, by $B(\Sigma^-_n, \mathcal{K}_n)$.

So, $\mathfrak{A} \models Th_{\Sigma_{n+1}}(B\Sigma_n)$. □

An application to Conservativity

Proposition ($n \geq 1$)

1. $I(\Sigma_n^-, \mathcal{K}_n) \vdash B(\Sigma_n^-, \mathcal{K}_n)$.
2. *Over* $I\Sigma_{n-1}^-$, $\quad I\Pi_n^- \equiv I(\Sigma_n^-, \mathcal{K}_n)$.

Proof: Usual proofs that $I\Sigma_n \vdash B\Sigma_n$ and $I\Sigma_n \equiv I\Pi_n$ 'localize.' □

Corollary ($n \geq 1$)

$B\Sigma_n$ *is* Σ_{n+1}*–conservative over* $I\Pi_n^-$.

Proof: $\varphi \in \Sigma_{n+1}$ and $B\Sigma_n \vdash \varphi$

$$\begin{aligned} &\Longrightarrow \quad B(\Sigma_n^-, \mathcal{K}_n) \vdash \varphi \\ &\Longrightarrow \quad I(\Sigma_n^-, \mathcal{K}_n) \vdash \varphi \\ &\Longrightarrow \quad I\Pi_n^- \vdash \varphi \end{aligned}$$

□

An application to the $I\Delta_n$ vs. $L\Delta_n$ Problem

- (Slaman'04) Over *exp*, $I\Delta_n \vdash B\Sigma_n \, (\equiv L\Delta_n)$.
- Parameter free version: $I\Delta_n^- \equiv L\Delta_n^-$?

Proposition ($n \geq 1$)

Over $I\Sigma_{n-1}$, $\quad L\Delta_n^- \equiv B(\Sigma_n^-, \mathcal{K}_n) \equiv Th_{\Sigma_{n+1}}(B\Sigma_n)$.

Corollary ($n \geq 1$)

Over $I\Sigma_{n-1} + exp$, $\quad I\Delta_n^- \equiv L\Delta_n^-$.

Proof: Assume $L\Delta_n^-$ fails.

1. Collection fails for $\theta \in \Sigma_n$ in $[0, a]$ with $a \in \mathcal{K}_n$.
2. By Slaman's proof, induction fails for $\varphi(x, a) \in \Delta_n$.
3. One can get rid of parameter a since $a \in \mathcal{K}_n$. □

What about $Th_{\Sigma_{n+1}}(I\Sigma_n)$?

- Natural candidate: $I(\Sigma_n^-, \mathcal{K}_n)$.
- Does $I(\Sigma_n^-, \mathcal{K}_n)$ axiomatize $Th_{\Sigma_{n+1}}(I\Sigma_n)$? NO
  Because...
  - $I(\Sigma_n^-, \mathcal{K}_n) \equiv I\Pi_n^-$.
  - $I\Pi_n^-$ is strictly weaker than $Th_{\Sigma_{n+1}}(I\Sigma_n)$
    e.g. $I\Sigma_n \vdash Con(I\Pi_n^-)$.
- **Question**:
  How can we extend $I(\Sigma_n^-, \mathcal{K}_n)$ to capture all the Σ_{n+1}–consequences of $I\Sigma_n$?

Iterating Σ_n–definability: $\mathcal{I}_n^\infty$

Definition

- $\mathcal{I}_n^0(\mathfrak{A}) = \mathcal{I}_n(\mathfrak{A})$
- For each k, $\mathcal{I}_n^{k+1}(\mathfrak{A}) = \mathcal{I}_n(\mathfrak{A}, \mathcal{I}_n^k(\mathfrak{A}))$
- $\mathcal{I}_n^\infty(\mathfrak{A}) = \bigcup_{k \geq 0} \mathcal{I}_n^k(\mathfrak{A})$

$$[\text{———})_{\mathcal{I}_n^0}\text{——})_{\mathcal{I}_n^1}\text{——}\ )_{\mathcal{I}_n^2}\text{————})_{\mathcal{I}_n^\infty}\text{————}\text{-})$$

