On Local Induction and Collection Principles. Part II: Inference rules and applications to parameter free induction.

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Contents

Fragments of Peano Arithmetic

- Peano Arithmetic is axiomatized over a basic theory (say, Robinson's Q theory) by the induction scheme:

$$I_{\varphi,x}: \quad \varphi(0,v) \wedge \forall x\,(\varphi(x,v) \rightarrow \varphi(x+1,v)) \rightarrow \forall x\,\varphi(x,v)$$

- Classical fragments:

$$I\Sigma_n = Q + \{I_{\varphi,x}:\ \varphi(x,v) \in \Sigma_n\}$$
$$I\Pi_n = Q + \{I_{\varphi,x}:\ \varphi(x,v) \in \Pi_n\}$$

- Well known fact: $I\Sigma_n \equiv I\Pi_n$.
- This equivalence fails for **Parameter free schemes**.
  - We write $\varphi(x) \in \Sigma_n^-$ if $\varphi(x) \in \Sigma_n$ and x is the only free variable of $\varphi(x)$.
  - $I\Sigma_n^- = Q + \{I_{\varphi,x}:\ \varphi(x) \in \Sigma_n^-\}$.
  - $I\Pi_n^-$ is defined accordingly.
- $(n \geq 1)$ $\quad I\Sigma_n^-$ is a proper extension of $I\Pi_n^-$.

Σ_n–Induction

$(n \geq 1)$ $I\Sigma_n$ is a well–behaved fragment with good conservation properties

- (Parsons) $I\Sigma_n$ is Π_{n+1}–conservative over $I\Delta_0 + \Sigma_n$–IR.
  - For every theory T, $T + \Sigma_n$–IR denotes the closure of T under first order logic and (nested) applications of Σ_n–induction rule, Σ_n–IR:

$$\frac{\varphi(0,v) \wedge \forall x\,(\varphi(x,v) \rightarrow \varphi(x+1,v))}{\forall x\,\varphi(x,v)}, \quad \varphi(x,v) \in \Sigma_n.$$

- (KPD) $I\Sigma_n$ is Σ_{n+2} conservative over $I\Sigma_n^-$.
- Elegant characterizations of its class of provably total computable functions are known.
- There is a host of both model theoretic and proof theoretic tools particularly suited for the study of $I\Sigma_n$.

Σ_n–Induction rules

Σ_n–induction rule expresses a very robust principle:

- For every theory T extending $I\Delta_0$, it holds that

  $$[T, \Sigma_n\text{–IR}] \equiv [T, \Sigma_n\text{–IR}_0] \equiv [T, \Sigma_n^-\text{–IR}] \equiv [T, \Pi_n\text{–IR}_0].$$

  where, for every rule R, $[T, R]$ is the closure of T under first order logic and unnested applications of R, and
  - Σ_n–IR$_0$ denotes the inference rule

    $$\frac{\forall x\,(\varphi(x,v) \to \varphi(x+1,v))}{\varphi(0,v) \to \forall x\,\varphi(x,v)}, \quad \varphi(x,v) \in \Sigma_n.$$

  - Σ_n^-–IR denotes the parameter free version of Σ_n–IR.
- There is a natural correspondance between applications of Σ_1–IR and iteration of a convenient function:

  $$[I\Delta_0, \Sigma_1\text{–IR}]_m \quad \equiv \quad I\Delta_0 + \forall x\,\exists y\,(F_m(x) = y)$$

  - $[T, R]_0 = T, \quad [T, R]_{k+1} = [[T, R]_k, R]$.
  - $F_0(x) = (x+1)^2$, $F_{k+1}(x) = F_k(x)^{x+1}$.

Parameter free Σ_n–Induction

- (Adamowicz–Bigorajska; Mints) For every $m \geq 1$, if $\varphi_1(x), \ldots, \varphi_m(x) \in \Sigma_1^-$ and $\psi \in \Pi_2$ then

$$I\Delta_0 + I_{\varphi_1} + \cdots + I_{\varphi_m} \vdash \psi \quad \Rightarrow \quad I\Delta_0 + \forall x\, \exists y\, (F_m(x) = y) \vdash \psi$$

- Z. Ratajczyk extended this result to provably total computable functions of $I\Sigma_n^-$, using the fast growing hierarchy. He also gave an independent proof of the following result.
- (Kaye) For every $m \geq 1$, $\varphi_1(x), \ldots, \varphi_m(x) \in \Sigma_{n+1}^-$ and $\psi \in \Pi_{n+2}$

$$I\Sigma_n + I_{\varphi_1} + \cdots + I_{\varphi_m} \vdash \psi \quad \Rightarrow \quad [I\Sigma_n, \Sigma_{n+1}\text{–IR}]_m \vdash \psi$$

Π_n–Induction rule

$(n \geq 1)$ Π_n–induction rule differs strongly from Σ_n–IR.

