Self-Embeddings of Models of Arithmetic, Redux

Ali Enayat (Report of Joint work with V. Yu. Shavrukov)

Model Theory and Proof Theory of Arithemtic
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- **1962.** In answer to a question of Dana Scott, in the mid 1950's Robert Vaught shows that there is a model of true arithmetic that is isomorphic to a proper initial segment of itself. This result is later included in a joint paper of Vaught and Morley.

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- **1973.** Harvey Friedman's landmark paper contains a proof of the striking result that *every* countable nonstandard model of PA is isomorphic to a proper initial segment of itself.

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- **1973.** Harvey Friedman's landmark paper contains a proof of the striking result that *every* countable nonstandard model of PA is isomorphic to a proper initial segment of itself.
- **1977.** Alex Wilkie shows that if $\mathcal{M}$ and $\mathcal{N}$ are countable nonstandard models of PA, then $\mathrm{Th}_{\Pi_2}(\mathcal{M}) \subseteq \mathrm{Th}_{\Pi_2}(\mathcal{N})$ iff there are arbitrarily high initial segment of $\mathcal{N}$ that are isomorphic to $\mathcal{M}$.

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- **1978.** Hamid Lessan shows that a countable model $\mathcal{M}$ of Π_2^{PA} is isomorphic to a proper initial segment of itself iff $\mathcal{M}$ is 1-tall and 1-extendible, where 1-tall means that the set of Σ_1-definable elements of $\mathcal{M}$ is not cofinal in $\mathcal{M}$, and 1-extendible means that there is an end extension $\mathcal{M}^*$ of $\mathcal{M}$ that satisfies $\mathsf{I}\Delta_0$ and $\mathrm{Th}_{\Sigma_1}(\mathcal{M}) = \mathrm{Th}_{\Sigma_1}(\mathcal{M}^*)$.

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- **1978.** With the introduction of the key concepts of recursive saturation and resplendence (in the 1970's), Vaught's result was reclothed by John Schlipf as asserting that every *resplendent* model of PA is isomorphic to a proper *elementary* initial segment of itself.

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- **1978.** Craig Smorynski's influential lectures and expositions systematize and extend Friedman-style embedding theorems around the key concept of (partial) recursive saturation.

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- **1979.** Leonard Lipshitz uses the Friedman embedding theorem and the MRDP theorem to show that a countable nonstandard model of PA is Diophantine correct iff it can be embedded into arbitrarily low nonstandard initial segments of itself (the result was suggested by Stanley Tennenbaum).

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- **1981.** Jeff Paris notes that an unpublished construction of Robert Solovay shows that every countable recursively saturated model of $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$ is isomorphic to a proper initial segment of itself.

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- **1983.** Žarko Mijajlović shows that if $\mathcal{M}$ is a countable model of PA and $a \notin \Delta_1^{\mathcal{M}}$, then there is a self-embedding of $\mathcal{M}$ onto a submodel $\mathcal{N}$ (where $\mathcal{N}$ is not necessarily an initial segment of $\mathcal{M}$) such that $a \notin N$. He also shows that $\mathcal{N}$ can be arranged to be an initial segment of $\mathcal{M}$ if there is no $b > a$ with $b \in \Delta_1^{\mathcal{M}}$ (he attributes this latter result to Marker and Wilkie).

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- **1985.** Costas Dimitracopoulos shows that every countable nonstandard model of $\mathsf{I}\Delta_0 + \mathrm{B}\Sigma_2$ is isomorphic to a proper initial segment of itself.
- **1986.** Aleksandar Ignjatović refines the aforementioned work of Mijajlović.

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- **1987.** Jean-Pierre Ressayre proves an optimal result: for every countable nonstandard model $\mathcal{M}$ of $I\Sigma_1$ and for every $a \in \mathcal{M}$ there is an embedding j of $\mathcal{M}$ onto a proper initial segment of itself such that $j(x) = x$ for all $x \leq a$; moreover, this property characterizes countable models of $I\Sigma_1$ among countable models of $I\Delta_0$.

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- **1987.** Bonnie Gold refines Lipshitz's aforementioned result by showing that if $\mathcal{M}$ and $\mathcal{N}$ are models of PA with $\mathcal{M} \subseteq_{\text{end}} \mathcal{N}$, then $\mathcal{N}$ is Diophantine correct relative to $\mathcal{M}$ iff for every $a \in N \backslash M$ there is an embedding $j : \mathcal{N} \to \mathcal{N}$ such that $j(N) < a$ and $j(m) = m$ for all $m \in M$.

