Recursively saturated real closed fields

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Model Theory and Proof Theory of Arithmetic

Bedlewo, 24 July 2012

Real closed fields

Definition

A real closed field is a model of the theory of the ordered field of real numbers in the language $\mathcal{L} = \{+, \cdot, 0, 1, <\}$.

Tarski:

1. An ordered field $\mathcal{R}$ is real closed iff every non-negative element is a square, and every odd degree polynomial has a root.
2. The theory of real closed fields admits elimination of quantifiers and it is decidable.
3. The theory of real closed fields is o-minimal, i.e. the 1-definable (with parameters) sets are finite unions of intervals and points.
4. If $f : (a, b) \to \mathcal{R}$ definable then there are $a_1, \ldots, a_k$ s.t. $f_{|_{(a_i, a_{i+1})}}$ is either constant, or a strictly monotone and continuous.

Integer parts

Definition

A *discrete ordered ring* I is an ordered ring in which 1 is the least positive element $(\neg\exists x(0 < x < 1))$.

Definition

Let R be an ordered field. An *integer part* (*IP*) for R is a discrete ordered subring I of R such that for each $r \in R$, there exists $i \in I$ such that $i \leq r < i + 1$.

If R is Archimedean, then $\mathbb{Z}$ is the unique integer part for R.

If R is non-archimedean there may be many different integer parts.

Shepherdson characterization '64

IOpen is the fragment of *PA* where the induction axiom is only for quantifier-free (open) formulas.

Theorem

Let I be a discrete ordered ring, $F(I)$ the fraction field of I, and $RC(I)$ the real closure of $F(I)$. $I^{\geq 0}$ is a model of Open Induction iff for all $\alpha \in RC(I)$ there is r in I such that $|r - \alpha| < 1$, i.e. I is an integer part of $RC(I)$. Moreover, $F(I)$ is dense in $RC(I)$.

The proof uses

1. kth root of a polynomial is 1st order property
2. Elimination of quantifiers for real closed fields

Integer parts

Theorem (Boughattas, 1993)

There exist ordered fields with no IP: a p-real closed field for any $p \in \mathbb{N}$
Every ordered field K has an ultrapower which admits an IP.

Theorem (Mourgues and Ressayre, 1993)

Every real closed fields has an integer part

Theorem (Berarducci and Otero, 1996)

There is a a real closed field which has a normal integer part, i.e. integrally closed in its fraction field.

$$\forall x, y, z_1, \ldots, z_n$$
$$(y \neq 0 \wedge x^n + z_1 x^{n-1} y + \ldots + z_{n-1} x y^{n-1} = 0 \to \exists z(yz = x))$$

Integer parts models of PA

Question: Which real closed fields have an IP which is a model of PA?

Answer: Recursively saturated countable real closed fields
(D'A-Knight-Starchenko 2010)

Definition

Let L be a computable language and $\mathcal{A}$ an L-structure. $\mathcal{A}$ is *recursively saturated* if for any computable set of L-formulas $\Gamma(\overline{u}, x)$, for all tuples $\overline{a}$ in $\mathcal{A}$ with $|\overline{a}| = |\overline{u}|$, if every finite subset of $\Gamma(\overline{a}, x)$ is satisfied in $\mathcal{A}$, then $\Gamma(\overline{a}, x)$ is satisfied in $\mathcal{A}$.

$\mathbb{N}$ is not recursively saturated because of the type $\{v > n : n \in \mathbb{N}\}$.

For each $\mathcal{A}$ there is $\mathcal{A}^\star$ such that $\mathcal{A} \preceq \mathcal{A}^\star$ and $\mathcal{A}^\star$ is recursively saturated.

Integer parts models of PA

Theorem (Barwise-Schlipf)

Suppose $\mathcal{A}$ is countable and recursively saturated. Let Γ be a c.e. set of sentences involving some new symbols. If Γ in the language of $\mathcal{A}$ is consistent with $\mathcal{A}$, then $\mathcal{A}$ can be expanded to $\mathcal{A}'$ satisfying Γ. Moreover, $\mathcal{A}'$ can be chosen recursively saturated.

Proposition

If R is a countable, recursively saturated real closed ordered field, then there is an integer part I satisfying PA. In fact, we may take the pair (R, I) to be recursively saturated.