- There is no nontrivial conservation between $I\Sigma_n$ and $I\Delta_0 + \Pi_n$–IR.
- $[I\Delta_0, \Pi_1\text{–IR}] \subset [I\Delta_0, \Pi_1^-\text{–IR}_0] \subset [I\Delta_0, \Pi_1\text{–IR}_0]$.
  - Recall that $[I\Delta_0, \Pi_1\text{–IR}_0] \equiv [I\Delta_0, \Sigma_1\text{–IR}]$.
- Over $I\Delta_0 + \exp$, (nested) applications of Π_{n+1}–IR corresponds to (iterated) n–consistency statements.
- (Beklemishev) $[I\Delta_0, \Pi_2\text{–IR}] \equiv [I\Delta_0, \Sigma_1\text{–IR}]$.

Parameter free Π_n–Induction

($n \geq 1$) $I\Pi_n^-$ also differs notably from $I\Sigma_n^-$.

- $I\Pi_n^- \subset I\Sigma_n^- \subset I\Sigma_n$
- $I\Pi_n^-$ is a very weak fragment, even w.r.t. $I\Sigma_n^-$.
  - As a matter of fact, it is closer to $I\Sigma_{n-1}$.
- It has been studied using ad hoc model theoretic constructions (Kaye-Paris–Dimitracopoulos, 1988).
- A more systematic study has been carry out by Beklemishev (1999) using an indirect approach through **Reflection principles**. The key ingredients are:
  - Results à la Kreisel–Levy, giving equivalences between parameter free induction and (relativized) local reflection principles.
  - Conservation results for reflection principles, obtained using methods from **provability logic**.
  - As an application, characterizations of the classes of provably total computable functions of $I\Pi_{n+1}^-$ and $I\Sigma_n + I\Pi_{n+1}^-$ are derived.

Some remarks

- The problems we find in the study of $I\Pi_n^-$ are particular cases of the more general problem of finding good (informative) descriptions or axiomatizations of the class of Σ_{n+1}–consequences of $I\Sigma_n$.
  - Observe that $I\Pi_n^-$ is Σ_{n+1}–axiomatizable.
- Local induction schemes allow us to address this question in a direct and systematic way.
- Our approach is model-theoretic, but inference rules play an important rol in our analysis.
- In this talk we restrict ourselves to $I\Pi_1^-$ and $I\Pi_2^-$. Most results can be generalized to $I\Pi_n^-$, $n > 2$, directly or by relativization.

Our starting point

Let θ be a sentence.

- Assume $I\Pi_1^- \vdash \theta$. Then

$\theta \in \Pi_2$	$I\Delta_0 + \exp \vdash \theta$ $(KPD'1988)$.
$\theta \in \mathcal{B}(\Sigma_1)$	$\textbf{?} \vdash \theta$
$\theta \in \Pi_1$	$\textbf{?} \vdash \theta$

- Assume $I\Pi_2^- \vdash \theta$. Then

$\theta \in \Pi_3$	$\textbf{?} \vdash \theta$
$\theta \in \mathcal{B}(\Sigma_2)$	$I\Sigma_1^- \vdash \theta$ $(Beklemishev, 1999)$
$\theta \in \Pi_2$	$I\Delta_0 + \Pi_2\text{–IR} \vdash \theta$ $(Beklemishev, 1999)$

A first step using rules

Let θ be a sentence.

