SEQUENTIAL THEORIES

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Overview

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What we want to do

We want to extend models of theories that contain a modicum of arithmetic with *a full satisfaction predicate*. This means a satisfaction predicate that works for all formulas that exists according to the model.

Of course, there is an issue about what 'according to the model' means.

Reflection on what is needed to build a satisfaction predicate shows that we need sequences that work for *all* objects of the theory. We also need projection of the sequences using the given numbers.

Historical Remarks

Research in this area started with Krajewski's paper of 1976. His base theories had full induction on the chosen natural numbers. Krajewski followed for a large part earlier work of Montague (1959).

After Krajewski's work, research shifted to models of PA.

Essentially we aim to do a sequel to Krajewski work using a more general idea of what a theory with sequences is, to wit *sequential theories*. This idea originate with Pavel Pudlák (1985), who acknowledges earlier ideas of Vaught. Harvey Friedman had a related notion which was slightly more restrictive.

One basic idea is that our arithmetic part satisfies a weak arithmetic like S^1_2.

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Aims and Claims

The notion of *sequential theory* is an explication of the idea of theory with coding. Sequential theories are precisely designed for the development as such metamathematical assets as partial satisfaction predicates *in* the language and satisfaction predicates *as an extension of* the language.

Sequential theories have very good properties w.r.t. interpretability. For finitely axiomatized sequential theories we have the Friedman Characterization and Friedman's Theorem on Faithful Interpretability.

Two Results of Friedman

Suppose A and B are sequential and finitely axiomatized. Let U have a p-time decidable axiomatization.

Friedman Characterization

$A \rhd B$ iff $\mathsf{EA} \vdash \mathsf{con}_{\rho(A)}(A) \to \mathsf{con}_{\rho(B)}(B)$.

Friedman's Theorem on Faithful Interpretability

If $A \rhd U$, then $A \rhd_{\mathsf{faith}} U$.

Definition

Sequential theories have a very simple definition. We call an interpretation *direct* if it is identity preserving and unrelativised.

A theory is *sequential* iff it directly interprets *Adjunctive Set Theory*, AS.

The theory AS is a one-sorted theory with a binary relation $\in$.

AS1 $\vdash \exists x \forall y \; y \notin x$,

AS2 $\vdash \forall x, y \, \exists z \forall u \, (u \in z \leftrightarrow (u \in x \vee u = y))$.

Note that we do not demand extensionality. The point of the directness of the interpretation is that we want *containers* for all the objects of the theory.

Encore

We may allow the witnessing interpretation to be many-dimensional. If we allow more dimensions we can always eliminate parameters.

We can develop a theory of numbers satisfying S_2^1 in any sequential theory. We develop sequences with projections in these numbers, etc. More below.

Examples

Examples of sequential theories are:

- GB.
- ZF.
- ACA_0.
- Peano Arithmetic PA.
- $\mathrm{I}\Sigma^0_1$.
- PRA.
- Elementary Arithmetic EA (aka Elementary Function Arithmetic EFA, or $I\Delta_0 + \exp$).
- Wilkie and Paris' theory $\mathrm{I}\Delta_0 + \Omega_1$.
- Buss' theory S^1_2 and bi-interpretable variants of it like a theory of strings due to Ferreira, and a theory of sets and numbers due to Zambella.
- PA^-, the theory of discretely ordered commutative semirings with a least element. This was recently shown by Emil Jeřábek.
- Adjunctive Set Theory AS.

Non-Examples

Examples of theories that are not sequential are:

- Presburger Arithmetic PresA.
- Various theories of pairing. E.g. the theory of Cantor pairing with successor.
- RCF the theory of Real Closed Fields.

AFC

For many reasons it is better to work with Adjunctive Frege Class Theory AFC: we have better formula classes, extensionality on classes, etc.

AFC is a two-sorted theory with sorts $\mathfrak{o}$ (objects) and $\mathfrak{c}$ (classes, concepts). This system is $\mathfrak{o}$-directly interpretable in AS. Conversely AS is directly interpretable in AFC.

$\mathfrak{o}$-directly interpretable is interpretable where the interpreted domain $\delta_{\mathfrak{o}}$ of the objects is all objects of the interpreting theory and where identity on objects is interpreted as identity in the interpreting theory.