Integer parts models of PA

Sketch of proof: Add to the language of ordered fields a unary predicate I. Let $\Gamma = Th(R) \cup \mathcal{I}$ where $\mathcal{I}$ says I is an integer part whose positive part satisfies PA. Γ is consistent with $Th(R) = Th(\mathbb{R})$ because of $(\mathbb{R}, \mathbb{Z})$. By Barwise-Schlipf we can expand R to (R, I) recursively saturated and having an integer part which is a model of PA.

Remark

If R is a countable recursively saturated real closed field not all the integer parts satify PA. Using Barwise-Schlipf theorem can obtain and integer part in which 2^x is not total. More generally, we can obtain an integer part which satisfies any property consistent with *IOpen*.

Models of PA

Let $\mathcal{A}$ is a nonstandard model of PA, and $a \in \mathcal{A}$,

$$A_a = \{n \in \omega : \mathcal{A} \models p_n | a\}$$

is the set coded by a in $\mathcal{A}$. Let $SS(\mathcal{A}) = \{A_a : a \in \mathcal{A}\}$

1. $SS(\mathcal{A})$ is closed under Turing reducibility and disjoint union,
2. for any infinite subtree T of $2^{<\omega}$ s.t. $T \in SS(\mathcal{A})$, there is a path in $SS(\mathcal{A})$

1+2 say that $SS(\mathcal{A})$ is a Scott set.

Can extend the notion of coded set also to a real closed field $\mathcal{R}$.

Models of PA

Proposition

Let $\mathcal{M}$ be a non standard model of PA. Then $\mathcal{M}$ is Σ_n-recursively saturated for each $n \in \mathbb{N}$.

Lemma

Let $\mathcal{A}$ be a nonstandard model of PA.

1. For any tuple $\overline{a}$ in $\mathcal{A}$, and any $n \in \omega$, the Σ_n type of $\overline{a}$ (with no parameters) is in $SS(\mathcal{A})$.
2. For any n, if $\Gamma(\overline{x}, \overline{w})$ is a consistent set of Σ_n-formulas belonging to $SS(\mathcal{A})$ and every finite subset of $\Gamma(\overline{x}, \overline{a})$ is satisfied in $\mathcal{A}$, then $\Gamma(\overline{x}, \overline{a})$ is satisfied in $\mathcal{A}$.

The proofs use partial satisfaction classes, i.e. satisfaction classes for Σ_n-formulas.

Integer parts models of PA

Theorem

If I is a non standard model of PA then $RC(I)$ is recursively saturated. $RC(I)$ is also ω-homogenous.

Lemma (1)

If $\overline{a}$ is in R, then $tp(\overline{a}) \in SS(I)$.

Lemma (2)

If $\overline{a}$ is in R, and $\Gamma(\overline{a}, x) \in SS(I)$ is a complete type realized in some elementary extension of R, then $\Gamma(\overline{a}, x)$ is realized in R.

Integer parts models of PA

The proofs of the lemmas use:

1. o-minimality of real closed fields;
2. Σ_n-recursive saturation of a non standard model of PA;

We show that there is a tuple $\bar{i}$ in I such that the quantifier free type realized by $\bar{a}$ in R is computable in the Σ_3 type realized by $\bar{i}$ in I. Then bounded recursive saturation of I implies that the type of $\bar{a}$ is coded in I, i.e. it belongs to $SS(I)$

Sketch of proof of Theorem:

Let $\overline{a}$ be a tuple in $RC(I)$, $\Gamma(\overline{u}, x)$ a computable set of formulas such that $\Gamma(\overline{a}, x)$ consistent with $RC(I)$. By Lemma 1 $tp(\overline{a}) \in SS(I)$. Then there is a completion $\Delta(\overline{a}, x)$ of $tp(\overline{a}) \cup \Gamma(\overline{a}, x)$ in $SS(I)$. By Lemma 2 this is realized in $RC(I)$. Therefore, $RC(I)$ is recursively saturated.

Remark

By inspection of the proofs of both lemmas we do not need full *PA* but $I\Sigma_4$ is enough.

Remark

Recently, Jeřabek and Kołodziejczyk have proved that real closed fields having integer parts which are models of some subsystems of Buss' bounded arithmetic (PV, $\Sigma_1^b - IND^{|x|_k}$).