Lemma

1. *If* $\mathfrak{A} \models I\Sigma_{n-1}$ *then* $\mathcal{I}_n^\infty(\mathfrak{A}) \prec_n^e \mathfrak{A}$.
2. $\mathcal{I}_n^\infty(\mathfrak{A})$ *is the least initial segment of* $\mathfrak{A}$ *closed under* Σ_n*–definability.*

Expressing "$\forall x \in \mathcal{I}_n^\infty$" in the language

- Recall $Def_\delta(x; v) \equiv \delta(x) \wedge \forall x_1, x_2\, (\delta(x_1) \wedge \delta(x_2) \rightarrow x_1 = x_2)$.

$$\text{“}\forall x \in \mathcal{I}_n^k\, \varphi(x)\text{”}$$

$$\Updownarrow$$

$$\forall \bar{a}, \bar{b}\, \left[\left\{ \begin{array}{lcl} Def_{\delta_0}(a_0) & \wedge & b_0 \leq a_0 \\ Def_{\delta_1}(a_1; b_0) & \wedge & b_1 \leq a_1 \\ & \vdots & \\ Def_{\delta_k}(a_k; b_{k-1}) & \wedge & b_k \leq a_k \end{array} \right\} \rightarrow \varphi(b_k) \right]$$

where $\delta_0, \ldots, \delta_k$ run over Σ_n.

An axiomatization of $Th_{\Sigma_{n+1}}(I\Sigma_n)$

Theorem ($n \geq 1$)

Over $I\Sigma_{n-1}$, $Th_{\Sigma_{n+1}}(I\Sigma_n) \equiv I(\Sigma_n^-, \mathcal{I}_n^\infty)$.

An axiomatization of $Th_{\Sigma_{n+1}}(I\Sigma_n)$

Theorem $(n \geq 1)$

Over $I\Sigma_{n-1}$, $Th_{\Sigma_{n+1}}(I\Sigma_n) \equiv I(\Sigma_n^-, \mathcal{I}_n^\infty)$.

Proof:

$(\vdash)$: By $Th_{\Sigma_{n+1}}(I\Sigma_n)$, $\mathcal{I}_n^{k+1}(\mathfrak{A})$ is bounded above in $\mathfrak{A}$.

$(\dashv)$: Assume $I(\Sigma_n^-, \mathcal{I}_n^\infty)$.

Case 1: $\mathcal{I}_n^\infty(\mathfrak{A}) = \mathfrak{A}$.

Then, $\mathfrak{A} \models I\Sigma_n^-$ and so $\mathfrak{A} \models Th_{\Sigma_{n+1}}(I\Sigma_n)$.

Case 2: $\mathcal{I}_n^\infty(\mathfrak{A}) \neq \mathfrak{A}$.

- $\mathcal{I}_n^\infty(\mathfrak{A}) \prec_n^e \mathfrak{A}$ and $\mathcal{I}_n^\infty(\mathfrak{A}) \models B\Sigma_{n+1}(\vdash I\Sigma_n)$.

So, $\mathfrak{A} \models Th_{\Sigma_{n+1}}(I\Sigma_n)$. □

Kaye–Paris–Dimitracopoulos' theories [JSL'88]

For each $k \geq 1$, $L\Sigma_n^{(k),-}$ denotes

$$\exists x_1, \dots, x_k \, \varphi(x_1, \dots, x_k)$$

$$\Downarrow$$

$$\exists x_1, \dots, x_k \left\{ \begin{array}{cc} x_1 = \mu t. \, \exists x_2, \dots, x_k \, \varphi(t, x_2, \dots, x_k) & \wedge \\ x_2 = \mu t. \, \exists x_3, \dots, x_k \, \varphi(x_1, t, \dots, x_k) & \wedge \\ \vdots & \\ x_k = \mu t. \, \varphi(x_1, x_2, \dots, t) & \end{array} \right\}$$

where $\varphi(x_1, \dots, x_k)$ runs over Σ_n.