- Assume $I\Pi_1^- \vdash \theta$. Then

$\theta \in \Pi_2$	$[I\Delta_0, \Sigma_1\text{–IR}] \vdash \theta$
$\theta \in \mathcal{B}(\Sigma_1)$	$\mathbf{?} \vdash \theta$
$\theta \in \Pi_1$	$\mathbf{?} \vdash \theta$

- Assume $I\Pi_2^- \vdash \theta$. Then

$\theta \in \Pi_3$	$\mathbf{?} \vdash \theta$
$\theta \in \mathcal{B}(\Sigma_2)$	$[I\Delta_0, \Pi_2^-\text{–IR}_0] \vdash \theta$
$\theta \in \Pi_2$	$I\Delta_0 + \Pi_2\text{–IR} \vdash \theta$

Fill in the blanks

Let θ be a sentence.

- Assume $I\Pi_1^- \vdash \theta$. Then

$\theta \in \Pi_2$	$[I\Delta_0, \Pi_1\text{–IR}_0] \vdash \theta \quad (0)$
$\theta \in \mathcal{B}(\Sigma_1)$	$[I\Delta_0, \Pi_1^-\text{–IR}_0] \vdash \theta \quad ? \quad (1)$
$\theta \in \Pi_1$	$I\Delta_0 + \Pi_1\text{–IR} \vdash \theta \quad ? \quad (2)$

- Assume $I\Pi_2^- \vdash \theta$. Then

$\theta \in \Pi_3$	$[I\Delta_0, \Pi_2\text{–IR}_0] \vdash \theta \quad ? \quad (3)$
$\theta \in \mathcal{B}(\Sigma_2)$	$[I\Delta_0, \Pi_2^-\text{–IR}_0] \vdash \theta$
$\theta \in \Pi_2$	$I\Delta_0 + \Pi_2\text{–IR} \vdash \theta$

Our goals

- We answer in the positive the open questions (1), (2) and (3).
  - Over $I\Delta_0 + \exp$ we can answer questions (1), (2) and (3) using an approach via (Local) Reflection principles.
  - We present here alternative techniques based on local induction principles that work over $I\Delta_0$ and avoid the use of the metamathematical machinery needed for an approach via reflection principles.
- Since $[I\Delta_0, \Sigma_2\text{–IR}] \equiv I\Sigma_1$, (3) can be formulated as

$$\text{Is } I\Pi_2^- \ \Pi_3\text{–conservative over } I\Sigma_1?$$

- We improve Kaye–Paris–Dimitracopoulos result (0) and obtain an explicit characterization of the set of Π_2–consequences of $I\Pi_1^-$.
- We also obtain additional refinements of these results in the spirit of Adamowicz-Bigorajska-Kaye-Ratajczyk theorem.

Induction up to Σ_n–definable elements

- We denote by $I(\Sigma_n, \mathcal{K}_n)$ the theory given by $I\Sigma_{n-1}^-$ together with the induction scheme

$$\varphi(0) \wedge \forall x\,(\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x \in \mathcal{K}_n\,\varphi(x)$$

  where $\varphi(x) \in \Sigma_n$ and $\delta(x) \in \Sigma_n^-$.

- $(\Sigma_n, \mathcal{K}_n)$–IR denotes the following inference rule:

$$\frac{\varphi(0) \wedge \forall x\,(\varphi(x) \rightarrow \varphi(x+1))}{\forall x \in \mathcal{K}_n\,\varphi(x)}$$

  where $\varphi(x) \in \Sigma_n$ and $\delta(x) \in \Sigma_n^-$.

- $I(\Sigma_n^-, \mathcal{K}_n)$ denotes the parameter free version.

Our results (I)

- Assume $I\Pi_1^- \vdash \theta$. Then

$\theta \in \Pi_2$	$[I\Delta_0, (\Sigma_1, \mathcal{K}_1)\text{–IR}] \vdash \theta$
$\theta \in \mathcal{B}(\Sigma_1)$	$[I\Delta_0, (\Sigma_1^-, \mathcal{K}_1)\text{–IR}] \vdash \theta$
$\theta \in \Pi_1$	$I\Delta_0 + \Pi_1\text{–IR} \vdash \theta$

- Some refinements: Let $\varphi_1(x), \ldots, \varphi_m(x) \in \Pi_1^-$ and assume that $I\Delta_0 + I_{\varphi_1} + \cdots + I_{\varphi_m} \vdash \theta$. Then

$\theta \in \Pi_2$	$[I\Delta_0, (\Sigma_1, \mathcal{K}_1)\text{–IR}] \vdash_m \theta$
$\theta \in \mathcal{B}(\Sigma_1)$	$[I\Delta_0, (\mathcal{B}(\Sigma_1)^-, \mathcal{K}_1)\text{–IR}] \vdash_m \theta$

  (where $\vdash_m$ expresses provability using at most m applications of the corresponding rule)

- Similar (weaker) results for $\Pi_1^-\text{–IR}_0$ and $\Pi_1\text{–IR}$ can also be proved.