Integer parts models of PA

Example

There is a non standard model of $I\Delta_0$ such that $RC(I)$ is not recursively saturated:

$J \models PA$, and $a \in J - \mathbb{N}$. Let

$$I = \{x \in J : x < a^n \text{ for some } n \in \mathbb{N}\}.$$

$I \models I\Delta_0$, $RC(I)$ is not recursively saturated since the type

$$\tau(v) = \{v > a^n : n \in \mathbb{N}\}$$

is not realized.

Integer parts models of PA

Theorem

Let R be a real closed field and I an integer part of R which is a model of PA. Then R and $RC(I)$ realize the same types.
R is ω-homogenous.

Sketch of proof:

1. $SS(RC(I)) = SS(I)$
2. For any $\overline{a} \in R$ there is $b \in R$ such that $b > RC(\overline{a})$ (*unbounded growth*).
3. R is ω-homogeneity since $RC(I)$ is ω-homogeneous and they realize the same types.

Integer parts models of PA

Theorem

Suppose R is a real closed field with integer part I, where I is a nonstandard model of PA. Then R is recursively saturated, and if R is countable $R \cong RC(I)$.

We have a kind of converse.

Theorem

Let R be a countable real closed ordered field. If R is recursively saturated, then there is an integer part I, satisfying PA, such that $R = RC(I)$.

Corollary

Two countable nonstandard models of PA have isomorphic real closures if and only if they have the same standard systems.

Integer parts models of *PA*

Question: Is the countability of the real closed field necessary?

Answer: YES **(Carl-D'A-Kuhlmann, Marker 2012)**

There are uncountable recursively saturated real closed fields with no integer part model of *PA*. These are constructed as power series fields.

Valuation theory notions

Natural valuation: Let R be a real closed field, $x, y \in R^*$,

$$x \sim y \quad \text{if there exist } m, n \in \mathbb{N} \quad n|x| > |y| \quad \text{and} \quad m|y| > |x|$$

The valuation rank of R is the linear ordered set $(R^*/\sim, <)$ where

$$[x] < [y] \quad \text{iff} \quad n|y| < |x| \quad \text{for all} \quad n \in \mathbb{N}$$

The value group G of R is the ordered group $(R^*/\sim, +, 0, <)$ where

$$[x] + [y] = [xy]$$

G is a divisible ordered abelian group.

$v : R^* \to G$ the **valuation map** $v(x) = [x]$

Valuation theory notions

$R_v = \{r \in R : v(r) \geq 0\}$ is the valuation ring of R, i.e. the finite elements of R

$\mu_v = \{r \in R : v(r) > 0\}$ is the maximal ideal of R, i.e. the infinitesimal elements of R

$\mathcal{U}_v^{>0} = \{r \in R : v(r) = 0, r > 0\}$ is the group of positive units in R_v and it is a subgroup of $(R^{>0}, \cdot, 1, <)$

$1 + \mu_v = \{r \in R^{>0} : v(r-1) > 0\}$ is the group of 1-units, and it is a subgroup of $\mathcal{U}_v^{>0}$

$k = R_v/\mu_v$ is the residue field of R, it is an archimedean real closed field

Valuation theory notions

Theorem

Let $(K,+,\cdot,0,1,<)$ be an ordered field. There is a group complement $\mathbf{A}$ of R_v in $(K,+,0,<)$ and a group complement $\mathbf{A}'$ of μ_v in R_v, i.e.

$$(K,+,0,<) = \mathbf{A} \oplus \mathbf{A}' \oplus \mu_v.$$

$\mathbf{A}$ and $\mathbf{A}'$ are unique up to order preserving isomorphism

Theorem

Let $(K,+,\cdot,0,1,<)$ be an ordered field, and assume that $(K^{>0},\cdot,1,<)$ is divisible. There is a group complement $\mathbf{B}$ of $\mathcal{U}_v^{>0}$ in $(K^{>0},\cdot,1,<)$ and a group complement $\mathbf{B}'$ of $1+\mu_v$ in $\mathcal{U}_v^{>0}$, i.e.

$$(K^{>0},\cdot,1,<) = \mathbf{B} \odot \mathbf{B}' \odot (1+\mu_v).$$

$\mathbf{B}$ and $\mathbf{B}'$ are unique up to order preserving isomorphism

Counterexample

Definition

An ordered field K is said to have left exponentiation iff there is an isomorphism from a group complement $\mathbf{A}$ of R_v in $(K,+,0,<)$ onto a group complement $\mathbf{B}$ of $\mathcal{U}_v^{>0}$ in $(K^{>0},\cdot,1,<)$.