- **Theorem(KPD):** $Th_{\Sigma_{n+1}}(I\Sigma_n) \equiv \bigcup_{k \geq 1} L\Sigma_n^{(k),-}$

Beklemishev–Visser's theories [APAL'05]

- The Σ_n^-–LIMR (*limit rule*) is given by:

$$\frac{\exists u\,\forall x > u\,(f(x+1) \leq f(x))}{\exists u\,\forall x > u\,(f(x) = f(u))},$$

where f runs over the Σ_n^-–total functions of $I\Sigma_{n-1}$.

- $[T, R]_0 = [T, R]$ = *non–nested* applications of the rule

$[T, R]_{k+1} = [\,[T, R]_k, R\,]$

- Theorem(BV):

Over $I\Sigma_{n-1}$, $Th_{\Sigma_{n+1}}(I\Sigma_n) \equiv \bigcup_{k \geq 1} [I\Sigma_{n-1}, \Sigma_n^-\text{–LIMR}]_k$.

The equivalence theorem

Theorem ($n \geq 1$, $k \geq 0$)

Over $I\Sigma_{n-1}$, the following theories are equivalent:

1. $I(\Sigma_n^-, \mathcal{I}_n^k)$.
2. $[I\Sigma_{n-1}, \Sigma_n^- \text{–}LIMR]_{k+1}$.
3. $L\Sigma_n^{(k+1),-}$.

- We obtain a hierarchy theorem for *local* induction:

$$\mathcal{K}_n(\mathfrak{A}, \mathcal{I}_n^k(\mathfrak{A})) \models I(\Sigma_n^-, \mathcal{I}_n^k) + \neg I(\Sigma_n^-, \mathcal{I}_n^{k+1})$$

- Kaye–Paris–Dimitracopoulos also gave a hierarchy theorem but needed involved arguments (*indicators, α–largeness*).

- Beklemishev–Visser posed the question of characterizing $[I\Sigma_{n-1}, \Sigma_n^-\text{–LIMR}]_k$ and left pending a hierarchy theorem.

Reflection Principles in Arithmetic

- Base theory: Elementary Arithmetic EA.
- $\Box_T(x) = \exists y \, \mathrm{Prf}_T(y, x)$
- Local Reflection for T, $\mathrm{Rfn}_\Gamma(T)$, consists of

$$\Box_T(\ulcorner \varphi \urcorner) \rightarrow \varphi,$$

  for all sentences $\varphi \in \Gamma$.
- Uniform Reflection for T, $\mathrm{RFN}_\Gamma(T)$, consists of

$$\forall x \, (\Box_T(\ulcorner \varphi(\dot{x})) \urcorner \rightarrow \varphi(x)),$$

  for all formulas $\varphi(x) \in \Gamma$.

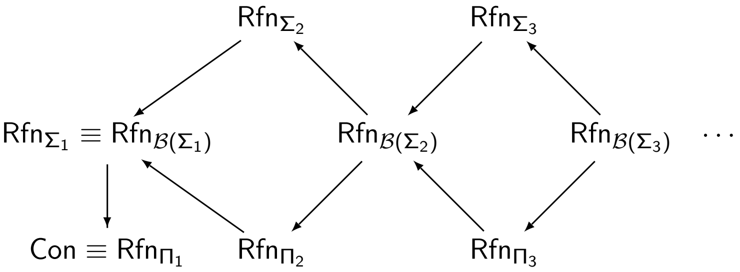
Uniform Reflection Hierarchy

$$\begin{array}{ccccccccc} \mathrm{Con} \equiv \mathrm{RFN}_{\Pi_1} & \longleftarrow & \mathrm{RFN}_{\Pi_2} & \longleftarrow & \mathrm{RFN}_{\Pi_3} & \longleftarrow & \mathrm{RFN}_{\Pi_4} & \longleftarrow & \dots \\ & & ||| & & ||| & & ||| & & \\ & & \mathrm{RFN}_{\Sigma_1} & & \mathrm{RFN}_{\Sigma_2} & & \mathrm{RFN}_{\Sigma_3} & & \end{array}$$

Induction and Uniform Reflection

- (Kreisel–Lévy) $\mathrm{EA} + \mathrm{RFN}(\mathrm{EA}) \equiv PA$.
- (Leivant–Ono) $\mathrm{EA} + \mathrm{RFN}_{\Sigma_{n+1}}(\mathrm{EA}) \equiv I\Sigma_n$ for $n > 0$.