Our results (II)

- Assume $I\Pi_2^- \vdash \theta$. Then

$\theta \in \Pi_3$	$I\Sigma_1^- + (\Sigma_2, \mathcal{K}_2)\text{–IR} \vdash \theta$
$\theta \in \mathcal{B}(\Sigma_2)$	$[I\Sigma_1^-, (\Sigma_2^-, \mathcal{K}_2)\text{–IR}] \vdash \theta$
$\theta \in \Pi_2$	$I\Delta_0 + \Pi_2\text{–IR} \vdash \theta$

- $I\Sigma_1$ extends $I\Sigma_1^- + (\Sigma_2, \mathcal{K}_2)$–IR.
- **Corollary**. $I\Pi_2^-$ is Π_3–conservative over $I\Sigma_1$.

Parameter free Π_1–induction

- $I\Pi_1^- \equiv I(\Sigma_1^-, \mathcal{K}_1)$
  - Analysis of the set of Π_2–consequences of $I(\Sigma_1, \mathcal{K}_1)$ is relevant in connection with $I\Pi_1^-$.
- Let us denote by Π_1^-–IR_0 the rule

$$\frac{\forall x\,(\varphi(x) \to \varphi(x+1))}{\varphi(0) \to \forall x\,\varphi(x)}, \qquad \varphi(x) \in \Pi_1^-$$

- For every theory T extending $I\Delta_0$,

$$[T, (\Sigma_1^-, \mathcal{K}_1)\text{–IR}] \equiv [T, \Pi_1^-\text{–IR}_0]$$

- Fact: $I\Delta_0 + \exp \equiv [I\Delta_0, \Sigma_1\text{–IR}]$.

Π_2–consequences of $I(\Sigma_1, \mathcal{K}_1)$

- Two key points:
  - A version of Parsons theorem holds for $I(\Sigma_1, \mathcal{K}_1)$.
  - The equivalence between applications of Σ_1–IR and iteration 'localizes".
- **(Local Parsons theorem)**
  $I(\Sigma_1, \mathcal{K}_1)$ is Π_2–conservative over $I\Delta_0 + (\Sigma_1, \mathcal{K}_1)$–IR.
- **(Local iteration theorem)** Let $f(x) = (x+1)^2$. Then the following theories are equivalent:
  - $I\Delta_0 + (\Sigma_1, \mathcal{K}_1)$–IR.
  - $[I\Delta_0, (\Sigma_1, \mathcal{K}_1)$–IR$]$.
  - $I\Delta_0 + \forall u \in \mathcal{K}_1 \, \forall x \, \exists y \, (f^u(x) = y)$.
- $I\Pi_1^-$ is Π_2–conservative over $[I\Delta_0, (\Sigma_1, \mathcal{K}_1)$–IR$]$.
  - As a corollary we get the result labelled with (0).
- Refinement: for every $\theta \in \Pi_2$

$$\begin{array}{rcl} I\Pi_1^- \vdash \theta & \Leftrightarrow & [I\Delta_0, (\Sigma_1, \mathcal{K}_1, \mathcal{I}_1^1)\text{–IR}] \vdash \theta \\ & \Leftrightarrow & I\Delta_0 + \forall u \in \mathcal{K}_1 \, \forall x \in \mathcal{I}_1^1 \, \exists y \, (f^u(x) = y) \vdash \theta \end{array}$$

Σ_{n+1}–closed models

- Σ_{n+1}–closed models provide a simple and clear method to obtain conservation results. The basic ideas were developed by J. Avigad working on previous ideas of D. Zambella and A. Visser.
- **Definition**. Let T be a theory. We say that $\mathfrak{A} \models T$ is a Σ_{n+1}–closed model of T if for each $\mathfrak{B} \models T$,

$$\mathfrak{A} \prec_n \mathfrak{B} \implies \mathfrak{A} \prec_{n+1} \mathfrak{B}$$

- It generalizes the notion of an *existentially closed model*.
- **Proposition**. (Existence)
  Let T be a Π_{n+2}–axiomatizable theory and $\mathfrak{A} \models T$ countable. Then there exists $\mathfrak{B} \models T$ such that $\mathfrak{A} \prec_n \mathfrak{B}$ and $\mathfrak{B}$ is Σ_{n+1}–closed for T.
- **Corollary**. Every consistent and Π_{n+2}–axiomatizable theory has a Σ_{n+1}–closed model.