Theorem

Let $(K,+,\cdot,0,1,<)$ be a real closed field and let Z be an integer part of K such that $Z^{\geq 0}$ is a nonstandard model of PA. Then K has left exponentiation.

Remark

Notice that $I\Delta_0 + exp$ is enough since we need only a weak fragment of PA in which exponentiation is defined and is provably total.

Counterexample

Power series fields:

Recall that given an ordered abelian group G and an ordered field k, we can form the Hahn series field, $k((G))$ of formal sums

$$f = \sum_{g \in G} a_g t^g$$

where the support of f, $supp(f) = \{g \in G : a_g \neq 0\}$ is well ordered. This has an ordered field structure. If G is divisible and k is real closed then $k((G))$ is real closed.

Counterexample

Theorem (F.-V. Kuhlmann, S. Kuhlmann, S. Shelah,1997)

For no nontrivial ordered abelian group G the field $\mathbb{R}((G))$ admits a left exponentiation.

Corollary

For any non trivial divisible ordered abelian group G the real closed field $\mathbb{R}((G))$ does not have an integer part which is a model of PA.

Uncontable case

Corollary

There exists an uncountable recursively saturated real closed field which does not have any integer part which is a model of PA.

Proof: Let G be a divisible ordered abelian group and suppose G is $\aleph_0$-saturated. Then by [KKMZ] also $\mathbb{R}((G))$ is $\aleph_0$-saturated, so in particular $\mathbb{R}((G))$ is recursively saturated. By previous corollary it cannot have an integer part which is a model of PA (or even of $I\Delta_0 + exp$).

Uncontable case

Remark (S. Kuhlmann)

If K is a non Archimedean real closed field and K admits left exponentiation then the value group of K is an exponential group in $(\overline{K}, +, 0, <)$.

Theorem

If $(K, +, \cdot, 0, 1, <)$ is a real closed field with an integer part model of PA then the value group of K is an exponential group in $(\overline{K}, +, 0, <)$, and in particular the rank of $v(K)$ is a dense linear order without endpoints.

Example

Let A be a countable divisible ordered abelian group and suppose A is archimedean (e.g $A = \mathbb{Q}$). Then G is an exponential group in A iff $G = \oplus_{\mathbb{Q}} A$.

Uncontable case

Question: Is there a natural characterization of the uncountable real closed fields with nonstandard models of PA for integer parts?
Answer: Work in progress (Marker and Steinhorn)

Definition

A structure $\mathcal{M}$ in a countable language $\mathcal{L}$ is resplendent if for any finite expansion $\mathcal{L}^* = \mathcal{L} \cup \{R_1, \ldots, R_k\}$ where R_i are new relational symbols and any $\mathcal{L}^*$-sentence ψ consistent with $Th(\mathcal{M})$ there is an expansion of $\mathcal{M}$ to $\mathcal{L}^*$ that is a model of ψ.

Remark

1. If $\mathcal{M}$ is resplendent the $\mathcal{M}$ is recursively saturated.
2. If $\mathcal{M}$ is countable and recursively saturated then $\mathcal{M}$ is replendent.

Uncontable case

Marker and Steinhorn (2012) showed that

1. The real closure of an ω_1-like model of *PA* is not replendent
2. If $\mathcal{M}$ and $\mathcal{N}$ are ω_1-like models of *PA* with the same standard system, then the value groups of their real closures (or any real closed field of which they are an integer part) are isomorphic.
3. ($\diamondsuit$) There are $2^{\aleph_1}$ elementarily equivalent ω_1-like recursively saturated models of *PA* with the same standard system such that their real closures are pairwise nonisomorphic.

Work in progress

D'A., Kuhlmann and Lange: look for a valuation theoretical characterization of recursively saturated real closed fields, in the spirit of that for $\aleph_\alpha$-saturation for real closed fields

Theorem (Kuhlmann, Kuhlmann, Marshall and Zekavat)

Let R be a real closed field, G and k the valu group and the residue field with respect to the natural valuation. Then R is $\aleph_\alpha$-saturated iff

1. G is $\aleph_\alpha$-saturated
2. $k \cong \mathbb{R}$
3. every pseudo Cauchy sequence in a subfield of absolute transcendence degree less than $\aleph_\alpha$ has a pseudolimit in R.