Local Reflection Hierarchy



Induction and Local Reflection

- (Beklemishev) Over EA, $\mathrm{Rfn}_{\Sigma_2}(\mathrm{EA}) \equiv I\Pi_1^-$.
- No other Kreisel–Lévy results are known.
- Conservativity is not completely understood yet.

A Kreisel–Lévy result for Local Reflection

Theorem ($n \geq 1$)

1. *Over EA,* $\mathrm{Rfn}_{\Sigma_{n+1}}(EA) \equiv I(\Sigma_n^-, \mathcal{K}_1)$.
2. *Over EA,* $\mathrm{Rfn}_{\Pi_{n+1}}(EA) \equiv [EA, (\Pi_{n+1}, \mathcal{K}_1)\text{–}IR]$.

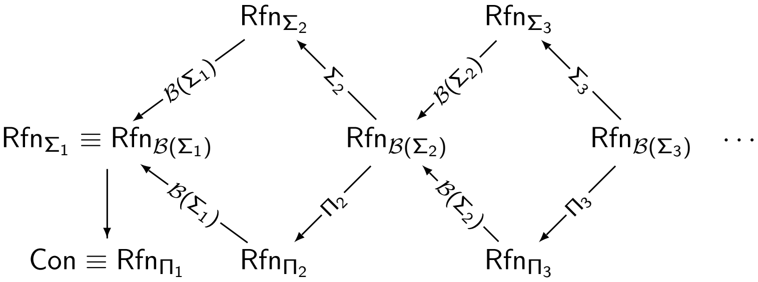
Proof: It suffices to prove Reflection for predicate calculus PC.

- By induction *up to the height* of a cut–free derivation, one can show that: $\mathrm{Prf}_{PC}(s, \ulcorner\varphi\urcorner) \to True(\ulcorner\varphi\urcorner)$.
- $\exists s\, \mathrm{Prf}_{PC}(s, \ulcorner\varphi\urcorner) \implies \exists s \in \mathcal{K}_1\, \mathrm{Prf}_{PC}(s, \ulcorner\varphi\urcorner)$. □

Remarks:

- Over EA, $\mathrm{Rfn}(EA) \equiv I(\Sigma_\infty^-, \mathcal{K}_1)$.
- $I(\Sigma_\infty^-, \mathcal{K}_1)$ is an analog of PA (*recent work by A. Visser*)
- $\mathrm{Rfn}^m_{\Sigma_{n+1}}(EA) \equiv I(\Sigma_n^-, \mathcal{K}_{m+1})$.

Conservativity for Local Reflection

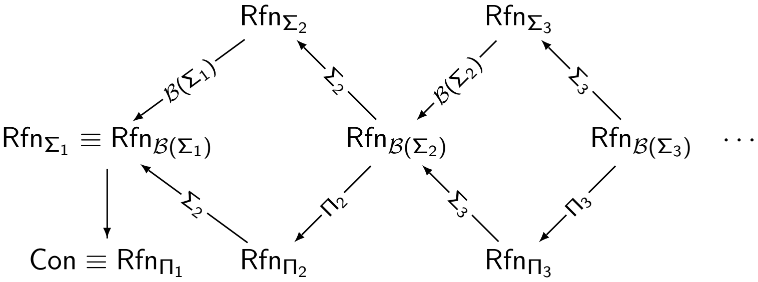


Theorem (Beklemishev)

Let $\Gamma = \Sigma_n/\Pi_n$ *with* $n \geq 2$, *or* $\Gamma = \mathcal{B}(\Sigma_k)$ *with* $k \geq 1$.