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- **1988.** Independently of Ressayre, Dimitracopoulos and Paris show that every countable nonstandard model of $\mathrm{I}\Sigma_1$ is isomorphic to a proper initial segment of itself. They also generalize Lessan's aforementioned result by weakening Π_2^{PA} to $\mathrm{I}\Delta_0 + \exp + \mathrm{B}\Sigma_1$.

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- A necessary and sufficient condition for the existence of a Σ_n-elementary embedding j of a countable model $\mathcal{M}$ onto an initial segment I between two prescribed elements $a < b$ of $\mathcal{M}$ such that $j(a) = a$;
- The existence of *continuum-many* initial segments of every countable nonstandard model of $\mathcal{M}$ of PA that are isomorphic to $\mathcal{M}$.
- **1997**. Kazuyuki Tanaka extends Ressayre's aforementioned result to countable nonstandard models of WKL_0.

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- **Definition.** A partial function f from M to M is a *partial $\mathcal{M}$-recursive function* if the graph of f is definable in $\mathcal{M}$ by a parameter-free Σ_1-formula.
- **Theorem.** (Sharpened Friedman Theorem) *Suppose $c \in M$, and $\{a, b\} \subseteq N$ with $a < b$. The following statements are equivalent:*
  (1) $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{N})$, *and for every Δ_0-formula $\delta(x, y)$ we have:*

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- **(4)** $f(a) < b$ *for all partial* $\mathcal{M}$*-recursive functions.*

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- **Theorem**. (Sharpened Hájek-Pudlák). *Suppose I is a cut shared by $\mathcal{M}$ and $\mathcal{N}$, and I is closed under exponentiation. Assume furthermore that $c \in \mathcal{M}$, with $I < c$, and $\{a, b\} \subseteq N$ with $I < a < b$. The following statements are equivalent:*

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- **(i)** $\mathrm{SSy}_I(\mathcal{M}) = \mathrm{SSy}_I(\mathcal{N})$, *and for every Δ_0-formula $\delta(x, y, z)$, and all $i \in I$ we have*:

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Total Recursive Functions and Embeddings (2)

- **Theorem** (Wilkie) **.** $\mathcal{M}$ *is isomorphic to arbitrarily high initial segments of* $\mathcal{N}$ *iff* $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{N})$ and $\mathrm{Th}_{\Pi_2}(\mathcal{M}) \subseteq \mathrm{Th}_{\Pi_2}(\mathcal{N})$.
- **Theorem.** *Suppose* $\{a, b\} \subseteq M$ *with* $a < b$. *The following statements are equivalent:*

- **(1)** *There is an initial embedding* $j : \mathcal{M} \to \mathcal{M}$ *with* $a < j(M) < b$.
- **(2)** *There is a cut* I *of* $\mathcal{M}$ *with* $a < I < b$ *and* $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(I)$.
- **(3)** $f(a) < b$ *for all* $\mathcal{M}$*-recursive functions* f.

New Proof of Tanaka's Theorem (1)

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- **Theorem** (Tanaka) *Every countable nonstandard model of* WKL_0 *has a nontrivial self-embedding in the following sense: given* $(\mathcal{M}, \mathcal{A}) \models \mathrm{WKL}_0$, *there is a proper initial segment* I *of* $\mathcal{M}$ *such that*

$$(\mathcal{M}, \mathcal{A}) \cong (I, \mathcal{A} \restriction I),$$

where $\mathcal{A} \restriction I := \{A \cap I : A \in \mathcal{A}\}$.

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- *Stage 2 Outline:* We build an *end extension* $\mathcal{N}$ of $\mathcal{M}$ such that (1) $\mathcal{N} \models \mathrm{B}\Sigma_1 + \exp$, (2) $\mathcal{N}$ is recursively saturated, and (3) $f(a) < b$ for all $\mathcal{N}$-partial recursive functions of $\mathcal{M}$, and (4) $\mathrm{SSy}_M(\mathcal{N}) = \mathcal{A}$.

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- *Stage 3 Outline*: We use a fine-tuned version of Solovay's embedding theorem to embed $\mathcal{N}$ onto a proper initial segment J of $\mathcal{M}$. By elementary considerations, this will yield a proper cut I of J with $(\mathcal{M}, \mathcal{A}) \cong (I, \mathcal{A} \upharpoonright I)$.