AFC

AFC1 $\vdash X = Y \leftrightarrow \forall z\,(z \in X \leftrightarrow z \in Y)$,

AFC2 $\vdash \exists X\, \forall x\; x \notin X$,

AFC3 $\vdash \exists X\, \forall z\,(z \in X \leftrightarrow (z \in Y \vee z = y))$,

AFC4 $\vdash \exists x \;\; X \mathrel{F} x$,

AFC5 $\vdash (Y \mathrel{F} x \wedge Z \mathrel{F} x) \to Y = Z$,

AFC6 $\vdash X \preceq \emptyset \to X = \emptyset$,

AFC7 $\vdash \emptyset \preceq X$,

AFC8 $\vdash (x \notin X \wedge y \notin Y) \to ((X \cup \{x\}) \preceq (Y \cup \{y\}) \leftrightarrow X \preceq Y)$.

Formulas

A formula is $\Delta_{0,\Sigma}^{\mathfrak{c}}$ if all quantifier-occurrences that bind a variable of sort $\mathfrak{c}$ are either $\preceq$-bounded or are of the form $\exists Y\,(X \cup \{x\} = Y \wedge \ldots)$ or $\forall Y\,(X \cup \{x\} = Y \rightarrow \ldots)$. We usually will suppress the Σ.

We form e.g. $\Delta_0^{\mathfrak{c}}(\cup)$-formulas, in the presence of an axiom that tells us that $\cup$ is total. This is by also allowing quantifiers of the form $\forall Z\,(X \cup Y = Z \rightarrow \ldots)$ and $\exists Z\,(X \cup Y = z \wedge \ldots)$.

Similarly for the Cartesian product modulo some details connected to the fact that the Cartesian product is not unique.

Extending AFC in a Systematic Way

By contracting the sort class of classes we can extend AFC with axioms that tell us that we have arbitrary unions and products. We can also add the $\Delta^{\mathfrak{c}}_{0,\Sigma}(\cup,\times)$-induction scheme: for ϕ in $\Delta^{\mathfrak{c}}_{0,\Sigma}(\cup,\times)$, possibly with further parameters,

$$\vdash [\phi\emptyset \wedge \forall X\, \forall x\, (\phi X \rightarrow \phi(X \cup \{x\}))] \rightarrow \phi Y.$$

The result of adding $\Delta^{\mathfrak{c}}_{0,\Sigma}(\cup,\times)$-induction to $\mathrm{ACF}_{\Sigma}(\cup,\times)$ is $\mathrm{I}\Delta^{\mathfrak{c}}_{0,\Sigma}(\cup,\times)$.

Note that we only get the full schemes *in the limit*: the theory with the full scheme and rule is generally only locally interpretable.

We can use this setting to develop e.g. syntax with domain constants in a non-trifling way.

Interpreting Union

We contract our classes to $\mathcal{X}$, the virtual class of all classes X such that for all Y, $X \cup Y$ exists. It is easy to see that the empty class and singletons are in $\mathcal{X}$. Suppose that $X_0, X_1 \in \mathcal{X}$. If follows that $(X_0 \cup X_1)$ exists. Moreover, for any Y, clearly $X_0 \cup (X_1 \cup Y)$ exists. But $(X_0 \cup X_1) \cup Y = X_0 \cup (X_1 \cup Y)$. So $(X_0 \cup X_1) \in \mathcal{X}$.

To show that all our previous properties are preserved we have to prove that $\mathcal{X}$ is downward closed under $\preceq$. We can indeed prove that using some further lemma's.

Our Provisional Extension AFC

We will work in the following extension of AFC that is $\mathfrak{o}$-directly interpretable in AFC.

First, we will assume that there is a sort $\mathfrak{n}$ of numbers and that for this sort we have S^1_2.

Secondly, we will assume that for any set of numbers coded as a number there is a corresponding class with the same elements.

Thirdly, we will assume elementary properties like closure of the classes under intersection, union, subtraction, cartesian product, taking domains and ranges of relations, domain-restriction, composition of relations.

Fourthly we will assume that $\preceq$ is equivalent to the existence of an injective embedding.