$T + \mathrm{Rfn}(T)$ *is* Γ*–conservative over* $T + \mathrm{Rfn}_\Gamma(T)$.

Conservativity for Local Reflection



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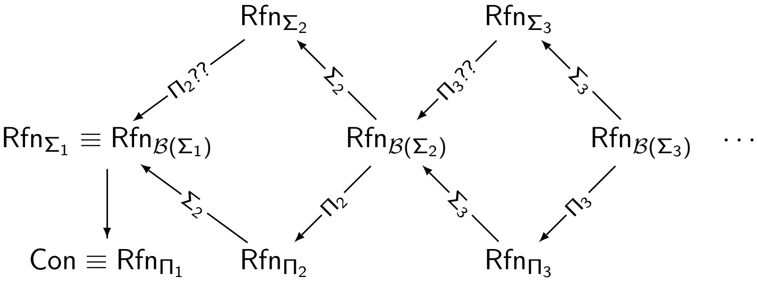
Conservativity for Local Reflection



Theorem (Beklemishev)

Let $\Gamma = \Sigma_n/\Pi_n$ *with* $n \geq 2$, *or* $\Gamma = \mathcal{B}(\Sigma_k)$ *with* $k \geq 1$.

$T + \mathrm{Rfn}(T)$ *is* Γ*–conservative over* $T + \mathrm{Rfn}_\Gamma(T)$.

Question (Beklemishev)

Is $T + \mathrm{Rfn}_{\Sigma_{n+1}}(T)$ Π_{n+1}*–conservative over* $T + \mathrm{Rfn}_{\mathcal{B}(\Sigma_n)}(T)$?

Answer to Beklemishev's Question

- We make use of $\mathrm{Rfn}_{\Sigma_{n+1}}(T) \equiv I(\Sigma_n^-, \mathcal{K}_1)$.
- We need a *sophisticated* separation property:

$$B(\Sigma_n, \mathcal{K}_1, \mathcal{K}_{n\,;\,1}) \;=\; \text{collection up to } \mathcal{K}_1 \text{ for } \Sigma_n\text{–formulas with parameters in } \mathcal{K}_n(\mathfrak{A}, \mathcal{I}_1).$$

Theorem ($n \geq 1$)

Suppose $T \subseteq \mathcal{B}(\Sigma_n)$ valid. Then, $T + \mathrm{Rfn}_{\Sigma_{n+1}}(T)$ is not Π_{n+1}–conservative over $T + \mathrm{Rfn}_{\mathcal{B}(\Sigma_n)}(T)$.

Proof:

- $T + \mathrm{Rfn}_{\Sigma_{n+1}}(T) \vdash I(\Sigma_n^-, \mathcal{K}_1) \vdash B(\Sigma_n, \mathcal{K}_1, \mathcal{K}_{n\,;\,1})$.
- Pick $a \in \mathcal{K}_n(\mathfrak{A}, \mathcal{I}_1) - \mathcal{I}_n(\mathfrak{A})$. Then:

$$\mathcal{K}_n(\mathfrak{A}, a) \models T + \mathrm{Rfn}_{\mathcal{B}(\Sigma_n)}(T).$$
$$\mathcal{K}_n(\mathfrak{A}, a) \not\models B(\Sigma_n, \mathcal{K}_1, \mathcal{K}_{n\,;\,1}). \qquad \square$$

Final Remarks

In this talk...

- We introduced axiom schemes restricted up to definable elements and presented applications:
  - *Induction up to Σ_n–definable elements* captures the Σ_{n+1}–consequences of $I\Sigma_n$ and $B\Sigma_n$.
  - *Induction up to Σ_1–definable elements* provides a Kreisel–Lévy theorem for the Local Reflection Hierarchy.
  - *Sophisticated* model–theoretic separation properties.

In the second talk...

- *Induction Rules* up to definable elements.
- Applications to Parameter free Π_n–induction.