The basics of the method

The basic device is the following result:

Theorem (Avigad,'02)

Let T_1 be a Π_{n+2}–axiomatizable theory such that every Σ_{n+1}–closed model for T_1 is a model of T_2. Then T_2 is Π_{n+1}–conservative over T_1.

Other key ingredient in most applications:

Lemma

Let $\mathfrak{A}$ be a Σ_{n+1}–closed model for T. Let $\varphi(\vec{x}) \in \Pi_{n+1}$ and $\vec{a} \in \mathfrak{A}$ such that $\mathfrak{A} \models \varphi(\vec{a})$. Then there exist $\theta(v, \vec{x}) \in \Pi_n$ and $b \in \mathfrak{A}$ such that

$$\mathfrak{A} \models \theta(b, \vec{a}) \quad \textit{and} \quad T \vdash \theta(v, \vec{x}) \to \varphi(\vec{x})$$

First applications

Let us prove results (0), (1) and (2).

- **Lemma 1**. Every Σ_2–closed model of $I\Delta_0 + (\Sigma_1, \mathcal{K}_1)$–IR is a model of $I(\Sigma_1, \mathcal{K}_1)$.
  - Local Parsons Theorem and result (0) follow from Lemma 1 and Local Iteration Theorem.
- A similar strategy fails for $I\Pi_1^-$ and $I\Delta_0 + \Pi_1$–IR, because of the following fact:
  - If T is recursive extension of $I\Delta_0$ and $\mathfrak{A}$ is Σ_1–closed model of T, then $\mathfrak{A}$ **is not** a model of $I\Pi_1^-$.
- **Lemma 2**. If $\mathfrak{A} \models [I\Delta_0, \Pi_1^-\text{–IR}_0]$ then $\mathcal{K}_1(\mathfrak{A}) \models [I\Delta_0, \Pi_1\text{–IR}_0]$.
  - As a corollary $[I\Delta_0, \Pi_1^-\text{–IR}_0]$ is Σ_2–conservative over $[I\Delta_0, \Pi_1\text{–IR}_0]$, and result (1) follows using result (0).
- **Lemma 3**. Every Σ_1–closed model of $I\Delta_0 + \Pi_1$–IR is model of $[I\Delta_0, \Pi_1^-\text{–IR}_0]$.
  - Result (2) follows from (1) and Lemma 3.

Parameter free Π_2–induction

In the case $n = 2$, we have:

1. $I\Pi_2^- \equiv I(\Sigma_2^-, \mathcal{K}_2)$.
2. $I(\Sigma_2, \mathcal{K}_2)$ is Π_3–conservative over $I\Sigma_1^- + (\Sigma_2, \mathcal{K}_2)$–IR.
3. $I\Sigma_1$ extends $I\Sigma_1^- + (\Sigma_2, \mathcal{K}_2)$–IR.
   - Reduction: $I\Sigma_1^- + (\Sigma_2, \mathcal{K}_2)\text{–IR} \equiv I\Sigma_1^- + (I\Delta_0 + (\Sigma_2, \mathcal{K}_2)\text{–IR})$.
   - A refinement of the (proof of) Local Iteration Theorem shows that $I\Sigma_1$ extends $I\Delta_0 + (\Sigma_2, \mathcal{K}_2)$–IR.

Theorem

$I\Pi_2^-$ is Π_3–conservative over $I\Sigma_1$.

- This improves a previous conservation result of L. Beklemishev.
- As corollary, we get that the class of provable total computable functions of $I\Pi_2^-$ is the class of primitive recursive functions.

Conditional axioms

Let L denote the language of First Order Arithmetic.

Definition

A set of L–formulas, E, is a set of **conditional axioms** if each element of E is a formula of the form $\alpha(\vec{v}) \rightarrow \beta(\vec{v})$.

Let T be an L–theory and E be a set of conditional axioms.