New Proof of Tanaka's Theorem (3)

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- *Stage 2 Details:* Fix some nonstandard $n^* \in M$ with $n^* >> b$ (e.g., $n^* = \mathrm{supexp}(b)$ is more than sufficient). Then by since $\mathcal{M}$ satisfies $\mathrm{I}\Sigma_1$ there is some element $c \in M$ that codes the fragment of $\mathbf{True}^{\mathcal{M}}_{\Pi_1}$ consisting of elements of $\mathbf{True}^{\mathcal{M}}_{\Pi_1}$ that are below n^*, i.e.,

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- We observe that c_E contains all sentences of the form $\exists y\ \delta(\overline{a}, y) \to \exists y < \overline{b}\ \delta(\overline{a}, y)$ that hold in $\mathcal{M}$, where δ is some Δ_0-formula and $\overline{a}$ and $\overline{b}$ are names for a and b. Within $\mathcal{M}$, we define the "theory" T_0 by:

$$T_0 := \mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1 + \ c.$$

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- Next we rely on a result of Clote-Hájek-Paris that says $\mathrm{I}\Sigma_1 \vdash \mathsf{Con}(\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1 + \textbf{True}_{\Pi_1})$ in order to conclude:
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- We observe that T_0 has a Δ_1-definition in $\mathcal{M}$. Hence by Δ_1^0-comprehension available in WKL_0 we also have:

  $(**) \quad T_0 \in \mathcal{A}$.

New Proof of Tanaka's Theorem (4)

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- We wish to build a chain $\langle \mathcal{N}_n : n \in \omega \rangle$ of internal models within $(\mathcal{M}, \mathcal{A})$, i.e., the elementary diagram $E_n := \mathrm{Th}(\mathcal{N}_n, a)_{a \in N_n}$ of each $\mathcal{N}_n$ is coded as a member of $\mathcal{A}$; note that E_n has all sorts of nonstandard sentences. Enumerate $\mathcal{A}$ as $\langle A_n : n \in \omega \rangle$. Our official requirements for $\langle \mathcal{N}_n : n \in \omega \rangle$ is that for each $n \in \omega$ we have:

(1) $\mathcal{N}_n \models T_0$.
(2) $E_n \in \mathcal{A}$.
(3) $\mathcal{M} \subset_{\mathrm{end}} \mathcal{N}_n \prec \mathcal{N}_{n+1}$.
(4) $A_n \in \mathrm{SSy}_M(\mathcal{N}_{n+1})$.

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$$T_{n+1} := E_n + \{\overline{t} \in_{\mathsf{Ack}} d : t \in A_n\} + \{\overline{t} \notin_{\mathsf{Ack}} d : t \notin A_n\}.$$

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- It is easy to see that T_{n+1} is consistent in the sense of $(\mathcal{M}, \mathcal{A})$ since $(\mathcal{M}, \mathcal{A})$ can verify that T_{n+1} is finitely interpretable in $\mathcal{N}_n$. This allows us to get hold of the desired $\mathcal{N}_{n+1}$ using the compacntess theorem for first order logic that is available in WKL_0. The recursive saturation of $\mathcal{N}_{n+1}$ follows immediately from (2), using a well-known argument.

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- **Theorem** *Suppose $\mathcal{N}$ is a countable model of $\mathrm{I}\Sigma_0 + \mathrm{B}\Sigma_1$ that is recursively saturated, and there are $a < b$ in $\mathcal{N}$ such that $f(a) < b$ for every $\mathcal{N}$-partial recursive function f.* **Then** *there is an initial embedding $\phi : \mathcal{N} \to \mathcal{N}$ with $\phi(a) = a$ and $a < \phi(N) < b$.*

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- Let $J := \phi(N)$, and $I := \phi(M)$. Then $I < J < M$. It is now easy to see that ϕ induces an embedding

$$\widehat{\phi} : (\mathcal{M}, \mathcal{A}) \to (I, \mathcal{A} \restriction I),$$

Controlling Fixed points (1)

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- **Theorem**. *Suppose I is proper cut of $\mathcal{M}$. The following conditions are equivalent.*
  (1) *There is an initial self-embedding $j : \mathcal{M} \to \mathcal{M}$ such that* $\mathrm{I}_{\mathrm{fix}}(j) = I$.

  (2) *I is closed under exponentiation.*

Controlling Fixed points (2)

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- **Theorem**. *Suppose I is proper initial segment of $\mathcal{M}$. The following conditions are equivalent.*

  (1) *There is an initial self-embedding $j : \mathcal{M} \to \mathcal{M}$ such that $\mathrm{Fix}(j) = I$.*

  (2) *I is a strong cut of $\mathcal{M}$, and $I \prec_{\Sigma_1} \mathcal{M}$.*

Controlling Fixed points (3)

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- **Theorem**. *The following conditions are equivalent.*

  (1) *There is an initial self-embedding* $j : \mathcal{M} \to \mathcal{M}$ *such that* $\mathrm{Fix}(j) = K^1(\mathcal{M})$.

  (2) $\mathbb{N}$ *is a strong cut of* $\mathcal{M}$.