- $T + E$ is obtained by adding to T the universal closure of each formula in E.
- **Example**: $T + E = I\Sigma_1$, for $T = I\Delta_0$ and

$$E = \{I_{\varphi,x}(\vec{v}) : \ \varphi(x,\vec{v}) \in \Sigma_1\}$$

where $I_{\varphi,x}(\vec{v})$ is the induction scheme

$$\underbrace{\varphi(0,\vec{v}) \wedge \forall x\,(\varphi(x,\vec{v}) \rightarrow \varphi(x+1,\vec{v}))}_{\alpha} \rightarrow \underbrace{\forall x\,\varphi(x,\vec{v})}_{\beta}$$

Conditional axioms (cont'd)

- We can associate to each set of conditional axioms, E, two auxiliary sets of conditional axioms:
  - $E^- = E \cap \mathrm{Sent}$, and
  - $UE = \{\forall \vec{v}\, \alpha(\vec{v}) \to \forall \vec{v}\, \beta(\vec{v}) :\ \alpha(\vec{v}) \to \beta(\vec{v}) \in E\}$
- The theories $T + UE$ and $T + E^-$ are obtained by adding to T the sentences in UE and E^- respectively.
- **Example**: For $E = I\Delta_1$ we have:

$$E = \{\underbrace{\forall x\, (\varphi(x, \vec{v}) \leftrightarrow \psi(x, \vec{v}))}_{\alpha(\vec{v})} \to \underbrace{I_{\varphi,x}(\vec{v})}_{\beta(\vec{v})} :\ \varphi \in \Sigma_1,\, \psi \in \Pi_1\}$$

$$UE = \{\forall \vec{v}\, (\forall x\, (\varphi(x, \vec{v}) \leftrightarrow \psi(x, \vec{v}))) \to \forall \vec{v}\, I_{\varphi,x}(\vec{v}) :\ \varphi \in \Sigma_1,\, \psi \in \Pi_1\}$$

$$E^- = \{\forall x\, (\varphi(x) \leftrightarrow \psi(x)) \to I_{\varphi,x} :\ \varphi(x) \in \Sigma_1^-,\, \psi(x) \in \Pi_1^-\}$$

Conditional axioms: Inference rules

We also define an inference rule, E-Rule, with instances

$$\frac{\forall \vec{v}\, \alpha(\vec{v})}{\forall \vec{v}\, \beta(\vec{v})}, \qquad \text{for each } \alpha(\vec{v}) \rightarrow \beta(\vec{v}) \in E$$

- $[T, E$–Rule$]$ denotes the closure of T under first order logic and *unnested* applications of E–Rule.
- $T + E$–Rule denotes the closure of T under first order logic and (nested) applications of E–Rule.
- We denote by E^-–Rule the inference rule associated to the set of conditional axioms E^-.

The basic reduction

- For each set of formulas Π, we introduce the rule E^{Π}–Rule given by the instances

$$\frac{\theta(\vec{v},\vec{z}) \rightarrow \alpha(\vec{v})}{\theta(\vec{v},\vec{z}) \rightarrow \beta(\vec{v})}$$

  for each $\alpha(\vec{v}) \rightarrow \beta(\vec{v}) \in E$ and $\theta(\vec{v},\vec{z}) \in \Pi$.
- A set of conditional axioms E is **normal set of conditional axioms w.r.t.** Π_n, if for every $\alpha(\vec{v}) \rightarrow \beta(\vec{v}) \in E, \quad \alpha(\vec{v}) \in \Pi_{n+1}$ and $\beta \in \Pi_{n+2}$.

Lemma

Let T be a Π_{n+2}–axiomatizable theory and E a set of normal conditional axioms w.r.t. Π_n. Then $T + E$ is Π_{n+1}–conservative over $T + E^{\Pi_n}$–Rule.

The basic reduction (cont'd)

- It holds that $[U, E\text{–Rule}] \subseteq [U, E^{\Pi_n}\text{–Rule}]$.
- E is Π_n**–reducible modulo** T if for every theory U extending T, it holds

$$[U, E^{\Pi_n}\text{–Rule}] \equiv [U, E\text{–Rule}]$$

Theorem

Let T be a Π_{n+2}–axiomatizable theory and E a set of normal conditional axioms w.r.t. Π_n. Assume that E is Π_n–reducible modulo T. Then

1. *$T + E$ is Π_{n+1}–conservative over $T + E$–Rule.*
2. *$T + E$ is Σ_{n+2}–conservative over $T + UE$.*
3. *If every Π_{n+2}–axiomatizable extension of $T + E^-$ is closed under E–Rule, then $T + E$ is Σ_{n+2}–conservative over $T + E^-$*

The finite case

Theorem

Let F be a finite set of normal conditional sentences w.r.t. Π_n. Then, for every Π_{n+2}–axiomatizable theory T it holds that

$$Th_{\Pi_{n+1}}(T + F) \subseteq [T, F^{\Pi_{n+1}}\text{–}Rule]_m$$

where m is the number of elements of F.

Corollary

Let E be a set of normal conditional axioms w.r.t. Π_n. Assume that E is Π_n–reducible modulo T. Then for every finite set of sentences $F \subseteq E$ with m elements, it holds that

$$Th_{\Pi_{n+1}}(T + F) \subseteq [T, E\text{–}Rule]_m.$$

The finite case (proof)

Lemma

Let $E = \{\psi_1, \ldots, \psi_m\}$ a finite set of normal conditional sentences w.r.t. Π_n. Then

$$T + E^{\Pi_n}\text{–}Rule \equiv [T, E^{\Pi_n}\text{–}Rule]_m$$

- If ψ is a sentence of the form $\alpha \to \beta$, with $\alpha \in \Pi_{n+1}$ and $\beta \in \Pi_{n+2}$, we define the rule

$$\psi^{\Pi_n}\text{–Rule}: \quad \frac{\theta(u) \to \alpha}{\theta(u) \to \beta}, \qquad (\theta(u) \in \Pi_n).$$

- $T + \psi^{\Pi_n}$–Rule $\equiv [T, \psi^{\Pi_n}$–Rule$]$.
- It holds that for each sentence $\varphi \in \Pi_{n+1}$, a proof of φ in $T + E^{\Pi_n}$–Rule only requires one application of each rule $\psi_j^{\Pi_n}$–Rule.

Adamowicz–Bigorajska–Kaye–Ratajczyk's Thm

Theorem

For every theory T extension of $I\Sigma_n$, $m \geq 1$ and $\varphi_1(x), \dots, \varphi_m(x) \in \Sigma^-_{n+1}$,

$$Th_{\Pi_{n+2}}(T + I_{\varphi_1} + \cdots + I_{\varphi_m}) \subseteq [T, \Sigma_{n+1}\text{–}IR]_m$$

- $I\Sigma^-_{n+1}$ is a set of normal conditional sentences w.r.t. Π_{n+1} .
- $I\Sigma_{n+1}$ is Π_{n+1}–reducible modulo $I\Sigma_n$.

Parameter free Π_1-Induction

- Let $\varphi_1(x), \ldots, \varphi_m(x) \in \Pi_1^-$ and $\theta \in \Pi_2$ such that

$$I\Delta_0 + I_{\varphi_1} + \cdots + I_{\varphi_m} \vdash \theta$$

  Then $[I\Delta_0, (\Sigma_1, \mathcal{K}_1)\text{–IR})]_m \vdash \theta$.

- Refinement: $[I\Delta_0, (\Sigma_1, \mathcal{K}_1)\text{–IR})] \vdash_m \theta$.
- If $\theta \in \mathcal{B}(\Sigma_1)$ then $[I\Delta_0, (\mathcal{B}(\Sigma_1)^-, \mathcal{K}_1)\text{–IR})] \vdash_m \theta$.
- If $\theta \in \mathcal{B}(\Sigma_1)$, then there exist sentences $\pi_1, \ldots, \pi_r \in \Pi_1$ and $\sigma_1, \ldots, \sigma_r \in \Sigma_1$ such that $I\Delta_0 \vdash \bigvee_{j=1}^{r}(\sigma_j \wedge \pi_j)$ and for each $j = 1, \ldots, r$,

$$[I\Delta_0 + \sigma_j \wedge \pi_j, \Pi_1^-\text{–IR}_0] \vdash_m \theta$$

- If in addition $\theta \in \Pi_1$, then

$$[I\Delta_0 + \sigma_j \wedge \pi_j, \Pi_1\text{–IR}]_m \vdash \theta$$